数理解析研究所講究録
第 862 巻 1994 年 237-248



# Free Boundary Problems for the Incompressible Euler Equations

Atusi TANI (谷 温之)
Department of Mathematics, Keio University

## § 1. Introduction

In this communication we are concerned with discussing the solvability of the non-stationary free boundary problems governing the motion of an incompressible inviscid fluid.

Let $\Omega(t)\subset\mathbf{R}^3$ be the domain occupied by a fluid at the moment $t>0$, which is bounded by a hard bottom $S_B$ and a free surface $S_F(t)$. The motion of such a fluid is described by the equations

$$(1)\qquad \frac{\partial v}{\partial t}+(v\cdot\nabla)v+\nabla p=g\nabla U+f,\quad \nabla\cdot v=0\quad (x\in\Omega(t),\ t>0).$$

The initial and boundary conditions are

$$v|_{t=0}=v_0(x)\ (x\in\Omega\equiv\Omega(0)),\quad F|_{t=0}=F_0(x)\ (x\in S_F\equiv S_F(0)),$$

$$(2)\qquad p=p_e,\quad \frac{\partial F}{\partial t}+(v\cdot\nabla)F=0\quad (x\in S_F(t),\ t>0),$$

$$v\cdot n=0\quad (x\in S_B,\ t>0).$$

Here $v(x,t)=(v_1,v_2,v_3)$ is a vector field of velocities; $p(x,t)$ is a pressure; $g$ is the gravitational constant; $U$ is a potential; $f$ is a vector field of external forces; $p_e$ is an atmospheric pressure; $F=F(x,t)$ is a function describing the free surface $S_F(t)$; $n=n(x)$ is the unit vector of the outward normal to $S_B$.

The density of mass is assumed to be equal to one.

Now we mention the following typical problems:

(I) Find $v$, $p$ and $\Omega(t)$ satisfying (1)-(2) under the assumptions that both $S_F(t)$ and $S_B$ are non-compact and $U=-x_3$ with coordinates $x=(x_1, x_2, x_3)$ taken so that $x_3$ is the vertical component.

(II) Find $v$, $p$ and $\Omega(t)$ satisfying (1)-(2) under the assumptions that both $S_F(t)$ and $S_B$ are compact, and $U=1/|x|$ with the origin of coordinates lying in $S_B$.

(III) Find $v$, $p$ and $\Omega(t)$ satisfying (1)-(2) under the assumptions that $S_F(t)$ is compact, $S_B=\phi$ and $U=\int_{\Omega(t)} \frac{dy}{|x-y|}$.

As usual, we introduce the characteristic transformation $\Pi_\xi^x : x \to \xi$,

$$(3) \quad x=\xi+\int_0^t u(\xi, \tau)\, d\tau \equiv X_u(\xi, t), \quad u(\xi, t)=\Pi_\xi^x v(x, t),$$

which is one-to-one mapping from $\Omega(t)$ onto $\Omega$ for each $t>0$ because of the boundary conditions in (2) under the suitable smoothness assumption of $v$. By this mapping $\Pi_\xi^x$ (1)-(2) is transformed into the equations

$$(4) \quad \begin{cases} \dfrac{\partial u}{\partial t}+\nabla_u q=g\nabla_u U^{(u)}, \quad \nabla_u \cdot u=0 \quad (\xi\in\Omega,\ t>0), \\ q=p_e^{(u)} \quad (\xi\in S_F,\ t>0), \quad u\cdot n^{(u)}=0 \quad (\xi\in S_B,\ t>0), \\ u|_{t=0}=v_0 \quad (\xi\in\Omega), \end{cases}$$

where $q(\xi, t)=\Pi_\xi^x p(x, t)$, $(p_e^{(u)}, U^{(u)})(\xi, t)=(p_e, U)(X_u(\xi, t), t)$, $n^{(u)}(\xi, t)=n(X_u(\xi, t))$; $\nabla_u=\mathcal{G}^{(u)}\nabla$, $\nabla=(\nabla_1, \nabla_2, \nabla_3)$, $\nabla_j=\partial/\partial\xi_j\ (j=1, 2, 3)$, $\mathcal{G}^{(u)}={}^t(\partial X_u/\partial\xi)^{-1}=(\delta_{jk}+\int_0^t \nabla_j u_k\, d\tau)^{-1}$ (cf. [5]).

