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Kyoto University
The Gradient Theory of the Phase Transitions in Cahn-Hilliard Fluids with the Dirichlet boundary conditions

Kazuhiro Ishige

Department of Mathematics, Faculty of Science Tokyo Institute of Technology
Oh-okayama, Meguro-ku, Tokyo, 152, Japan

1. Introduction

In this note we will investigate the asymptotic behavior of minimizer \( \{u_\epsilon\}_{\epsilon>0} \) (as \( \epsilon \to 0 \)) of the following variational problem:

\[
(P_\epsilon) \quad \inf \left\{ \int_\Omega \left[ \epsilon |\nabla u|^2 + \frac{1}{\epsilon} W(x, u) \right] dx \right\} u \in W^{1,2}(\Omega : \mathbb{R}^n), u = g \text{ on } \partial \Omega,
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with \( C^2 \) smooth boundary \( \partial \Omega \) and \( g \) is a Lipschitz continuous function from \( \partial \Omega \) into \( \mathbb{R}^n \). Here \( W(x, \cdot) \) is a nonnegative continuous function which has two potential wells with equal depth. This type of problem is related to the study of the phase transitions of the Cahn-Hilliard fluids. See [8] and [9].

In [7] R.V.Kohn & P.Sternberg conjectured that minimizer of the variational problem, which is special case of \( (P_\epsilon) \),

\[
(SP_\epsilon) \quad \inf \left\{ \int_\Omega \left[ \epsilon |\nabla u|^2 + \frac{1}{\epsilon} (u^2 - 1)^2 \right] dx \right\} u \in W^{1,2}(\Omega), u|_{\partial \Omega} = g
\]

converges to a solution of

\[
\inf \left\{ \frac{8}{3} P_\Omega \{u = 1\} + 2 \int_{\partial \Omega} |d(u) - d(g)| d\mathcal{H}_{N-1} \right\} u \in BV(\Omega), |u| = 1 \text{ a.e.}
\]

where \( d(t) = \int_{-1}^{t} |s^2 - 1| ds \). Here \( \mathcal{H}_{N-1} \) is the \( N-1 \) dimensional Hausdorff measure.

In this note, we will study the asymptotic behavior of minimizer of \( (P_\epsilon) \), and as a byproduct, we will state the affirmative results to the conjecture in [7].

Recently using the theory of Gamma-convergence, several authors studied the asymptotic behavior of the minimizer of the problem:

\[
(E_\epsilon) \quad \inf \left\{ \int_\Omega \left[ \epsilon |\nabla u|^2 + \frac{1}{\epsilon} W(u) \right] dx \right\} u \in W^{1,2}(\Omega : \mathbb{R}^n), \int_\Omega u(x) dx = m
\]
where \(m\) is a constant vector in \(\mathbb{R}^n\). For the scalar case (i.e. \(n = 1\)), see [8] and [9]. For the vector case (i.e. \(n \geq 2\)), see [1] and [4]. Our results on the problem \((P_\epsilon)\) depend mainly on the study of asymptotic behavior of minimizer of \((E_\epsilon)\). However there are several different aspects between the asymptotic behavior of minimizer of \((P_\epsilon)\) and that of \((E_\epsilon)\). In fact, minimizer of \((E_\epsilon)\) generates the only interior layer, but minimizer of \((P_\epsilon)\) generates both the interior and the boundary layers as \(\epsilon \to 0\).

On the other hand, we can easily see that minimizer of \((SP_\epsilon)\) satisfies the equation:

\[
(CP_\epsilon) \quad \begin{cases} 
\epsilon^2 \Delta u - u(u - 1)(u + 1) = 0 & \text{in } \Omega, \\
u(x) = g(x) & \text{on } \partial \Omega.
\end{cases}
\]

Then there exist several results for the solutions of \((CP_\epsilon)\) obtained by using the method of matched expansion. Our results also seem to be closely related to [2] and [3].

