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The Gradient Theory of the Phase Transitions in Cahn-Hilliard Fluids with the Dirichlet boundary conditions

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1. Introduction

In this note we will investigate the asymptotic behavior of minimizer \( \{u_\epsilon\}_{\epsilon > 0} \) (as \( \epsilon \to 0 \)) of the following variational problem:

\[
(P_\epsilon) \quad \inf \left\{ \int_\Omega \left[ \epsilon |\nabla u|^2 + \frac{1}{\epsilon} W(x, u) \right] dx \middle| u \in W^{1,2}(\Omega : \mathbb{R}^n), u = g \text{ on } \partial \Omega \right\},
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with \( C^2 \) smooth boundary \( \partial \Omega \) and \( g \) is a Lipschitz continuous function from \( \partial \Omega \) into \( \mathbb{R}^n \). Here \( W(x, \cdot) \) is a nonnegative continuous function which has two potential wells with equal depth. This type of problem is related to the study of the phase transitions of the Cahn-Hilliard fluids. See [8] and [9].

In [7] R.V. Kohn & P. Sternberg conjectured that minimizer of the variational problem, which is special case of \( (P_\epsilon) \),

\[
(SP_\epsilon) \quad \inf \left\{ \int_\Omega \left[ \epsilon |\nabla u|^2 + \frac{1}{\epsilon} (u^2 - 1)^2 \right] dx \middle| u \in W^{1,2}(\Omega), u|_{\partial \Omega} = g \right\}
\]

converges to a solution of

\[
\inf \left\{ \frac{8}{3} P_\Omega \{u = 1\} + 2 \int_{\partial \Omega} |d(u) - d(g)|d\mathcal{H}_{N-1} \middle| u \in BV(\Omega), |u| = 1 \text{ a.e.} \right\},
\]

where \( d(t) = \int_{-1}^t |s^2 - 1|ds \). Here \( \mathcal{H}_{N-1} \) is the \( N-1 \) dimensional Hausdorff measure.

In this note, we will study the asymptotic behavior of minimizer of \( (P_\epsilon) \), and as a byproduct, we will state the affirmative results to the conjecture in [7].

Recently using the theory of Gamma-convergence, several authors studied the asymptotic behavior of the minimizer of the problem:

\[
(E_\epsilon) \quad \inf \left\{ \int_\Omega \left[ \epsilon |\nabla u|^2 + \frac{1}{\epsilon} W(u) \right] dx \middle| u \in W^{1,2}(\Omega : \mathbb{R}^n), \int_\Omega u(x)dx = m \right\},
\]
where $m$ is a constant vector in $\mathbb{R}^n$. For the scalar case (i.e. $n = 1$), see [8] and [9]. For the vector case (i.e. $n \geq 2$), see [1] and [4]. Our results on the problem $(P_\epsilon)$ depend mainly on the study of asymptotic behavior of minimizer of $(E_\epsilon)$. However there are several different aspects between the asymptotic behavior of minimizer of $(P_\epsilon)$ and that of $(E_\epsilon)$. In fact, minimizer of $(E_\epsilon)$ generates the only interior layer, but minimizer of $(P_\epsilon)$ generates both the interior and the boundary layers as $\epsilon \to 0$.

On the other hand, we can easily see that minimizer of $(SP_\epsilon)$ satisfies the equation:

\[
(CP_{\epsilon}) \quad \begin{cases} 
\epsilon^2 \Delta u - u(u-1)(u+1) = 0 & \text{in } \Omega, \\
 u(x) = g(x) & \text{on } \partial \Omega.
\end{cases}
\]

Then there exist several results for the solutions of $(CP_{\epsilon})$ obtained by using the method of matched expansion. Our results also seem to be closely related to [2] and [3].

We will give the precise conditions of the functions $W(x, u)$ and $g(x)$. Let $W(x, u) : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous nonnegative function, and for any $x \in \overline{\Omega}$, $W(x, u) = 0$ if and only if $u = \alpha$ or $\beta$. Here we note $\alpha$ and $\beta$ are constant vectors independent of $x$. We assume that there exist two constants $K_1$ and $K_2$ such that

\begin{equation}
(1.1) \quad \sup_{u \in [K_1, K_2]^n} W(x, u) \leq W(x, v) \quad \text{for all } x \in \overline{\Omega}, \; v \not\in [K_1, K_2]^n
\end{equation}

and

\begin{equation}
(1.2) \quad g(x) \in [K_1, K_2]^n \quad \text{for all } x \in \partial \Omega.
\end{equation}

Moreover we set $W_{\infty}(\cdot) = \inf_{x \in \overline{\Omega}} W(x, \cdot)$ and assume that for any $\epsilon > 0$ there exists a positive constant $\delta$ such that

\begin{equation}
(1.3) \quad |W^{1/2}(x, u) - W^{1/2}(y, u)| \leq \epsilon W^{1/2}_{\infty}(u)
\end{equation}

for all $x, y \in \overline{\Omega}$ with $|x - y| \leq \delta$ and all $u \in \mathbb{R}^n$. Here from the definition of $W_{\infty}(u)$ and (1.3) we have the following relation

\begin{equation}
(1.3') \quad |W^{1/2}(x, u) - W^{1/2}(y, u)| \leq \epsilon W^{1/2}(x, u)
\end{equation}

for all $x, y \in \overline{\Omega}$ with $|x - y| \leq \delta$ and for all $u \in \mathbb{R}^n$.