Now we introduce the function spaces. Let $G$ be a domain in $\mathbf{R}^3$ and $l$ be non-negative. By $C^l(G)$ and $W_p^l(G)$ $(p>1)$ we mean the usual Hölder and Sobolev-Slobodetskiĭ spaces, respectively. Let $X$ be a Banach space, $k$ be a non-negative integer and $T>0$. By $C^k([0, T]; X)$ and $L^\infty(0, T; X)$ we mean, respectively, the spaces of all $X$-valued functions $u(t)$ which belong to $C^k([0, T])$ and $L^\infty(0, T)$. The same notation will be used for the spaces of vector fields, the norms of a vector supposed to be equal to the sum of the norms of all its components.

Since the problems (I)–(III) can be discussed in a similar way, here we confine ourselves to the problem of type (I).

The following is our main result:

Theorm: Let $g$ and $p_e$ be positive constants, $T>0$ and $0<\alpha<1$. Suppose that $S_F\in C^{3+\alpha}$ and $S_B\in C^{4+\alpha}$ be located in $\{x\in\mathbf{R}^3 \mid x_3\geqq 0\}\cap\{(x', b) \mid |x'|>K\}$ and $\{x\in\mathbf{R}^3 \mid x_3\leqq -h\}\cap\{(x', -h') \mid |x'|>K\}$ for some non-negative constant $b$ and positive constants $h$, $h'$ and $K$, respectively. Then for any $v_0\in C^{2+\alpha}(\Omega)$ with $\nabla\cdot v_0=0$ in $\Omega$ and $v_0\cdot n|_{S_B}=0$, there exists a solution $(u, q)$ to the problem (4) such that $u\in C([0, T^*]; C^{2+\alpha}(\Omega))$, $u_t\in L^\infty(0, T^*; C^{1+\alpha}(\Omega))$ and $q\in L^\infty(0, T^*; C^{2+\alpha}(\Omega))$ for some $T^*\in(0, T]$.

The solution to the free boundary problem (1)-(2) is given by

(5) $(v, p)(x, t)=(u, q)(X_u^{-1}(x, t), t),\quad \Omega(t)=X_u\Omega.$

Remarks: (i) The before-mentioned theorem with the Hölder spaces $C^{k+\alpha}(\Omega)$ replaced by the Sobolev-Slobodetskiĭ spaces $W_r^{k+l}(\Omega)$ $(r>1,$ $l>\max(0, 3/r-1))$ can be proved only in the case $S_B=\phi$. When $S_B\neq\phi$, the problem is open. On the other hand for the problems of type (II) and (III) the similar results as above hold not only in the Hölder spaces but also in the Sobolev-Slobodetskiĭ spaces with respect to the space variables.

(ii) The problem of the uniqueness of the solution is still open.

In 1982 Ton obtained in [6] the same result as the above theorem in case $S_B=\{x\in\mathbf{R}^3 \mid x_3=0\}$ by using Nalimov's result([2]). The aim of the present paper is to improve his result. In 1987 Ebin ([1]) gave an example of the initial data for which the problem (1)-(2) of type (III) with $g=0$ is not well-posed. However, the solution that he constructed does not belong to our solution space. Therefore our result does not contradict his.

## § 2. Linearized problem

In this section we consider the linearized problem of (4):

$$(6)\begin{cases} \dfrac{\partial u}{\partial t}+\nabla_w q=-g\boldsymbol{e}, \quad \nabla_w\cdot u=0 \quad \text{in} \quad Q_T\equiv\Omega\times(0,T), \\ q=p_e \ \text{ on } \ S_{F,T}\equiv S_F\times(0,T), \quad u\cdot n^{(w)}=0 \ \text{ on } \ S_{B,T}\equiv S_B\times(0,T), \\ \qquad u|_{t=0}=v_0 \ \text{ on } \ \Omega, \end{cases}$$

for a suitably given $w$ (see below), where $\boldsymbol{e}={}^t(0,0,1)$.