We will give the precise conditions of the functions \(W(x, u)\) and \(g(x)\). Let \(W(x, u) : \Omega \times \mathbb{R}^n \to \mathbb{R}\) be a continuous nonnegative function, and for any \(x \in \Omega\) \(W(x, u) = 0\) if and only if \(u = \alpha\) or \(\beta\). Here we note \(\alpha\) and \(\beta\) are constant vectors independent of \(x\). We assume that there exist two constants \(K_1\) and \(K_2\) such that

\[
(1.1) \quad \sup_{u \in \partial[K_1,K_2]^n} W(x, u) \leq W(x, v) \quad \text{for all } x \in \Omega, \quad v \not\in [K_1,K_2]^{n}
\]

and

\[
(1.2) \quad g(x) \in [K_1, K_2]^{n} \quad \text{for all } x \in \partial \Omega.
\]

Moreover we set \(W_\infty(\cdot) = \inf_{x \in \Omega} W(x, \cdot)\) and assume that for any \(\epsilon > 0\) there exists a positive constant \(\delta\) such that

\[
(1.3) \quad |W^{1/2}(x, u) - W^{1/2}(y, u)| \leq \epsilon W^{1/2}_\infty(u)
\]

for all \(x, y \in \Omega\) with \(|x - y| \leq \delta\) and all \(u \in \mathbb{R}^n\). Here from the definition of \(W_\infty(u)\) and (1.3) we have the following relation

\[
(1.3') \quad |W^{1/2}(x, u) - W^{1/2}(y, u)| \leq \epsilon W^{1/2}(x, u)
\]

for all \(x, y \in \Omega\) with \(|x - y| \leq \delta\) and for all \(u \in \mathbb{R}^n\).

We think that the conditions (1.1) and (1.3) are not restrictive. In fact, consider continuous functions \(W(u), h(x)\), where \(W(u)\) satisfies the condition (1.1) and where \(h(x)\) is positive function in \(\Omega\). If the function \(W(x, u)\) has a form of \(h(x)W(u)\), then we can see that \(W(x, u)\) satisfies the conditions (1.1) and (1.3).
In order to state the main theorem, we will introduce a Riemannian metric on $\mathbb{R}^n$, $d(x, a, b)$ which depends on $x \in \overline{\Omega}$. For $x \in \overline{\Omega}$ and $a, b \in \mathbb{R}^n$, let $d(x, a, b)$ be the metric defined by

\[
d(x, a, b) = \inf \left\{ \int_0^1 W^{1/2}(x, \gamma(t)) |\dot{\gamma}(t)| \, dt \mid \gamma \in C^1([0,1] : \mathbb{R}^n), \gamma(0) = a, \gamma(1) = b \right\}.
\]

For example, in the case of $W(x, u) = (u^2 - 1)^2$ and $n = 1$, we have

\[
d(x, -1, b) = \int_{-1}^{b} |s^2 - 1| \, ds \quad \text{for} \quad b \geq -1.
\]

We now state our main theorem of this note.

**Theorem 1.** (See [6].) Suppose that function $W$ satisfies (1.1) and (1.3) and that $g$ satisfies (1.2). For $\epsilon > 0$, let $u_\epsilon$ be a solution of the variational problem:

\[
\inf \left\{ \int_{\Omega} \left[ \epsilon |\nabla u|^2 + \frac{1}{\epsilon} W(x, u) \right] \, dx \mid u \in W^{1,2}(\Omega : \mathbb{R}^n), u|_{\partial\Omega}(x) = g(x) \right\}.
\]

If there exist a positive sequence $\{\epsilon_i\}_{i=1}^\infty$ and a function $u_0(x) \in L^1(\Omega : \mathbb{R}^n)$ such that

\[
\lim_{i \to \infty} \epsilon_i = 0 \quad \text{and} \quad \lim_{i \to \infty} u_{\epsilon_i} = u_0 \quad \text{in} \quad L^1(\Omega : \mathbb{R}^n),
\]

then the function $u_0$ is characterized by

\[
W(x, u_0(x)) = 0 \quad \text{for almost all} \quad x \in \Omega, \quad \text{that is,} \quad u_0(x) = \alpha \text{ or } \beta \quad \text{for almost all} \quad x \in \Omega.
\]

Moreover the set $E_0 = \{ x \in \Omega \mid u_0(x) = \alpha \}$ is a solution of the variational problem $(P_0)$:

\[
(P_0) \quad \inf \left\{ \int_{\Omega \cap \partial^* E} d(x, \alpha, \beta) d\mathcal{H}_{N-1} + \int_{\partial \Omega} d(x, v|_{\partial \Omega}(x), g(x)) d\mathcal{H}_{N-1} \mid E \subset \Omega, P_\Omega(E) < \infty, v = \alpha \chi_E + \beta \chi_{\Omega \setminus E} \right\},
\]

where $P_\Omega(E)$ is a perimeter of $E$ in $\Omega$ and $v|_{\partial \Omega}$ is the trace of $v$ to $\partial \Omega$. Furthermore we have