We think that the conditions (1.1) and (1.3) are not restrictive. In fact, consider continuous functions $W(u)$, $h(x)$, where $W(u)$ satisfies the condition (1.1) and where $h(x)$ is positive function in $\overline{\Omega}$. If the function $W(x, u)$ has a form of $h(x)W(u)$, then we can see that $W(x, u)$ satisfies the conditions (1.1) and (1.3).
In order to state the main theorem, we will introduce a Riemannian metric on $\mathbb{R}^n$, $d(x, a, b)$ which depends on $x \in \Omega$. For $x \in \Omega$ and $a, b \in \mathbb{R}^n$, let $d(x, a, b)$ be the metric defined by

\[
(1.4) \quad d(x, a, b) = \inf \left\{ \int_0^1 W^{1/2}(x, \gamma(t))|\dot{\gamma}(t)| dt \left| \gamma \in C^1([0,1] : \mathbb{R}^n), \right. \right. \\
\left. \left. \gamma(0) = a, \gamma(1) = b \right\} \right.
\]

For example, in the case of $W(x, u) = (u^2 - 1)^2$ and $n = 1$, we have

\[
d(x, -1, b) = \int_{-1}^{b} |s^2 - 1| ds \quad \text{for} \quad b \geq -1.
\]

We now state our main theorem of this note.

**Theorem 1.** (See [6].) Suppose that function $W$ satisfies (1.1) and (1.3) and that $g$ satisfies (1.2). For $\epsilon > 0$, let $u_\epsilon$ be a solution of the variational problem:

\[
\inf \left\{ \int_\Omega \left[ \epsilon |\nabla u|^2 + \frac{1}{\epsilon} W(x, u) \right] dx \right| \quad u \in W^{1,2}(\Omega : \mathbb{R}^n), \ u|_{\partial\Omega}(x) = g(x) \right\}.
\]

If there exist a positive sequence $\{\epsilon_i\}_{i=1}^{\infty}$ and a function $u_0(x) \in L^1(\Omega : \mathbb{R}^n)$ such that

\[
(1.5) \quad \lim_{i \to \infty} \epsilon_i = 0 \quad \text{and} \quad \lim_{i \to \infty} u_{\epsilon_i} = u_0 \quad \text{in} \ L^1(\Omega : \mathbb{R}^n),
\]

then the function $u_0$ is characterized by

\[
W(x, u_0(x)) = 0 \quad \text{for almost all} \ x \in \Omega, \ \text{that is,} \ u_0(x) = \alpha \text{ or } \beta \text{ for almost all } x \in \Omega.
\]

Moreover the set $E_0 = \{x \in \Omega \mid u_0(x) = \alpha\}$ is a solution of the variational problem $(P_0)$:

\[
(P_0) \quad \inf \left\{ \int_{\Omega \cap \partial^* E} d(x, \alpha, \beta)d\mathcal{H}_{N-1} + \int_{\partial\Omega} d(x, v|_{\partial\Omega}(x), g(x))d\mathcal{H}_{N-1} \left| \right. \right. \\
\left. \left. E \subset \Omega, \ P_{\Omega}(E) < \infty, \ v = \alpha\chi_E + \beta\chi_{\Omega \setminus E} \right\},
\]

where $P_{\Omega}(E)$ is a perimeter of $E$ in $\Omega$ and $v|_{\partial\Omega}$ is the trace of $v$ to $\partial\Omega$. Furthermore we have

\[
\lim_{i \to \infty} \int_\Omega \left[ \epsilon_i |\nabla u_{\epsilon_i}|^2 + \frac{1}{\epsilon_i} W(x, u_{\epsilon_i}) \right] dx = 2 \int_{\Omega \cap \partial^* E_0} d(x, \alpha, \beta)d\mathcal{H}_{N-1} \\
+ 2 \int_{\partial\Omega \cap \partial^* E_0} d(x, \alpha, g(x))d\mathcal{H}_{N-1} + 2 \int_{\partial\Omega \setminus \partial^* E_0} d(x, \beta, g(x))d\mathcal{H}_{N-1}.
\]
Here $\partial^*E_0$ is the reduced boundary of $E_0$.

**Remark.** It is not restrictive to assume that there exists a subsequence $\{u_{\epsilon_i}\}_{i=1}^{\infty}$ satisfying (1.5). In fact, the following is proved in [4] and [5]: if there exist constants $C$ and $R$ such that

\[(1.6) \quad W_\infty(u) \geq C|u| \quad \text{for} \quad |u| \geq R,\]

then there exists a subsequence $\{u_{\epsilon_i}\}_{i=1}^{\infty}$ satisfying (1.5).

It is worth noting that the study of asymptotic behavior of minimizer of $(P_\epsilon)$ occurs a completely different difficulty from that of $(SP_\epsilon)$. One of the difficulties is that the selection of minimizing sequence $\{\gamma_k\}_{k=1}^{\infty}$ achieving the value of $d(x, \alpha, g(x))$ depends on the space variable $x$. In order to overcome this difficulty, we approximate $W(\cdot, u)$ and $g(\cdot)$ by suitable piecewise smooth functions near the transition layer and the boundary $\partial \Omega$.