It is easily seen that if $w\in C([0,T];C^{1+\alpha}(\Omega))$ satisfy

$$(7) \qquad T\|w\|_{C([0,T];C^{1+\alpha}(\Omega))}\leqq\delta<\delta_0$$

and $\tilde{w}\cdot n(X_w^{-1})=0$ on $S_{B,T}$, $\tilde{w}(x,t)=w(X_w^{-1}(\xi,t),t)$, where $\delta_0$ being the positive root of $1-3x-6x^2-6x^3=0$, then $\mathcal{G}^{(w)}$ is well-defined and the following inequalities hold for some constant $C_1$ independent of $t$ (see, [5]):

$$(8) \qquad \begin{cases} \|\mathbf{I}-\mathcal{G}^{(w)}\|_{C([0,t];C^{\alpha}(\Omega))}\leqq C_1\delta^2, \\ \|\dfrac{\partial}{\partial t}\mathcal{G}^{(w)}\|_{C([0,t];C^{\alpha}(\Omega))}\leqq C_1\|w\|_{C([0,t];C^{1+\alpha}(\Omega))} \end{cases}$$

$$(\forall t\in(0,T)).$$

Theorem A: Let $v_0$, $p_e$, $S_F$ and $S_B$ be as in Theorem. Suppose that $w\in C([0,T];C^{3+\alpha}(\Omega))\cap C^1([0,T];C^{2+\alpha}(\Omega))$ satisfy $w|_{t=0}=$ $=v_0$, $\tilde{w}\cdot n(X_w^{-1})|_{S_{B,T}}=0$ and the inequality

$$(9) \quad T\{\|w\|_{C([0,T];C^{3+\alpha}(\Omega))}+\|w_t\|_{C([0,T];C^{2+\alpha}(\Omega))}\}\leqq\delta<\delta_0.$$

Then there exists a unique solution $(u,q)$ to the initial-boundary value problem (6) such that

$$u\in C^1([0,T];C^{2+\alpha}(\Omega)), \quad q\in C([0,T];C^{3+\alpha}(\Omega)).$$

As preliminaries we consider the linear problem:

$$(10)\quad \begin{cases} \dfrac{\partial u}{\partial t} + \nabla q = f, \quad \nabla\cdot u = \rho \quad \text{in} \quad Q_T, \\ q\,|_{S_{F,T}} = q_e, \quad u\cdot n\,|_{S_{B,T}} = d, \quad u\,|_{t=0} = u_0 \text{ on } \Omega. \end{cases}$$

Lemma: Let $S_F$ and $S_B$ be as in Theorem. Suppose that $f\in C([0,T];C^{2+\alpha}(\Omega))$, $\rho\in C^1([0,T];C^{1+\alpha}(\Omega))$, $q_e\in C([0,T];C^{3+\alpha}(S_F))$ and $d\in C^1([0,T];C^{1+\alpha}(S_B))$. Then for any $u_0\in C^{2+\alpha}(\Omega)$ satisfying the conditions $u_0\cdot n|_{S_B} = d|_{t=0}$ and $\nabla\cdot u_0 = \rho|_{t=0}$ there exists a unique solution $(u, q)$ of (10) such that

$$\begin{aligned} \mathscr{P}_l(u, q; t) &\equiv \|u\|_{C^1([0,t];C^{l+1+\alpha}(\Omega))} + \|q\|_{C([0,t];C^{l+2+\alpha}(\Omega))} \leqq \\ &\leqq C_2\{\|f\|_{C([0,t];C^{l+1+\alpha}(\Omega))} + \|\rho\|_{C^1([0,t];C^{l+\alpha}(\Omega))} + \\ &\quad + \|q_e\|_{C([0,t];C^{l+2+\alpha}(S_F))} + \|d\|_{C^1([0,t];C^{l+1+\alpha}(S_B))} + \\ &\quad + \|u_0\|_{C^{l+1+\alpha}(\Omega)}\} \quad (l=0,1;\ \forall t\in(0,T)). \end{aligned}$$

The constant $C_2$ is independent of $t$.