\[
\lim_{i \to \infty} \int_{\Omega} \left[ \epsilon_i |\nabla u_{\epsilon_i}|^2 + \frac{1}{\epsilon_i} W(x, u_{\epsilon_i}) \right] \, dx = 2 \int_{\Omega \cap \partial^* E_0} d(x, \alpha, \beta) d\mathcal{H}_{N-1} + 2 \int_{\partial \Omega \cap \partial^* E_0} d(x, \alpha, g(x)) d\mathcal{H}_{N-1} + 2 \int_{\partial \Omega \setminus \partial^* E_0} d(x, \beta, g(x)) d\mathcal{H}_{N-1}.
\]
Here $\partial^* E_0$ is the reduced boundary of $E_0$.

**Remark.** It is not restrictive to assume that there exists a subsequence $\{u_{\epsilon_i}\}_{i=1}^{\infty}$ satisfying (1.5). In fact, the following is proved in [4] and [5]: if there exist constants $C$ and $R$ such that

\[
W_\infty(u) \geq C|u| \quad \text{for } |u| \geq R,
\]

then there exists a subsequence $\{u_{\epsilon_i}\}_{i=1}^{\infty}$ satisfying (1.5).

It is worth noting that the study of asymptotic behavior of minimizer of $(P_\epsilon)$ occurs a completely different difficulty from that of $(SP_\epsilon)$. One of the difficulties is that the selection of minimizing sequence $\{\gamma_k\}_{k=1}^{\infty}$ achieving the value of $d(x, \alpha, g(x))$ depends on the space variable $x$. In order to overcome this difficulty, we approximate $W(\cdot, u)$ and $g(\cdot)$ by suitable piecewise smooth functions near the transition layer and the boundary $\partial \Omega$.

2. The Main Propositions

At first, we will give functionals $F_\epsilon$ and $F_0$ from $L^1(\Omega : \mathbb{R}^N)$ into $[0, \infty]$. For $u \in L^1(\Omega : \mathbb{R}^n)$ and $\epsilon > 0$, we define $F_\epsilon(u)$, $F_0(u)$ by

\[
F_\epsilon(u) = \begin{cases} 
\int_{\Omega} \left[ \epsilon |\nabla u|^2 + \frac{1}{\epsilon} W(x, u) \right] dx, & \text{if } u \in W^{1,2}(\Omega : \mathbb{R}^n) \text{ and } u = g \text{ on } \partial \Omega, \\
+\infty, & \text{otherwise ,} 
\end{cases}
\]

\[
F_0(u) = \begin{cases} 
2 \int_{\Omega} d(x, \alpha, \beta)|\nabla \chi_{\{u(x) = \alpha\}}| + 2 \int_{\partial \Omega} d(x, u|_{\partial \Omega}(x), g(x))d\mathcal{H}_{N-1}, & \text{if } u \in BV(\Omega : \mathbb{R}^n) \text{ and } W(x, u(x)) = 0 \text{ for almost all } x \in \Omega, \\
+\infty, & \text{otherwise .}
\end{cases}
\]

In order to prove our main theorem, we need the following two propositions which are crucial in our analysis.

**Proposition A.** Suppose that $\{v_\epsilon\}_{\epsilon>0}$ is a sequence in $L^1(\Omega : \mathbb{R}^n)$ which converges in $L^1(\Omega : \mathbb{R}^n)$ as $\epsilon \to 0_+$ to a function $v_0$. If

\[
\liminf_{\epsilon \to 0_+} F_\epsilon(v_\epsilon) < +\infty,
\]

then $v_0$ is a function in $BV(\Omega : \mathbb{R}^n)$ such that

\[
F_0(v_0) \leq \liminf_{\epsilon \to 0_+} F_\epsilon(v_\epsilon).
\]
Proposition B. Suppose that $w_0 \in L^1(\Omega : \mathbb{R}^n)$ is a function with $w_0 = \alpha \chi_E + \beta \chi_{\Omega \setminus E}$ where $E$ is a measurable subset in $\Omega$ with finite perimeter. Then there exists a sequence $\{w_\epsilon\}_{\epsilon > 0}$ in $W^{1,2}(\Omega : \mathbb{R}^n)$ which converges in $L^1(\Omega : \mathbb{R}^n)$ as $\epsilon \to 0_+$ to $w_0$ such that

$$\limsup_{\epsilon \to 0_+} F_\epsilon(w_\epsilon) \leq F_0(w_0).$$

Using Propositions A and B, we can prove Theorem 1 as in the same manner with in [8]. Therefore we have only to prove Proposition A and B. In this note, we will only prove Proposition B for the special case.