### 2. The Main Propositions

At first, we will give functionals $F_\epsilon$ and $F_0$ from $L^1(\Omega : \mathbb{R}^N)$ into $[0, \infty]$. For $u \in L^1(\Omega : \mathbb{R}^n)$ and $\epsilon > 0$, we define $F_\epsilon(u)$, $F_0(u)$ by

\[
F_\epsilon(u) = \begin{cases} 
\int_{\Omega} [\epsilon|\nabla u|^2 + \frac{1}{\epsilon}W(x, u)] \, dx, & \text{if } u \in W^{1,2}(\Omega : \mathbb{R}^n) \text{ and } u = g \text{ on } \partial \Omega, \\
\infty, & \text{otherwise},
\end{cases}
\]

\[
F_0(u) = \begin{cases} 
2 \int_{\Omega} d(x, \alpha, \beta)|\nabla \chi_{\{u(x) = 0\}}| + 2 \int_{\partial \Omega} d(x, u|_{\partial \Omega}(x), g(x))d\mathcal{H}_{N-1}, & \text{if } u \in BV(\Omega : \mathbb{R}^n) \text{ and } W(x, u(x)) = 0 \text{ for almost all } x \in \Omega, \\
\infty, & \text{otherwise}.
\end{cases}
\]

In order to prove our main theorem, we need the following two propositions which are crucial in our analysis.

**Proposition A.** Suppose that $\{v_\epsilon\}_{\epsilon>0}$ is a sequence in $L^1(\Omega : \mathbb{R}^n)$ which converges in $L^1(\Omega : \mathbb{R}^n)$ as $\epsilon \to 0_+$ to a function $v_0$. If

\[
\liminf_{\epsilon \to 0_+} F_\epsilon(v_\epsilon) < +\infty,
\]

then $v_0$ is a function in $BV(\Omega : \mathbb{R}^n)$ such that

\[
F_0(v_0) \leq \liminf_{\epsilon \to 0_+} F_\epsilon(v_\epsilon).
\]
Proposition B. Suppose that $w_0 \in L^1(\Omega : \mathbb{R}^n)$ is a function with $w_0 = \alpha \chi_E + \beta \chi_{\Omega \setminus E}$ where $E$ is a measurable subset in $\Omega$ with finite perimeter. Then there exists a sequence $\{w_\epsilon\}_{\epsilon>0}$ in $W^{1,2}(\Omega : \mathbb{R}^n)$ which converges in $L^1(\Omega : \mathbb{R}^n)$ as $\epsilon \to 0_+$ to $w_0$ such that

\begin{equation}
\limsup_{\epsilon \to 0_+} F_\epsilon(w_\epsilon) \leq F_0(w_0).
\end{equation}

Using Propositions A and B, we can prove Theorem 1 as in the same matter with in [8]. Therefore we have only to prove Proposition A and B. In this note, we will only prove Proposition B for the special case.

On the other hand, in Theorem 1, the minimizers $\{u_\epsilon\}_{\epsilon>0}$ do not always generate interior layers. For example, if we consider the problem $(SP_\epsilon)$ with $g \equiv 0$, we have $E_0 = \Omega$ or $\emptyset$. In contrast, considering the family of local minimizers, from Theorem 1 and the results of [7], we obtain the following theorem.

Theorem 2. Let $u_0 \in L^1(\Omega : \mathbb{R}^n)$ be a isolated $L^1$-local minimizer of $F_0$, that is,

there exists a positive constant $\delta$ such that $F_0(u_0) < F_0(v)$

whenever $u \neq v$ and $\|u_0 - v\|_{L^1(\Omega : \mathbb{R}^n)} \leq \delta$.

Then there exist a constant $\epsilon_0 > 0$ and a sequence $\{u_\epsilon\}_{\epsilon<\epsilon_0}$ such that $u_\epsilon$ is a local minimizer of $F_\epsilon$ and $u_\epsilon \to u_0$ in $L^1(\Omega : \mathbb{R}^n)$ as $\epsilon \to 0$.

3. Proof of Proposition B

In this section, we will only prove Proposition B for the special case that $w_0 \equiv \alpha$ in $\Omega$. In order to prove Proposition B for the case of $w_0 \equiv \alpha$, we need the following two lemmas. The first lemma is obtained easily by the inverse mapping theorem.

Lemma 3-1. Let $\Omega$ be a bounded domain with $C^2$-smooth boundary $\partial \Omega$. For $x \in \partial \Omega$ let $\nu(x)$ be a inner normal vector to $\partial \Omega$ at $x$. Define a mapping $\pi : \partial \Omega \times [0, \infty) \to \mathbb{R}^N$ by

\begin{equation}
\pi(x, t) = \pi_t(x) = x + t \nu(x).
\end{equation}

Then there exists a constant $s_0$ such that the image of $\pi$ in $\partial \Omega \times (0, s_0]$ is contained in $\Omega$ and the $C^1$-smooth inverse mapping $\pi^{-1}$ of $\pi$ exists in $\pi(\partial \Omega \times [0, s_0])$.