Proof. In the same way as in [6], the solution $(u, q)$ of (10) is given in the form $u = u' + u''$, $u'(\xi, t) = \nabla\varphi(\xi, t) - \nabla\varphi(\xi, 0) + u_0(\xi)$, $u''(\xi, t) = \int_0^t (f - \nabla q - \frac{\partial u'}{\partial \tau})\, d\tau$, where $\varphi$ and $q$ are, respectively, solutions to the boundary value problems

$$(11)\quad \Delta\varphi = \rho \text{ in } \Omega, \quad \varphi|_{S_F} = 0, \quad n\cdot\nabla\varphi|_{S_B} = d,$$

and

$$(11)'\quad \Delta q = \nabla\cdot f - \rho_t \text{ in } \Omega, \quad q|_{S_F} = q_e, \quad n\cdot\nabla q|_{S_B} = f\cdot n - d_t.$$

The unique existence of the solutions to (11) and (11)′ and the estimates for $\mathscr{P}_l$ ($l = 0, 1$) are well-known from the general boundary value problem for the elliptic system (cf. [3]). □

Proof of Theorem A. We use the successive approximation method. Let $(u^0, q^0)$ be the solution of (10) with $(f, \rho, q_e, d, u_0) \equiv (0, 0, p_e, 0, v_0)$ and $(u^k, q^k)$ be the solution of the initial-boundary value problem

$$(12)^k \begin{cases} \dfrac{\partial u^k}{\partial t} + \nabla q^k = -g\boldsymbol{e} + (\mathrm{I} - \mathscr{G}^{(w)})\nabla q^{k-1} \equiv f^k \quad \text{in} \quad Q_T, \\ \nabla \cdot u^k = (\mathrm{I} - \mathscr{G}^{(w)})\nabla \cdot u^{k-1} \equiv \rho^k \quad \text{in} \quad Q_T, \\ q^k = p_e^{(w)} \quad \text{on} \quad S_{F,T}, \\ u^k \cdot n(\xi) = u^{k-1} \cdot (n(X_w(\xi, t)) - n(\xi)) \equiv d^k \quad \text{on} \quad S_{B,T}, \\ u^k|_{t=0} = v_0 \quad \text{on} \quad \Omega \ (k = 1, 2, 3, \cdots). \end{cases}$$

Lemma guarantees the existence of $(u^k, q^k)$ $(k = 0, 1, 2, \cdots)$. Evaluating each term in the right hand side with the help of (8) and

$$\|w\|_{C([0,t];C^{2+\alpha}(\Omega))}\|u\|_{C([0,t];C^{2+\alpha}(\Omega))} \leqq$$
$$\leqq C_3\, t\, \{\|w\|_{C([0,t];C^{3+\alpha}(\Omega))} + \|w_t\|_{C([0,t];C^{2+\alpha}(\Omega))}\} \times$$
$$\times \{\|u_t\|_{C([0,t];C^{2+\alpha}(\Omega))} + \|v_0\|_{C^{2+\alpha}(\Omega)}\} + C_4 \|v_0\|^2_{C^{2+\alpha}(\Omega)}$$

obtained from the initial conditions $w|_{t=0} = u|_{t=0} = v_0$, we get

$$(13) \qquad \mathscr{P}_1(u^k, q^k; t) + \|u_0\|_{C^{2+\alpha}(\Omega)} \leqq$$
$$\leqq C_5\, t\, \{\|w\|_{C([0,t];C^{3+\alpha}(\Omega))} + \|w_t\|_{C([0,t];C^{2+\alpha}(\Omega))}\} \times$$
$$\times \{\mathscr{P}_1(u^{k-1}, q^{k-1}; t) + \|v_0\|_{C^{2+\alpha}(\Omega)}\} + C_6 \quad (\forall t \in (0, T)).$$

Then the sequence $\{(u^k, q^k)\}_{k=0}^{\infty}$ is well-defined and converges provided that $T$ is chosen so small (say, $T_0$) that the inequality $C_5\delta \leqq 1 - C_6/M$ for any fixed $M > C_6$. In fact, the convergence results from the inequality derived from (13) with $(g, p_e) = (0, 0)$ and $v_0 \equiv 0$:

$$(14)\ \mathscr{P}_1(u^k - u^{k-1}, q^k - q^{k-1}; t) \leqq C_5\,\delta\,\mathscr{P}_1(u^{k-1} - u^{k-2}, q^{k-1} - q^{k-2}; t).$$

Hence $\sum_{k=1}^{\infty}\mathscr{P}_1(u^k-u^{k-1}, q^k-q^{k-1}; t)<\infty$ $(0<t<T_0)$ because $C_5\delta<1$. Namely, we can find the limit function $(u, q)$ such that $u^k\to u$ in $C^1([0, t]; C^{2+\alpha}(\Omega))$, $q^k\to q$ in $C([0, t]; C^{3+\alpha}(\Omega))$ as $k\to\infty$ for $0<t<T_0$.