On the other hand, in Theorem 1, the minimizers $\{u_\epsilon\}_{\epsilon > 0}$ do not always generate interior layers. For example, if we consider the problem $(SP_\epsilon)$ with $g \equiv 0$, we have $E_0 = \Omega$ or $\emptyset$. In contrast, considering the family of local minimizers, from Theorem 1 and the results of [7], we obtain the following theorem.

Theorem 2. Let $u_0 \in L^1(\Omega : \mathbb{R}^n)$ be an isolated $L^1$-local minimizer of $F_0$, that is,

there exists a positive constant $\delta$ such that $F_0(u_0) < F_0(v)$

whenever $u \neq v$ and $||u_0 - v||_{L^1(\Omega : \mathbb{R}^n)} \leq \delta$.

Then there exist a constant $\epsilon_0 > 0$ and a sequence $\{u_\epsilon\}_{\epsilon < \epsilon_0}$ such that $u_\epsilon$ is a local minimizer of $F_\epsilon$ and $u_\epsilon \to u_0$ in $L^1(\Omega : \mathbb{R}^n)$ as $\epsilon \to 0$.

3. Proof of Proposition B

In this section, we will only prove Proposition B for the special case that $w_0 \equiv \alpha$ in $\Omega$. In order to prove Proposition B for the case of $w_0 \equiv \alpha$, we need the following two lemmas. The first lemma is obtained easily by the inverse mapping theorem.

Lemma 3–1. Let $\Omega$ be a bounded domain with $C^2$-smooth boundary $\partial \Omega$. For $x \in \partial \Omega$ let $\nu(x)$ be a inner normal vector to $\partial \Omega$ at $x$. Define a mapping $\pi : \partial \Omega \times [0, \infty) \to \mathbb{R}^N$ by

$$\pi(x,t) = \pi_t(x) = x + t\nu(x).$$

Then there exists a constant $s_0$ such that the image of $\pi$ in $\partial \Omega \times (0, s_0]$ is contained in $\Omega$ and the $C^1$-smooth inverse mapping $\pi^{-1}$ of $\pi$ exists in $\pi(\partial \Omega \times [0, s_0])$.

Lemma 3–2. (See [8] and [9].) Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$ with Lipschitz-continuous boundary. Let $A$ be an open subset of $\mathbb{R}^N$ with $C^2$, compact, nonempty
boundary such that $\mathcal{H}_{N-1}(\partial A \cap \partial \Omega) = 0$. Define a distance function to $\partial A$, $d_{\partial A} : \Omega \to \mathbb{R}$, by $d_{\partial A}(x) = \text{dist}(x, A)$. Then, for some $s_1 > 0$, $d_{\partial A}$ is a $C^2$-function in $\{0 < d_{\partial A}(x) < s_1\}$ with

$$\nabla d_{\partial A} = 1.$$  

Furthermore, $\lim_{s \to 0} \mathcal{H}_{N-1}(\{d_{\partial A}(x) = s\}) = \mathcal{H}_{N-1}(\partial A \cap \Omega)$ and

$$|\{x \mid |d_{\partial A}(x)| < s\}| = O(s).$$

By $d_{\partial \Omega}(x)$ we denote a function $\text{dist}(x, \partial \Omega)$. From Lemma 3-2, we can see that $d_{\partial \Omega}$ is a $C^2$-function. We set $s^* = \min\{s_0, s_1\}$. For any $\nu \in S^{N-1}$ we denote $Q_{\nu}$ the open unit cube centered at the origin with two of its surfaces normal to $\nu$. Furthermore for $x \in \partial \Omega$, $\eta > 0$, and sufficiently small $\delta$ with $0 < \delta < s^*$, we set $\partial \Omega_{\eta}(x) = \partial \Omega \cap (x + \eta Q_{\nu(x)})$ and $\Omega_{\eta}^\delta(x) = \bigcup_{\delta<s^*} \pi_t(\partial \Omega_{\eta}(x))$.

We will start to prove Proposition $B$ for the special case $w_0 \equiv \alpha$. The proof of Proposition $B$ for the case of $w_0 = \alpha$ requires three steps.