Lemma 3-2. (See [8] and [9].) Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$ with Lipschitz-continuous boundary. Let $A$ be an open subset of $\mathbb{R}^N$ with $C^2$, compact, nonempty
boundary such that $\mathcal{H}_{N-1}(\partial A \cap \partial \Omega) = 0$. Define a distance function to $\partial A$, $d_{\partial A} : \Omega \to \mathbb{R}$, by $d_{\partial A}(x) = \text{dist}(x, A)$. Then, for some $s_1 > 0$, $d_{\partial A}$ is a $C^2$-function in $\{0 < d_{\partial A}(x) < s_1\}$ with

$$|\nabla d_{\partial A}| = 1.$$  \hspace{1cm} (3.2)

Furthermore, $\lim_{s \to 0} \mathcal{H}_{N-1}(\{d_{\partial A}(x) = s\}) = \mathcal{H}_{N-1}(\partial A \cap \Omega)$ and

$$|\{x \mid d_{\partial A}(x) < s\}| = O(s).$$  \hspace{1cm} (3.3)

By $d_{\partial \Omega}(x)$ we denote a function $\text{dist}(x, \partial \Omega)$. From Lemma 3-2, we can see that $d_{\partial \Omega}$ is a $C^2$-function. We set $s^* = \min\{s_0, s_1\}$. For any $\nu \in S^{N-1}$ we denote by $Q_{\nu}$ the open unit cube centered at the origin with two of its surfaces normal to $\nu$. Furthermore for $x \in \partial \Omega$, $\eta > 0$, and sufficiently small $\delta$ with $0 < \delta < s^*$, we set $\partial \Omega_{\eta}(x) = \partial \Omega \cap (x + \eta Q_{\nu(x)})$ and $\Omega^\delta_{\eta}(x) = \cup_{\delta < t < s^*} \pi_t(\partial \Omega_{\eta}(x))$.

We will start to prove Proposition 3 for the special case $w_0 \equiv \alpha$. The proof of Proposition 3 for the case of $w_0 = \alpha$ requires three steps.

**The First Step:** Let $x_0$ be any point in $\partial \Omega$. In this step, for any sufficiently small $\eta > 0$ we will construct a family $\{w^\delta_{\epsilon}\}_{\epsilon, \delta > 0} \subset W^{1,2}(\Omega^\delta_{\eta}(x_0) : \mathbb{R}^n)$ such that

$$\lim_{\epsilon, \delta \to 0} \sup_{0} \int_{\Omega^\delta_{\eta}} \left[ \epsilon |\nabla w^\delta_{\epsilon}|^2 + \frac{1}{\epsilon} W(x_0, w^\delta_{\epsilon}) \right] dx \leq 2d(x_0, \alpha, g(x_0)) \mathcal{H}_{N-1}(\partial \Omega_{\eta}(x_0)).$$  \hspace{1cm} (3.4)

In this step, for simplicity, we set $\Omega^\delta_{\eta} = \Omega^\delta_{\eta}(x_0)$.

In order to construct $\{w^\delta_{\epsilon}\}_{\epsilon, \delta > 0}$, we fix $\epsilon, \delta > 0$, and consider the following ordinary differential equation:

$$\begin{align*}
\frac{d}{dt} y_{\epsilon}(t) &= \frac{[\epsilon^{1/2} + W(x_0, \gamma(y_{\epsilon}(t)))]^{1/2}}{\epsilon |\dot{\gamma}(y_{\epsilon}(t))|}, \\
y_{\epsilon}(\delta) &= 0.
\end{align*}$$  \hspace{1cm} (3.5)

Here by $\dot{\gamma}$ we denote $d\gamma(t)/dt$, and assume that $\gamma \in C^1([0,1] : [K_1, K_2]^n)$, $\gamma(0) = \alpha$, $\gamma(1) = g(x_0)$. We set

$$\psi_{\epsilon}(t) = \int_{0}^{t} \frac{\epsilon |\dot{\gamma}(t)|}{[\epsilon^{1/2} + W(x_0, \gamma(t))]^{1/2}} dt$$

for $t \in (0, 1)$. Then $\psi_{\epsilon}(t)$ is a monotone increasing function and

$$\tau_{\epsilon} \equiv \psi_{\epsilon}(1) \leq \epsilon^{3/4}. \quad \text{length of } \gamma.$$  \hspace{1cm} (3.6)
Here we set $\tilde{y}_\epsilon(t) = \psi_\epsilon^{-1}(t - \delta)$, and we can see that $\tilde{y}_\epsilon(t)$ satisfies (3.5) in $[\delta, \delta + \tau_\epsilon]$ and we define $y_\epsilon(t)$ by

$$y_\epsilon(t) \equiv \max\{0, \min\{1, \tilde{y}_\epsilon(t)\}\}.$$  

We separate $\Omega_\eta^\delta$ to three domains $\Omega_{\eta,i}^\delta$, $i = 1, 2, 3$ as follows:

$$\Omega_{\eta,1}^\delta \equiv \{x \in \Omega_\eta^\delta : d_{\partial \Omega}(x) < \delta + \tau_\epsilon, d_S(x) \leq \eta \tau_\epsilon\};$$
$$\Omega_{\eta,2}^\delta \equiv \{x \in \Omega_\eta^\delta : d_{\partial \Omega}(x) < \delta + \tau_\epsilon, d_S(x) \geq \eta \tau_\epsilon\};$$
$$\Omega_{\eta,3}^\delta \equiv \{x \in \Omega_\eta^\delta : d_{\partial \Omega}(x) \geq \delta + \tau_\epsilon\},$$

where $d_S(x)$ is a distance function to $\bigcup_{\delta < t < s^*} \pi_t[\partial \Omega \cap (x_0 + \eta \partial Q_{\nu(x_0)})]$. Here we define $w_\epsilon(x)$ on $\bigcup_{i=2,3} \Omega_{\eta,i}^\delta$ as follows:

$$w_\epsilon(x) = \begin{cases} \gamma(y_\epsilon(d_{\partial \Omega}(x))), & \text{if } x \in \Omega_{\eta,2}^\delta; \\ \alpha, & \text{if } x \in \Omega_{\eta,3}^\delta. \end{cases}$$

and extend $w_\epsilon$ to $\Omega_{\eta,1}^\delta$ such that for any $x \in \Omega_\eta^\delta$ with $d_S(x) = 0$ or $d_{\partial \Omega}(x) = \delta + \tau_\epsilon$, $w_\epsilon(x) = \alpha$ and

$$|\nabla w_\epsilon| \leq 2/(K_2 - K_1) \eta \tau_\epsilon + C/\epsilon \leq C(\eta \tau_\epsilon)^{-1} + C\epsilon^{-1}.$$