It is trivial to check that $(u, q)$ is a solution to (6) on $(0, T_0)$. This solution can be extended on any time interval $(0, T)$ by the standard method (cf. [4]). The uniqueness follows from (14) as usual. □

Remark: Under the weaker condition $T\{\|w\|_{C([0,T];C^{2+\alpha}(\Omega))}+$ $+\|w_t\|_{C([0,T];C^{1+\alpha}(\Omega))}\}\leqq\delta$, similar inequality to (13) holds:

$$(13)'\qquad \mathscr{P}_0(u^k, q^k; t)+\|v_0\|_{C^{1+\alpha}(\Omega)}\leqq$$
$$\leqq C_5' t\{\|w\|_{C([0,t];C^{2+\alpha}(\Omega))}+\|w_t\|_{C([0,t];C^{1+\alpha}(\Omega))}\}\times$$
$$\times\{\mathscr{P}_0(u^{k-1}, q^{k-1}; t)+\|v_0\|_{C^{1+\alpha}(\Omega)}\}+C_6'.$$

§ 3. A priori estimates

For the nonlinear problem it is necessary to get the a priori estimates for the solution to problem (6) under the weaker hypothesis on $w$ (cf. [6]).

Theorem B: Let $(v_0, p_e, S_F, S_B)$ be as in Theorem A. Suppose that

(i) $T\{\|w\|_{C([0,T];C^{2+\alpha}(\Omega))}+\|w_t\|_{L^\infty(0,T;C^{1+\alpha}(\Omega))}\}\leqq\delta<\delta_0$,

(ii) $\tilde{w}\cdot n(X_w^{-1})|_{S_{B,T}}=0$, $w|_{t=0}=v_0$ on $\Omega$.

Let $(u, q)$ be a solution to the initial-boundary value problem (6) with $u\in C([0,T];C^{2+\alpha}(\Omega))$, $u_t\in L^\infty(0,T;C^{1+\alpha}(\Omega))$ and $q\in L^\infty(0,T;C^{2+\alpha}(\Omega))$. Then the following estimates hold:

$$(15)\quad \begin{cases} \|u(\cdot,t)\|_{C^{2+\alpha}(\Omega)}\leqq(1+\|v_0\|_{C^{2+\alpha}(\Omega)})\exp[t\,\mathcal{U}(t)], \\ \|u_t(\cdot,t)\|_{C^{1+\alpha}(\Omega)}+\|q(\cdot,t)\|_{C^{2+\alpha}(\Omega)}\leqq \\ \qquad\leqq C_7(1+\|v_0\|_{C^{2+\alpha}(\Omega)})\,\mathcal{U}(t)\exp[t\,\mathcal{U}(t)], \end{cases}$$

where $\mathcal{U}(t)=C_8(1+\|w\|_{C([0,t];C^{2+\alpha}(\Omega))})$, and $C_7$, $C_8$ are constants depending on $\delta$ but not on $t<T$.

Proof. First of all, let $w$ satisfy the condtion

(i)′ $T\{\|w\|_{C([0,T];C^{3+\alpha}(\Omega))}+\|w_t\|_{L^\infty(0,T;C^{2+\alpha}(\Omega))}\}\leqq\delta<\delta_0$,

in place of (i). By going back to the $x$-variables and then taking the divergence, we get

$$(16)\begin{cases}\Delta p=-\sum_{i,j=1}^{3}\nabla_i\tilde{w}_j\cdot\nabla_j v_i \quad (x\in X_w\Omega\equiv\Omega(t)),\\ p=p_e \quad (x\in X_w S_F\equiv S_F(t)),\quad n\cdot\nabla p=(\tilde{w}\cdot\nabla)n-g n_3 \quad (x\in S_B),\end{cases}$$

where $(v,p)(x,t)=(u,q)(X_w^{-1}(x,t),t)$, because $\tilde{w}\cdot\nabla$ is the tangential derivative on $S_B$ according to the hypothesis (ii). For this problem it is well-known (cf. [3]) that the following inequality holds with the help of (8) and (i)′:

$$(17)\quad \|p(\cdot,t)\|_{C^{3+\alpha}(\Omega(t))}+\|q(\cdot,t)\|_{C^{2+\alpha}(\Omega)}\leqq$$
$$\leqq C_9(\delta)\{1+\|w(\cdot,t)\|_{C^{2+\alpha}(\Omega)}\|v(\cdot,t)\|_{C^{2+\alpha}(\Omega)}\}.$$