**The First Step:** Let $x_0$ be any point in $\partial \Omega$. In this step, for any sufficiently small $\eta > 0$ we will construct a family $\{w_\epsilon^\delta\}_{\epsilon, \delta > 0} \subset W^{1,2}(\Omega_{\eta}^\delta(x_0) : \mathbb{R}^n)$ such that

$$\lim_{\epsilon, \delta \to 0} \sup_{0} \int_{\Omega_{\eta}^\delta} \left[ \epsilon |\nabla w_\epsilon^\delta|^2 + \frac{1}{\epsilon} W(x_0, w_\epsilon^\delta) \right] dx \leq 2d(x_0, \alpha, g(x_0)) \mathcal{H}_{N-1}(\partial \Omega_{\eta}(x_0)).$$

In this step, for simplicity, we set $\Omega_{\eta}^\delta = \Omega_{\eta}^\delta(x_0)$.

In order to construct $\{w_\epsilon^\delta\}_{\epsilon, \delta > 0}$, we fix $\epsilon, \delta > 0$, and consider the following ordinary differential equation:

$$\begin{cases}
  \frac{d}{dt} y_\epsilon(t) = \frac{[\epsilon^{1/2} + W(x_0, \gamma(y_\epsilon(t)))]^{1/2}}{\epsilon|\dot{\gamma}(y_\epsilon(t))|}, \\
  y_\epsilon(\delta) = 0.
\end{cases}$$

Here by $\dot{\gamma}$ we denote $d\gamma(t)/dt$, and assume that $\gamma \in C^1([0,1] : [K_1, K_2]^n)$, $\gamma(0) = \alpha$, $\gamma(1) = g(x_0)$. We set

$$\psi_\epsilon(t) = \int_0^t \frac{\epsilon|\dot{\gamma}(t)|}{[\epsilon^{1/2} + W(x_0, \gamma(t))]^{1/2}} dt$$

for $t \in (0, 1)$. Then $\psi_\epsilon(t)$ is a monotone increasing function and

$$\tau_\epsilon = \psi_\epsilon(1) \leq \epsilon^{3/4}. \text{ length of } \gamma.$$
Here we set \( \tilde{y}_{\epsilon}(t) = \psi_{\epsilon}^{-1}(t - \delta) \), and we can see that \( \tilde{y}_{\epsilon}(t) \) satisfies (3.5) in \([\delta, \delta + \tau_{\epsilon}]\) and we define \( y_{\epsilon}(t) \) by

\[
y_{\epsilon}(t) \equiv \max\{0, \min\{1, \tilde{y}_{\epsilon}(t)\}\}.
\]

We separate \( \Omega_{\eta}^{\delta} \) to three domains \( \Omega_{\eta, i}^{\delta}, i = 1, 2, 3 \) as follows:

\[
\begin{align*}
\Omega_{\eta, 1}^{\delta} & \equiv \{x \in \Omega_{\eta}^{\delta} : d_{\partial \Omega}(x) < \delta + \tau_{\epsilon}, d_{S}(x) \leq \eta \tau_{\epsilon}\}; \\
\Omega_{\eta, 2}^{\delta} & \equiv \{x \in \Omega_{\eta}^{\delta} : d_{\partial \Omega}(x) < \delta + \tau_{\epsilon}, d_{S}(x) \geq \eta \tau_{\epsilon}\}; \\
\Omega_{\eta, 3}^{\delta} & \equiv \{x \in \Omega_{\eta}^{\delta} : d_{\partial \Omega}(x) \geq \delta + \tau_{\xi}\},
\end{align*}
\]

where \( d_{S}(x) \) is a distance function to \( \bigcup_{\delta < t < s^{*}} \pi_{t}[\partial \Omega \cap (x_{0} + \eta \partial Q_{\nu(x_{0})})] \). Here we define \( w_{\epsilon}(x) \) on \( \bigcup_{i=2,3} \Omega_{\eta, i}^{\delta} \) as follows:

\[
\begin{align*}
w_{\epsilon}(x) & = \begin{cases} 
\gamma(y_{\epsilon}(d_{\partial \Omega}(x))), & \text{if } x \in \Omega_{\eta, 2}^{\delta}, \\
\alpha, & \text{if } x \in \Omega_{\eta, 3}^{\delta}.
\end{cases}
\end{align*}
\]

and extend \( w_{\epsilon} \) to \( \Omega_{\eta, 1}^{\delta} \) such that for any \( x \in \Omega_{\eta}^{\delta} \) with \( d_{S}(x) = 0 \) or \( d_{\partial \Omega}(x) = \delta + \tau_{\epsilon} \), \( w_{\epsilon}(x) = \alpha \) and

\[
|\nabla w_{\epsilon}| \leq 2/(K_{2} - K_{1}) \eta \tau_{\epsilon} + C/\epsilon \leq C(\eta \tau_{\epsilon})^{-1} + C\epsilon^{-1}.
\]