For sufficiently small $\epsilon > 0$, we have the length of $\gamma < \epsilon^{-1/8}$ and $\tau_\epsilon \leq \epsilon^{5/8}$. Therefore we obtain

$$\int_{\Omega_{\eta,1}^\delta} \left[ \epsilon|\nabla w_\epsilon|^2 + \frac{1}{\epsilon}W(x_0, w_\epsilon) \right] dx \leq C[\epsilon/\eta^2 \tau_\epsilon^2 + 1/\epsilon] \tau_\epsilon N \mathcal{H}_{N-1}(\partial \Omega_\eta) \leq C(\epsilon/\eta^2 + \epsilon^{1/4}) \tau_\epsilon N \mathcal{H}_{N-1}(\partial \Omega_\eta).$$

Here we note that constants $C$ are independent of $\epsilon$ and $\eta$. On the other hand, for sufficiently small $\delta > 0$ and $\epsilon > 0$ we have $\delta + \tau_\epsilon < s^* \equiv \min\{s_0, s_1\}$ and obtain from Lemma 3-2 and (3.9)

$$\int_{\bigcup_{i=2,3} \Omega_{\eta,i}^\delta} \left[ \epsilon|\nabla w_\epsilon|^2 + \frac{1}{\epsilon}W(x_0, w_\epsilon) \right] dx \leq \int_{\Omega_{\eta,2}^\delta} \frac{2}{\epsilon}[\epsilon^{1/2} + W(x_0, \gamma(y_\epsilon(d_{\partial \Omega}(x))))]|\nabla d_{\partial \Omega}(x)| dx,$$

and from the co-area formula in $BV$ functions, we get
\[
\leq 2 \int_{\delta}^{\tau_{\epsilon}+\delta} dt \int_{\Omega_{\eta}^{\delta} \cap \{d_{\partial\Omega}(x) = t\}} \epsilon^{-1} \left[ \epsilon^{1/2} + W(x_0, \gamma(y_{\epsilon}(t))) \right] d\mathcal{H}_{N-1}
\leq 2 \kappa_{\epsilon}^{\delta} \int_{\delta}^{\tau_{\epsilon}+\delta} \epsilon^{-1} (\epsilon^{1/2} + W(x_0, \gamma(y_{\epsilon}(t)))) dt,
\]
where \( \kappa_{\epsilon}^{\delta} = \sup_{\delta \leq d(x) \leq \delta + \epsilon} (\Omega_{\eta}^{\delta} \cap \pi_{t}(\partial\Omega)) \).

Then from (3.5) we obtain

\[
(3.11) \quad \int_{\bigcup_{i=1,2} \Omega_{\eta,i}^{\delta}} \left[ \epsilon |\nabla w_{\epsilon}|^2 + \frac{1}{\epsilon} W(x_0, w_{\epsilon}) \right] dx \leq 2 \kappa_{\epsilon}^{\delta} \int_{0}^{1} \left( \epsilon^{1/2} + W(x_0, \gamma(t)) \right)^{1/2} |\dot{\gamma}(t)| dt.
\]

From the regularity of \( \partial\Omega \) and the definition of \( \Omega_{\eta}^{0}(x_0) \), there exist a constant \( \eta_0 \) independent of \( x_0 \) (dependent only on \( \partial\Omega \)) such that for any \( 0 < \eta < \eta_0 \), we have \( \mathcal{H}_{N-1}(\partial\Omega_{\eta}^{0}(x_0) \cap \partial\Omega) = 0 \). So from Lemma 3–2 we have \( \lim_{\epsilon, \delta \to 0} \kappa_{\epsilon}^{\delta} = \mathcal{H}_{N-1}(\partial\Omega_{\eta}(x_0)) \) for any \( \eta \in (0, \eta_0) \). Here we set \( w_{\epsilon}^{\delta, \gamma} = w_{\epsilon} \). Therefore from (3.10) and (3.11), for any \( \eta \in (0, \eta_0) \) we obtain

\[
(3.12) \quad \int_{\Omega_{\eta}^{\delta}(x_0)} \left[ \epsilon |\nabla w_{\epsilon}^{\delta, \gamma}|^2 + \frac{1}{\epsilon} W(x_0, w_{\epsilon}^{\delta, \gamma}) \right] dx \leq 2 \mathcal{H}_{N-1}(\partial\Omega_{\eta}) \int_{0}^{1} W^{1/2}(x_0, \gamma(t)) |\dot{\gamma}(t)| dt + \mathcal{H}_{N-1}(\partial\Omega_{\eta}) \left[ O(\epsilon/\eta^2) + O(\epsilon^{1/4}) + O_{\sqrt{\epsilon^2 + \delta^2}}(1) \right].
\]

Here by \( O(1) \) we mean \( \lim_{\epsilon \to 0} O(\epsilon)(1) = 0 \). Since for any \( \epsilon > 0 \) there exist a sequence of \( C^1 \)-curves \( \{\gamma_i\}_{i=1}^{\infty} \) such that the length of \( \gamma_i \leq \epsilon^{-1/8} \) and

\[
\lim_{i \to \infty} \int_{0}^{1} W^{1/2}(x_0, \gamma_i(t)) |\dot{\gamma}_i(t)| dt = d(x_0, a, b),
\]
by the diagonal argument and (3.12), we can construct a sequence \( \{w_{\epsilon}^{\delta}\}_{\epsilon, \delta > 0} \) satisfying (3.4). Therefore the aim of the first step is completed.