Next extending $w$ to $\mathbf{R}^3$, we define $w^\varepsilon$ as the average of $w$ with an infinitely differentiable kernel depending only on $\xi$ (cf. [4]). Then $w^\varepsilon\in C([0,T];C^{3+\alpha}(\Omega))$, $w_t^\varepsilon\in L^\infty(0,T;C^{2+\alpha}(\Omega))$. Moreover, $w^\varepsilon\to w$ in $C([0,T];C^{2+\alpha}(\Omega))$, $w_t^\varepsilon\to w_t$ in $L^\infty(0,T;C^{1+\alpha}(\Omega))$. We take $T_\varepsilon>0$ so that

$$T_\varepsilon\{\|w^\varepsilon\|_{C([0,T_\varepsilon];C^{3+\alpha}(\Omega))}+\|w_t^\varepsilon\|_{L^\infty(0,T_\varepsilon;C^{2+\alpha}(\Omega))}\}\leqq\delta<\delta_0.$$

Then the equation corresponding to (6) becomes

$$\begin{cases}\dfrac{\partial v^\varepsilon}{\partial t}+(\tilde{w}^\varepsilon\cdot\nabla)v^\varepsilon=-\nabla p^\varepsilon-g\boldsymbol{e}\equiv f^\varepsilon,\quad \nabla\cdot v^\varepsilon=0 \ (x\in X_{w^\varepsilon}\Omega,\ t>0),\\ v^\varepsilon|_{t=0}=v_0 \quad (x\in\Omega).\end{cases}$$

Here $p^{\varepsilon}$ is the solution of (16) with $w=w^{\varepsilon}$.

Returning to the $\xi$-variables, we obtain

$$u^{\varepsilon}(\xi, t)=v_0(\xi)+\int_0^t F^{\varepsilon}(\xi, \tau)\, d\tau, \quad F^{\varepsilon}(\xi, t)=f^{\varepsilon}(X_{w^{\varepsilon}}, t),$$

which implies, with the help of (17) and the Gronwall inequality,

$$(18) \quad \|u^{\varepsilon}(\cdot, t)\|_{C^{2+\alpha}(\Omega)} \leqq$$

$$\leqq (1+\|v_0\|_{C^{2+\alpha}(\Omega)}) \exp\{C_{10}(\delta)\, t\, (1+\|w\|_{C([0, t]; C^{2+\alpha}(\Omega))})\}$$

for any $t \in (0, T_{\varepsilon}]$. Here $C_{10}$ does not depend on $\varepsilon$ and $t$. Now we do the same argument on $[T_{\varepsilon}, T_{2\varepsilon}]$ and after a finite number of steps $N_{\varepsilon}$ $(\leqq [T/T_{\varepsilon}]+1)$, we finally arrive at the estimate (18) on $[0, T]$. Letting $\varepsilon \to 0$ we obtain the desired estimate (15)'.

From (15)' and (17) we have

$$(19) \quad \|q(\cdot, t)\|_{C^{2+\alpha}(\Omega)} \leqq$$

$$\leqq C_9(\delta)\{1+\|w\|_{C([0, T]; C^{2+\alpha}(\Omega))}(1+\|v_0\|_{C^{2+\alpha}(\Omega)}) \exp[t\, \mathcal{U}(t)]\}$$

$$\leqq C_{11}(\delta)(1+\|v_0\|_{C^{2+\alpha}(\Omega)})\, \mathcal{U}(t) \exp[t\, \mathcal{U}(t)]\}.$$

The estimate for $\|u(\cdot, t)\|_{C^{1+\alpha}(\Omega)}$ easily follows from (6), (8) and (19). □

§ 4. Proof of Theorem

We begin with preparing the extended version of Theorem A:

Theorem C: Under the hypotheses in Theorem B, there exists a unique solution $(u, q)$ to the initial-boundary value problem (6). Moreover, the solution satisfies the inequality (15) for any $t \in (0, T)$.