For sufficiently small \( \epsilon > 0 \), we have the length of \( \gamma < \epsilon^{-1/8} \) and \( \tau_{\epsilon} \leq \epsilon^{5/8} \). Therefore we obtain

\[
\int_{\Omega_{\eta, 1}^{\delta}} [\epsilon|\nabla w_{\epsilon}|^{2} + \frac{1}{\epsilon} W(x_{0}, w_{\epsilon})]dx \leq C[\epsilon/\eta^{2} \tau_{\epsilon}^{2} + 1/\epsilon] \tau_{\epsilon}^{N-1} \mathcal{H}_{N-1}(\partial \Omega_{\eta})
\]

\[
\leq C(\epsilon/\eta^{2} + \epsilon^{1/4}) \tau_{\epsilon}^{N-2} \mathcal{H}_{N-1}(\partial \Omega_{\eta}).
\]

Here we note that constants \( C \) are independent of \( \epsilon \) and \( \eta \). On the other hand, for sufficiently small \( \delta > 0 \) and \( \epsilon > 0 \) we have \( \delta + \tau_{\epsilon} < s^{*} \equiv \min\{s_{0}, s_{1}\} \) and obtain from Lemma 3–2 and (3.9)

\[
\int_{\bigcup_{i=2,3} \Omega_{\eta, i}^{\delta}} [\epsilon|\nabla w_{\epsilon}|^{2} + \frac{1}{\epsilon} W(x_{0}, w_{\epsilon})]dx \leq \int_{\Omega_{\eta, 2}^{\delta}} \frac{2}{\epsilon} [\epsilon^{1/2} + W(x_{0}, \gamma(y_{\epsilon}(d_{\partial \Omega}(x))))]|\nabla d_{\partial \Omega}(x)|dx,
\]

and from the co-area formula in \( BV \) functions, we get
\[
\leq 2 \int_{\delta}^{\tau_{\epsilon}+\delta} dt \int_{\Omega_{\eta}^{\delta} \cap \{d_{\partial\Omega}(x)=t\}} \epsilon^{-1} [\epsilon^{1/2} + W(x_{0}, \gamma(y_{\epsilon}(t)))] d\mathcal{H}_{N-1}
\]
\[
\leq 2 \kappa_{\epsilon}^{\delta} \int_{\delta}^{\tau_{\epsilon}+\delta} \epsilon^{-1} (\epsilon^{1/2} + W(x_{0}, \gamma(y_{\epsilon}(t)))) dt,
\]
where \(\kappa_{\epsilon}^{\delta} = \sup_{\delta \leq d_{S}(x) \leq \delta+\epsilon} (\Omega_{\eta}^{\delta} \cap \pi_{t}(\partial\Omega))\). Then from (3.5) we obtain
\[
(3.11) \quad \int_{\cup_{i=1,2} \Omega_{\eta,i}^{\delta}} [\epsilon |\nabla w_{\epsilon}|^{2} + \frac{1}{\epsilon} W(x_{0}, w_{\epsilon})] dx \leq 2 \kappa_{\epsilon}^{\delta} \int_{0}^{1} (\epsilon^{1/2} + W(x_{0}, \gamma(t)))^{1/2} |\dot{\gamma}(t)| dt.
\]
From the regularity of \(\partial\Omega\) and the definition of \(\Omega_{\eta}^{0}(x_{0})\), there exist a constant \(\eta_{0}\) independent of \(x_{0}\) (dependent only on \(\partial\Omega\)) such that for any \(0 < \eta < \eta_{0}\), we have \(H_{N-1}(\partial\Omega_{\eta}^{0}(x_{0}) \cap \partial\Omega) = 0\). So from Lemma 3–2 we have \(\lim_{\epsilon,\delta \to 0} \kappa_{\epsilon}^{\delta} = H_{N-1}(\partial\Omega_{\eta}(x_{0}))\) for any \(\eta \in (0, \eta_{0})\). Here we set \(w_{\epsilon}^{\delta,\gamma} = w_{\epsilon}\). Therefore from (3.10) and (3.11), for any \(\eta \in (0, \eta_{0})\) we obtain
\[
(3.12) \quad \int_{\Omega_{\eta}^{\delta}(x_{0})} [\epsilon |\nabla w_{\epsilon}^{\delta,\gamma}|^{2} + \frac{1}{\epsilon} W(x_{0}, w_{\epsilon}^{\delta,\gamma})] dx \leq 2 H_{N-1}(\partial\Omega_{\eta}) \int_{0}^{1} W^{1/2}(x_{0}, \gamma(t)) |\dot{\gamma}(t)| dt
\]
\[
+ H_{N-1}(\partial\Omega_{\eta})[0(\epsilon/\eta^{2}) + 0(\epsilon^{1/4}) + 0_{\sqrt{\epsilon^{2}+\delta^{2}}}(1)].
\]
Here by \(0_{\epsilon}(1)\) we mean \(\lim_{\epsilon \to 0} 0_{\epsilon}(1) = 0\). Since for any \(\epsilon > 0\) there exist a sequence of \(C^{1}\)-curves \(\{\gamma_{i}\}_{i=1}^{\infty}\) such that the length of \(\gamma_{i} \leq \epsilon^{-1/8}\) and
\[
\lim_{i \to \infty} \int_{0}^{1} W^{1/2}(x_{0}, \gamma_{i}(t)) |\dot{\gamma}_{i}(t)| dt = d(x_{0}, a, b),
\]
by the diagonal argument and (3.12), we can construct a sequence \(\{w_{\epsilon}^{\delta}\}_{\epsilon,\delta > 0}\) satisfying (3.4). Therefore the aim of the first step is completed.