**The Second Step:** Let \( \Omega_{\delta} \) be a domain \( \{x \in \Omega : \delta < d_{\partial\Omega}(x) < s^*\} = \bigcup_{\delta < s^*} \pi_t(\partial\Omega) \). At the second step, we construct a sequence \( \{w_{\epsilon}^{\delta}\}_{\epsilon, \delta > 0} \) in \( W^{1,2}(\Omega_{\delta}, \mathbb{R}^n) \) such that

\[
(3.13) \quad \limsup_{\delta, \epsilon \to \infty} \int_{\Omega_{\delta}} \left[ \epsilon |\nabla w_{\epsilon}^{\delta}|^2 + \frac{1}{\epsilon} W(x, w_{\epsilon}^{\delta}) \right] dx \leq 2 \int_{\partial\Omega} d(x, a, g(x)) d\mathcal{H}_{N-1}.
\]

In order to construct a sequence \( \{w_{\epsilon}^{\delta}\}_{\epsilon, \delta > 0} \), we will separate \( \partial\Omega \) into small pieces. From the regularity of \( \partial\Omega \), for sufficiently small \( \eta > 0 \), there exist \( p \) points \( \{x_i\}_{i=1}^{p} \subset \partial\Omega \) and a subset \( \Omega_{\eta} \) of \( \partial\Omega \) such that

\[
(3.14) \quad \partial\Omega \setminus \bigcup_{1 \leq i \leq p} \partial\Omega_{\eta}(x_i) \subset \omega_{\eta}, \quad \partial\Omega_{\eta}(x_i) \cap \partial\Omega_{\eta}(x_j) = 0, \ i \neq j, \ i, j = 1, 2, \ldots, p.
\]
and \( \lim_{\eta \to 0} \mathcal{H}_{N-1}(\omega_{\eta}) = 0 \). Here we note that \( p \) depends on \( \eta \) and \( \lim_{\eta \to 0} p(\eta) = \infty \).

For any \( \eta, \delta, \epsilon > 0 \), fix \( \eta, \delta, \) and \( \epsilon \). Then for any \( i \in \{1, 2, \cdots, p\} \), from (3.10) we can construct functions \( u_{\epsilon}^{i, \delta, \eta} \in W^{1,2}(\Omega_{\eta}^\delta(x_{i})) \) such that

\[
(3.15) \quad \int_{\Omega_{\eta}^\delta(x_{i})} \left[ \epsilon |\nabla w_{\epsilon}^{i, \delta, \eta}|^2 + \frac{1}{\epsilon} W(x_{i}, w_{\epsilon}^{i}) \right] dx 
\leq 2 \mathcal{H}_{N-1}(\partial \Omega_{\eta}(x_{i})) d(x_{i}, \alpha, g(x_{i})).
\]

Then we note that \( p \) depends on \( \eta \) and \( \lim_{\eta \to 0} p(\eta) = \infty \).

For any \( \eta, \delta, \epsilon > 0 \), fix \( \eta, \delta \), and \( \epsilon \).

Then we define \( w_{\epsilon}^{\delta, \eta} \in W^{1,2}(\Omega_{\delta} : \mathbb{R}^{n}) \) as follows:

\[
w_{\epsilon}^{\delta, \eta} = \begin{cases} w_{\epsilon}^{i, \delta, \eta}, & \text{if } x \in \Omega_{\eta}^\delta(x_{i}), \\ \alpha, & \text{otherwise}. \end{cases}
\]

By the argument of Step 1, we can see \( w_{\epsilon}^{\delta, \eta} \in W^{1,2}(\Omega_{\delta} : \mathbb{R}^{n}) \) easily. Then we have

\[
(3.16) \quad \int_{\Omega_{\delta}} \left[ \epsilon |\nabla w_{\epsilon}^{\delta, \eta}|^2 + \frac{1}{\epsilon} W(x, w_{\epsilon}^{\delta, \eta}) \right] dx = \sum_{i=1}^{p(\eta)} \int_{\Omega_{\delta}(x_{i})} \left[ \epsilon |\nabla w_{\epsilon}^{i, \delta, \eta}|^2 + \frac{1}{\epsilon} W(x, w_{\epsilon}^{i}) \right] dx.
\]

On the other hand, we have (for simplicity we omit the index \( \delta, \eta \) of \( w_{\epsilon}^{i, \delta, \eta} \))

\[
\int_{\Omega_{\eta}^\delta(x_{i})} \left[ \epsilon |\nabla w_{\epsilon}^{i, \delta, \eta}|^2 + \frac{1}{\epsilon} W(x, w_{\epsilon}^{i}) \right] dx = \int_{\Omega_{\eta}^\delta(x_{i})} \left[ \epsilon |\nabla w_{\epsilon}^{i}|^2 + \frac{1}{\epsilon} W(x, w_{\epsilon}^{i}) \right] dx + \int_{\Omega_{\eta}^\delta(x_{i})} \frac{1}{\epsilon} \left( W(x, w_{\epsilon}^{i}) - W(x_{i}, w_{\epsilon}^{i}) \right) dx 
\equiv I_{1}^{i} + I_{2}^{i}.
\]