Proof. Extend $w \in C([0, T]; C^{2+\alpha}(\Omega))$ with $w_t \in L^{\infty}(0, T; C^{1+\alpha}(\Omega))$ to $\mathbf{R}^3$. Let $w^{\varepsilon}$ be the same function as that in the proof of Theorem B. According to Theorem A, there exists a unique solution $(u^{\varepsilon}, q^{\varepsilon})$ to the problem (6) with $w$ replaced by $w^{\varepsilon}$. Theorem B implies that the inequality (15) holds for $(u^{\varepsilon}, q^{\varepsilon})$ with $w$ replaced by $w^{\varepsilon}$, namely,

$$\|u^{\varepsilon}\|_{C([0,t];C^{2+\alpha}(\Omega))}+\|u_t^{\varepsilon}\|_{L^{\infty}(0,t;C^{1+\alpha}(\Omega))}+ \\ +\|q^{\varepsilon}\|_{L^{\infty}(0,t;C^{2+\alpha}(\Omega))}\leqq M \quad (\forall t\in(0,T))$$

for some positive constant $M$ independent of $t$ and $\varepsilon$. This yields

$$\|u^{\varepsilon}\|_{C^{\beta}([0,t];C^{1+\alpha}(\Omega))}\leqq M' \quad (\forall t\in(0,T)) \tag{20}$$

for any $\beta$, $0<\beta<1$, and some positive constant $M'$ indedependent of $t$ and $\varepsilon$. Taking a subsequence if necessary, we have $u^{\varepsilon}\to u$, $q^{\varepsilon}\to q$ in the weak* topology of $L^{\infty}(0,T;C^{2+\alpha}(\Omega))$, $u_t^{\varepsilon}\to u_t$ in the weak* topology of $L^{\infty}(0,T;C^{1+\alpha}(\Omega))$ as $\varepsilon\to 0$. It is obvious that

$$\begin{cases} \|u(\cdot,t)\|_{C^{2+\alpha}(\Omega)}\leqq(1+\|v_0\|_{C^{2+\alpha}(\Omega)})\exp[t\,\mathcal{U}(t)], \\ \|u_t(\cdot,t)\|_{C^{1+\alpha}(\Omega)}+\|q(\cdot,t)\|_{C^{2+\alpha}(\Omega)}\leqq \\ \qquad\leqq C_7(1+\|v_0\|_{C^{2+\alpha}(\Omega)})\,\mathcal{U}(t)\exp[t\,\mathcal{U}(t)]. \end{cases}$$

From (20) $u^{\varepsilon}$ also converges to $u$ in $C^{\beta'}([0,t];C^{1+\alpha'}(\Omega))$ for any $\beta'\in(0,\beta)$ and $\alpha'\in(0,\alpha)$. Using this and the interpolation inequality in the Hölder spaces, we obtain $u^{\varepsilon}\to u$ in $C([0,T];C^{2+\alpha'}(\Omega))$. Therefore, by passing to the limit, it is clear that $(u,q)$ is a solution of (6). The convergence of $(u^{\varepsilon},q^{\varepsilon})$ to $(u,q)$ as $\varepsilon\to 0$ is indeed valid without taking a subsequence, since the uniqueness of the solution can be established in the following way. Let $(u,q)$ be a solution of (6) with $(g,p_e)=(0,0)$, $v_0\equiv 0$. Then (13)′ yields that $\mathcal{P}_0(u,q;t)\leqq\mathcal{P}_0(u,q;t)/2$ $(0<t<T_0')$ for some constant $T_0'>0$. This means $(u,q)=0$ on $(0,T_0')$. After repeating this step, we finally conclude $(u,q)=0$ on $(0,T)$. □

Now we turn to prove our main theorem. Let $(u^0,q^0)$ be the solution of (10) with $(f, \rho, q_e, d, u_0)=(-g\boldsymbol{e}, 0, p_e, 0, v_0)$ given in Lemma. Lemma also says that there exists a positive constant $K(>1)$ such that

$$(21) \qquad \|u^0\|_{C([0,t];C^{2+\alpha}(\Omega))} \leqq K \quad (\forall t \in (0,T)).$$

Let $T^*$ be a positive number such that

$$(22) \quad 2T^*(1+\|v_0\|_{C^{2+\alpha}(\Omega)})^2 \exp\{C_8[(2+K)T^*+C_{10}+\delta_0]\} \leqq \delta < \delta_0.$$

