COMMUTING CONTRACTIONS の SIMULTANEOUS UNITARY DILATION

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The following matter is really fundamental:

Sz.-Nagy's Unitary Dilation Theorem. Let $T$ be a contraction on a Hilbert space $\mathcal{H}$. Then, there exist an enlarged Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a unitary $U$, called a unitary dilation of $T$, on $\mathcal{K}$, such that

$$T^m = P U^m |\mathcal{H} \text{ for } m = 0, 1, 2, \cdots,$$

where $P$ is the projection on $\mathcal{K}$ onto $\mathcal{H}$.

This yields, and, is yielded by, the so-called

von Neumann Inequality. Let $T$ be a contraction on a Hilbert space. Then,

$$||p(T)|| \leq ||p|| = \sup_{z \in T} |p(z)|$$

holds for any polynomial $p$ with complex coefficients.

The "logical equivalence" is accompanied by the following
**Theorem** [6]. If a set of commuting contractions on a Hilbert space $\mathcal{H}$, $T_1, T_2, \cdots, T_n$, admits a simultaneous unitary dilation, namely, there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and commuting unitaries $U_1, U_2, \cdots, U_n$ on $\mathcal{K}$, such that

$$T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n} = P U_1^{m_1} U_2^{m_2} \cdots U_n^{m_n} |\mathcal{H}$$

for $m_1, m_2, \cdots, m_n = 0, 1, 2, \cdots$, where $P$ is the projection on $\mathcal{K}$ onto $\mathcal{H}$, then $T_1, T_2, \cdots, T_n$ enjoys the von Neumann inequality, namely,

$$\|(p_{ij}(T_1, T_2, \cdots, T_n))\| \leq \|(p_{ij})\| = \sup_{z_1, z_2, \cdots, z_n \in T} \|(p_{ij}(z_1, z_2, \cdots, z_n))\|$$

holds for any $m \times m$ matrix $(p_{ij})$ whose entries are polynomials with complex coefficients; and *vice versa*.

On the other hand, the following theorems are known:

**Andô's Theorem** [1]. Any pair of commuting contractions on a Hilbert space admits a simultaneous unitary dilation.

**Andô's Theorem** [2]. Any triple of commuting contractions on a Hilbert space, one of which double commutes with others, admits a simultaneous unitary dilation.

We, aside, have examples of triples of commuting contractions which do not admit a simultaneous unitary dilation, [4], [8] and [9].

In [6] we gave the following theorem and corollary:

**Theorem.** Suppose each of sets of commuting contractions, $S_1, S_2, \cdots, S_m$ and $T_1, T_2, \cdots, T_n$, on a Hilbert space, admits a simultaneous unitary dilation, and every $S_j$ double commutes with all $T_k$. If the set $S_1, S_2, \cdots, S_m$ generates a nuclear $C^*$ algebra, then the set $S_1, S_2, \cdots, S_m, T_1, T_2, \cdots, T_n$ admits a simultaneous unitary dilation.

**Corollary.** Suppose $S$ is a GCR contraction, i.e., a contraction which generates a GCR (postliminal) algebra, $T_1, T_2, \cdots, T_n$ commuting contractions, on a Hilbert space, the set $T_1, T_2, \cdots, T_n$ admits a simultaneous unitary dilation.
dilation. and $S$ double commutes with all $T_k$. Then the set $S, T_1, T_2, \ldots, T_n$ admits a simultaneous unitary dilation.

The following, furthermore, turned out to be true [7]:

**Theorem.** Suppose each of sets of commuting contractions, $S_1, S_2, \ldots, S_m$ and $T_1, T_2, \ldots, T_n$, on a Hilbert space, admits a simultaneous unitary dilation, and every $S_j$ double commutes with all $T_k$. If the set $S_1, S_2, \ldots, S_m$ generates an injective von Neumann algebra, then the set $S_1, S_2, \ldots, S_m, T_1, T_2, \ldots, T_n$ admits a simultaneous unitary dilation.

**Corollary.** Suppose $S$ is a type I contraction, i.e., a contraction which generates a type I von Neumann algebra, $T_1, T_2, \ldots, T_n$ commuting contractions, on a Hilbert space, the set $T_1, T_2, \ldots, T_n$ admits a simultaneous unitary dilation and $S$ double commutes with all $T_k$. Then, the set $S, T_1, T_2, \ldots, T_n$ admits a simultaneous unitary dilation.

We here will improve the theorem, by making the assumption thin as the following

**Theorem.** Suppose each of sets of commuting contractions, $S_1, S_2, \ldots, S_m$ and $T_1, T_2, \ldots, T_n$, on a Hilbert space, admits a simultaneous unitary dilation, and every $S_j$ double commutes with all $T_k$. Then, the set $S_1, S_2, \ldots, S_m, T_1, T_2, \ldots, T_n$ admits a simultaneous unitary dilation.

This is the aimed theorem of ours. A proof of this is given, on account of the Steinspring representation of completely positive maps, by the preceding theorem and the

**Arveson Theorem** [3, Theorem 1.3.1]. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, $V$ a bounded operator from $\mathcal{H}$ into $\mathcal{K}$, and $B$ a *subalgebra of $\mathcal{B}(\mathcal{K})$, the full operator algebra, which satisfies that $[BV\mathcal{H}] = \mathcal{K}$. Then, for every $T \in (V^*BV)'$ there exists a unique $\tilde{T} \in B'$ such that $\tilde{T}V = VT$, and the mapping ( )$^\sim$: $(V^*BV)' \rightarrow B'$ is a $\sigma$ weakly continuous *homomorphism.
We have as well

**Corollary.** Suppose each of pairs of commuting contractions, $S_1$, $S_2$, and $T_1, T_2$, on a Hilbert space, admits a simultaneous unitary dilation, and each of $S_1$, $S_2$ double commutes with $T_1, T_2$. Then, the set $S_1, S_2, T_1, T_2$ admits a simultaneous unitary dilation.

Our theorem, of course, gives a good understanding to Andô's "triple" assertion; on the Andô's "pair" assertion, the next matter sheds light:

**Theorem** [5, Theorem 6]. Let $T$ be a contraction on a Hilbert space $\mathcal{H}$, $U$ the minimal unitary dilation of $T$. Then for every $S \in \{T\}'$ there exists $\tilde{S} \in \{U\}'$ such that $S = P\tilde{S}|\mathcal{H}$ and $||\tilde{S}|| = ||S||$.

**References**