**The Second Step:** Let \(\Omega_{\delta}\) be a domain \(\{x \in \Omega : \delta < d_{\partial\Omega}(x) < s^{*}\} = \bigcup_{\delta < t < s^{*}} \pi_{t}(\partial\Omega)\). At the second step, we construct a sequence \(\{w_{\epsilon}^{\delta}\}_{\epsilon,\delta > 0}\) in \(W^{1,2}(\Omega_{\delta}, \mathbb{R}^{n})\) such that
\[
(3.13) \quad \limsup_{\delta,\epsilon \to 0} \int_{\Omega_{\delta}} [\epsilon |\nabla w_{\epsilon}^{\delta}|^{2} + \frac{1}{\epsilon} W(x, w_{\epsilon}^{\delta})] dx \leq 2 \int_{\partial\Omega} d(x, a, g(x)) dH_{N-1}.
\]
In order to construct a sequence \(\{w_{\epsilon}^{\delta}\}_{\epsilon,\delta > 0}\), we will separate \(\partial\Omega\) into small pieces. From the regularity of \(\partial\Omega\), for sufficiently small \(\eta > 0\), there exist points \(x_{i}\) such that
\[
(3.14) \quad \partial\Omega \setminus \bigcup_{1 \leq i \leq p} \Omega_{\eta}(x_{i}) \subset \omega_{\eta}, \quad \Omega_{\eta}(x_{i}) \cap \Omega_{\eta}(x_{j}) = \emptyset, \quad i \neq j, \quad i, j = 1, 2, \ldots, p.
\]
and \( \lim_{\eta \to 0} \mathcal{H}_{N-1}(\omega_{\eta}) = 0 \). Here we note that \( p \) depends on \( \eta \) and \( \lim_{\eta \to 0} p(\eta) = \infty \).

For any \( \eta, \delta, \epsilon > 0 \), fix \( \eta, \delta, \) and \( \epsilon \). Then for any \( i \in \{1, 2, \cdots, p\} \), from (3.10) we can construct functions \( w_{\epsilon}^{i, \delta, \eta} \in W^{1,2}(\Omega_{\eta}(x_i)) \) such that

\[
\int_{\Omega_{\eta}(x_i)} \left[ \epsilon |\nabla w_{\epsilon}^{i}|^2 + \frac{1}{\epsilon} W(x, w_{\epsilon}^{i}) \right] dx \\
\leq 2\mathcal{H}_{N-1}(\partial \Omega_{\eta}(x_i))d(x_i, \alpha, g(x_i)) \\
+ \mathcal{H}_{N-1}(\partial \Omega_{\eta}(x_i))\left[ 0(\epsilon/\eta^2) + 0(\epsilon^{1/4}) + \sqrt{\epsilon^{2} + \delta^{2}}(1) \right].
\]

Then we define \( w_{\epsilon}^{\delta, \eta} \in W^{1,2}(\Omega_{\delta} : \mathbb{R}^n) \) as follows:

\[
w_{\epsilon}^{\delta, \eta} = \begin{cases} &w_{\epsilon}^{i, \delta, \eta}, \quad \text{if } x \in \Omega_{\eta}(x_i), \\ &\alpha, \quad \text{otherwise}. \end{cases}
\]