Such a $T^*$ really exists. Then let $(u^k, q^k)$ be the solution to the initial-boundary value problem (6) with $w=u^{k-1}$ and $T=T^*$ $(k=1, 2, 3, \cdots)$. Theorem C implies, by virtue of (21) and (22), that $(u^1, q^1)$ exists and satisfies the inequalities

$$\begin{cases} \|u^1\|_{C([0,t];C^{2+\alpha}(\Omega))} \leqq (1+\|v_0\|_{C^{2+\alpha}(\Omega)}) \exp[C_8(1+K)t], \\ \|u^1_t\|_{L^\infty(0,t;C^{1+\alpha}(\Omega))} + \|q^1\|_{L^\infty(0,t;C^{2+\alpha}(\Omega))} \leqq \\ \qquad \leqq C_8 C_{10}(1+\|v_0\|_{C^{2+\alpha}(\Omega)})(1+K) \exp[C_8(1+K)t]. \end{cases}$$

Therefore from (22) it follows the hypothesis (i) in Theorem B for $u^1$ and $T=T^*$. Then we may apply Theorem C again so that $(u^2, q^2)$ exists and satisfies the inequalities

$$\begin{aligned} &\|u^2\|_{C([0,t];C^{2+\alpha}(\Omega))} \leqq \\ &\quad \leqq (1+\|v_0\|_{C^{2+\alpha}(\Omega)}) \exp[C_8(1+\|u^1\|_{C([0,t];C^{2+\alpha}(\Omega))})t] \\ &\quad \leqq (1+\|v_0\|_{C^{2+\alpha}(\Omega)}) \exp[C_8(t+\delta)], \end{aligned}$$

$$\begin{aligned} &\|u^2_t\|_{L^\infty(0,t;C^{1+\alpha}(\Omega))} + \|q^2\|_{L^\infty(0,t;C^{2+\alpha}(\Omega))} \leqq \\ &\leqq C_8 C_{10}(1+\|v_0\|_{C^{2+\alpha}(\Omega)})(1+\|u^1\|_{C([0,t];C^{2+\alpha}(\Omega))})\exp[C_8(t+\delta)] \\ &\leqq C_8 C_{10}(1+\|v_0\|_{C^{2+\alpha}(\Omega)})^2 \exp\{C_8[(2+K)t+\delta_0]\} \quad (\forall t \in (0,T^*)). \end{aligned}$$

Whence again for $t<T^*$ with the same $T^*$ as before the hypothesis (i)

in Theorem B is satisfied for $u^2$. By induction $(u^k, q^k)$ for all $k$ exist and satisfy the inequality

$$\|u^k\|_{C([0,t];C^{2+\alpha}(\Omega))}+\|u_t^k\|_{L^\infty(0,t;C^{1+\alpha}(\Omega))}+$$
$$+\|q^k\|_{L^\infty(0,t;C^{2+\alpha}(\Omega))}\leqq K' \quad (\forall t\in(0,T^*)).$$

The constant $K'$ is independent of $t$ and $k$. Letting $k\to\infty$ and an argument as in the proof of Theorem C give the existence of a solution $(u, q)$ of (4) with all stated properties. □

## References


[1] Ebin, D.G., The equations of motion of a perfect fluid with free boundary are not well-posed. Comm. in Part. Diff. Eq. 12 (1987), 1175-1201.

[2] Налимов, В.И., Априорные оценки решений эллиптических уравнений в классе аналитических функций и их приложения к задаче Коши-Пуассона. ДАН СССР 189 (1969), 45-49.

[3] Солонников, В.А., Об общих краевых задачах для систем, эллиптических в смысле А. Дагласа-Л. Ниренберга. I. Изв. АН СССР, сер. матем. 28 (1964), 665-706; II. Труды МИАН 92 (1966), 233-297.

[4] Солонников, В.А., Разрешимость задачи о движении вязкой несжимаемой жидкости, ограниченной свободной поверхностью. Изв. АН СССР, сер. матем. 41 (1977), 1388-1424.

[5] Tani, A., On the first initial-boundary value problem of compressible viscous fluid motion. Publ. RIMS, Kyoto Univ. 13 (1977), 193-253.

[6] Ton, B.A., On a free boundary problem for an inviscid incompressible fluid. Nonlinear Anal. Theo. Math. Appl. 6 (1982), 335-347.