From (3.15) we obtain

\[
(3.17) \quad \sum_{i=1}^{p(\eta)} I_{1}^{i} \leq 2 \sum_{i=1}^{p(\eta)} \left[ d(x_{i}, \alpha, g(x_{i})) \mathcal{H}_{N-1}(\partial \Omega_{\eta}(x_{i})) \right] + 0(\epsilon/\eta^2) + 0(\epsilon^{1/4}) + 0(\sqrt{\epsilon^{2} + \delta^{2}})(1),
\]

and from (1.2) and (3.15)

\[
\sum_{i=1}^{p(\eta)} |I_{2}^{i}| \leq \sum_{i=1}^{p(\eta)} \int_{\Omega_{\eta}^\delta(x_{i})} 0_{|x-x_{i}|}(1) \frac{1}{\epsilon} W(x, w_{\epsilon}^{i}) dx \leq 0_{\eta}(1) \sum_{i=1}^{p(\eta)} I_{1}^{i}.
\]

We set \( \eta^2 = \epsilon^{3/4} \). Then combinating (3.16) and (3.17), we obtain

\[
(3.18) \quad \limsup_{\delta, \epsilon \to 0} \int_{\Omega_{\delta}} \left[ \epsilon |\nabla w_{\epsilon}^{\delta, \eta(\epsilon)}|^2 + \frac{1}{\epsilon} W(x, w_{\epsilon}^{\delta, \eta(\epsilon)}) \right] dx 
\leq \limsup_{\epsilon \to 0} 2 \sum_{i=1}^{p(\eta)} d(x_{i}, \alpha, g(x_{i})) \mathcal{H}_{N-1}(\partial \Omega_{\eta}(x_{i})).
\]
From the continuity of the function \(d(x, \alpha, g(x))\), we obtain
\[
\sum_{i=1}^{p(\eta)} d(x_{j}, \alpha, g(x_{j}))\mathcal{H}_{N-1}(\partial\Omega_{\eta}(x_{i})) \leq \int_{1 \leq \cup \partial\Omega_{\eta}(x_{\mathfrak{i}})} d(x, \alpha, g(x))d\mathcal{H}_{N-1} + o_{\eta}(1)
\]
Therefore combining (3.18), we can see that the sequence \(\{w_{\delta,\eta(\epsilon)}^{\epsilon}\}_{\epsilon,\delta>0}\) satisfies (3.13).
Hence we set \(w_{\epsilon}^{\delta} = w_{\delta,\eta(\epsilon)}^{\epsilon}\), and the purpose of Step 2 is completed.

**The Third Step:** In this step, we will complete the proof of Proposition B for the special case \(w_{0} \equiv \alpha\). For any \(\delta, \epsilon > 0\) we define \(w_{\epsilon}^{\delta}\) as follows:
\[
w_{\epsilon}^{\delta} = \begin{cases} 
\alpha, & \text{if } x \in \Omega \setminus \Omega_{0}, \\
w_{\epsilon}^{\ast,\delta}, & \text{if } x \in \Omega_{\delta}, 
\end{cases}
\]
where \(\Omega_{0} = \bigcup_{0 < t < s^{\ast}} \pi_{t}(\partial\Omega)\) and where \(w_{\epsilon}^{\ast,\delta}\) is a function constructed in Step 2. In \(\Omega_{\delta} \equiv \Omega_{0} \setminus \Omega_{\delta}\), we construct \(w_{\epsilon}^{\delta}\) by combining between \(g(x)\) and \(w_{\epsilon}^{\ast,\delta}(\pi_{\delta}(x))\) i.e. for \(x \in \Omega_{0} \setminus \Omega_{\delta}\),
\[
w_{\epsilon}^{\delta}(x) = \frac{d_{\partial\Omega}(x)}{\delta}w_{\epsilon}^{\ast,\delta}|_{(\partial\Omega)_{\delta}}(\pi_{\delta}0\pi_{d_{\partial\Omega}(x)}^{-1}(x)) + \left(1 - \frac{d_{\partial\Omega}(x)}{\delta}\right)g(\pi_{d_{\partial\Omega}(x)}^{-1}(x)).
\]
Here \(\pi_{\delta}(x)\) and \(\pi_{d_{\partial\Omega}}(x)\) are functions appearing in Lemma 3–1. Then we can see easily \(w_{\epsilon}^{\delta} \in W^{1,2}(\Omega)\) and \(w_{\epsilon}^{\delta}(x) = g(x)\) for all \(x \in \partial\Omega\).

In order to estimate the gradient of \(w_{\epsilon}^{\delta}\), we fix \(\epsilon, \delta\), and fix \(\{\Omega_{\eta,i}^{\delta}(x_{i})\}_{i=1}^{p}\) and \(\omega_{\eta}\). Then we set
\[
\Omega_{1}^{\delta} = \{x \in \Omega^{\delta} : \pi_{\delta} \circ \pi_{d_{\partial\Omega}}^{-1}(x) \in \bigcup_{1 \leq i \leq p} \partial(\Omega_{\eta,1}^{\delta}(x_{i}))\},
\]
\[
\Omega_{2}^{\delta} = \{x \in \Omega^{\delta} : \pi_{\delta} \circ \pi_{d_{\partial\Omega}}^{-1}(x) \in \bigcup_{1 \leq i \leq p} \partial(\Omega_{\eta,2}^{\delta}(x_{i}))\},
\]
where \(\Omega_{\eta,\delta}^{\delta}(x)\) is a domain appearing in Step 1.
Furthermore for simplicity, we set
\[
\hat{g}(x) = g(\pi_{d_{\partial\Omega}}^{-1}(x)) \quad \text{and} \quad \hat{w}_{\epsilon}^{\delta}(x) = w_{\epsilon}^{\ast,\delta}|_{(\partial\Omega)_{\delta}}(\pi_{\delta} \circ \pi_{d_{\partial\Omega}}^{-1}(x)).
\]
for \(x \in \Omega^{\delta}\). Then from Lemma 3–1 we can see that there exists a constant \(C\) such that \(|\nabla \hat{g}(x)| \leq C\) for almost all \(x \in \Omega^{\delta}\).
Now in the domains $\Omega_{\eta,1}^\delta$, $\Omega_{\eta,2}^\delta$, and $\Omega^\delta$, we will estimate the gradient of $w_\epsilon^\delta$, and obtain the inequality (2.1). If $x \in \omega_\eta^\delta$, then from the construction of $w_\epsilon$ in Step 2 we see $v_\epsilon^\delta = \alpha$ in a neighborhood of $x$, and so for almost all $x \in \omega_\eta^\delta$ we have

$$|\nabla w_\epsilon^\delta| \leq C(1 + 1/\delta).$$

So we obtain

$$\int_{\omega_\eta^\delta} [\epsilon|\nabla w_\epsilon^\delta| + \frac{1}{\epsilon}W(x, w_\epsilon^\delta)] dx \leq C\left(\frac{\epsilon}{\delta^2} + \epsilon + \frac{1}{\epsilon}\right)\delta H_{N-1}(\omega).$$

For almost all $x \in \Omega_{\eta,1}^\delta(x_i)$, then we have

$$|\nabla w_\epsilon^\delta| \leq \frac{|\nabla d_{\partial\Omega}(x)|}{\delta} w_\epsilon^\delta(x) + \frac{d_{\partial\Omega}(x)}{\delta} |\nabla w_\epsilon^\delta(x)| + \left(1 - \frac{d_{\partial\Omega}(x)}{\delta}\right) |\nabla \hat{g}(x)|.$$

Here from the argument in Step 1, there exists a constant $C_2$ such that $|\nabla v_\epsilon^\delta(x)| \leq C/(\epsilon^{5/8} \eta)$ for all $x \in \Omega_{\eta,1}^\delta$. Moreover we have $|\Omega_{\eta,1}^\delta| \leq C\delta(\epsilon^{5/8} \eta^{N-1})(H_{N-1}(\partial\Omega)/\eta^{N-1}) \leq C\delta^{5/8}$. So we obtain

$$\int_{\Omega_{\eta,1}^\delta} [\epsilon|\nabla w_\epsilon^\delta| + \frac{1}{\epsilon}W(x, w_\epsilon^\delta)] dx \leq C\left(\frac{\eta}{\delta^{1/4}} + \frac{\delta}{\epsilon}\right)\epsilon^{5/8}.$$

For any $x \in \Omega_{\eta,2}^\delta(x_i)$, from Step 1 we see $w_\epsilon^*(x) \equiv g(x_i)$ in a neighborhood of $x$. Then from the Lipschitz continuity of $g(x)$ on $\partial\Omega$ and (3.19) we have

$$|\nabla w_\epsilon^\delta| \leq \frac{|\nabla d_{\partial\Omega}(x)|}{\delta} |g(x_i) - \hat{g}(x)| + \left(1 - \frac{d_{\partial\Omega}(x)}{\delta}\right) |\nabla \hat{g}(x)| \leq \frac{C}{\delta} |g(x_i) - \hat{g}(x)| + C \leq \frac{C\eta}{\delta} + C.$$

So we obtain

$$\int_{\Omega_{\eta,2}^\delta} [\epsilon|\nabla w_\epsilon^\delta| + \frac{1}{\epsilon}W(x, w_\epsilon^\delta)] dx \leq C\left(\frac{\eta}{\delta} + \frac{1}{\epsilon}\right)\delta H_{N-1}(\partial\Omega).$$

Let $\sigma(\cdot)$ be a positive function with $\sigma(0) = 0$ such that $\lim_{\epsilon \to 0} H_{N-1}(\omega_{\eta(\epsilon)})/\sigma(\epsilon) = 0$ and $\lim_{\epsilon \to 0} \epsilon^{5/8}/\sigma(\epsilon) = 0$. Here we set $\delta_\epsilon = \epsilon\sigma(\epsilon)$, and define $w_\epsilon = w_\epsilon^\delta$. Then from (3.21)-(3.23) we obtain

$$\lim_{\epsilon \to 0} \int_{\Omega^\epsilon\sigma(\epsilon)} [\epsilon|\nabla w_\epsilon| + \frac{1}{\epsilon}W(x, w_\epsilon)] dx = 0.$$
Therefore from (3.13) and (3.24) we obtain
\[
\limsup_{\epsilon \to 0} \int_{\Omega} [\epsilon|\nabla w_{\epsilon}|^2 + \frac{1}{\epsilon} W(x, w)] \, dx \leq 2 \int_{\partial\Omega} d(x, \alpha, g(x)) d\mathcal{H}_{N-1}.
\]
Hence the proof of Proposition B for the special case \( w_0 = \alpha \) is completed.

Finally we remark that the proof of this section is an essential part of complete proof of Proposition B.

REFERENCES