By the argument of Step 1, we can see \( w_{\epsilon}^{\delta, \eta} \in W^{1,2}(\Omega_{\delta} : \mathbb{R}^n) \) easily. Then we have

\[
\int_{\Omega_{\delta}} \left[ \epsilon |\nabla w_{\epsilon}^{\delta, \eta}|^2 + \frac{1}{\epsilon} W(x, w_{\epsilon}^{\delta, \eta}) \right] dx = \sum_{i=1}^{p(\eta)} \int_{\Omega_{\eta}^{\delta}(x_i)} \left[ \epsilon |\nabla w_{\epsilon}^{i, \delta, \eta}|^2 + \frac{1}{\epsilon} W(x, w_{\epsilon}^{i, \delta, \eta}) \right] dx
\]

On the other hand, we have (for simplicity we omit the index \( \delta, \eta \) of \( w_{\epsilon}^{i, \delta, \eta} \))

\[
\int_{\Omega_{\eta}^{\delta}(x_i)} \left[ \epsilon |\nabla w_{\epsilon}^{i}|^2 + \frac{1}{\epsilon} W(x, w_{\epsilon}^{i}) \right] dx = \sum_{i=1}^{p(\eta)} \int_{\Omega_{\eta}^{\delta}(x_i)} \left[ \epsilon |\nabla w_{\epsilon}^{i}|^2 + \frac{1}{\epsilon} W(x, w_{\epsilon}^{i}) \right] dx
\]

From (3.15) we obtain

\[
\sum_{i=1}^{p(\eta)} I_{1}^{i} \leq 2 \sum_{i=1}^{p(\eta)} \left[ d(x_i, \alpha, g(x_i))\mathcal{H}_{N-1}(\partial \Omega_{\eta}(x_i)) \right] + 0(\epsilon/\eta^2) + 0(\epsilon^{1/4}) + \sqrt{\epsilon^{2} + \delta^{2}}(1),
\]

and from (1.2) and (3.15)

\[
\sum_{i=1}^{p(\eta)} |I_{2}^{i}| \leq \sum_{i=1}^{p(\eta)} \int_{\Omega_{\eta}^{\delta}(x_i)} 0|_{x \to x_i}(1)\frac{1}{\epsilon} W(x, w_{\epsilon}^{i}) dx \leq 0(1) \sum_{i=1}^{p(\eta)} I_{1}^{i}.
\]

We set \( \eta^2 = \epsilon^{3/4} \). Then combining (3.16) and (3.17), we obtain

\[
\lim_{\delta, \epsilon \to 0} \sup \int_{\Omega_{\delta}} \left[ \epsilon |\nabla w_{\epsilon}^{\delta, \eta(\epsilon)}|^2 + \frac{1}{\epsilon} W(x, w_{\epsilon}^{\delta, \eta(\epsilon)}) \right] dx \\
\leq \lim_{\epsilon \to 0} \sup 2 \sum_{i=1}^{p(\eta)} d(x_i, \alpha, g(x_i))\mathcal{H}_{N-1}(\partial \Omega_{\eta}(x_i))
\]
From the continuity of the function $d(x, \alpha, g(x))$, we obtain

$$
\sum_{i=1}^{p(\eta)} d(x_{j}, \alpha, g(x_{j}))\mathcal{H}_{N-1}(\partial\Omega_{\eta}(x_{i})) \leq \int_{\bigcup_{1 \leq i \leq p} \partial\Omega_{\eta}(x_{i})} d(x, \alpha, g(x))d\mathcal{H}_{N-1} + o_{\eta}(1)
$$

$$
\leq \int_{\partial\Omega} d(x, \alpha, g(x))d\mathcal{H}_{N-1} + o_{\eta}(1)
$$

Therefore combining (3.18), we can see that the sequence $\{w_{\epsilon, \eta(\epsilon)}^\delta\}_{\epsilon, \delta > 0}$ satisfies (3.13). Hence we set $w_{\epsilon}^\delta = w_{\epsilon, \eta(\epsilon)}^\delta$, and so the purpose of Step 2 is completed. 

**The Third Step:** In this step, we will complete the proof of Proposition B for the special case $w_{0} \equiv \alpha$. For any $\delta, \epsilon > 0$ we define $w_{\epsilon}^\delta$ as follows:

$$
w_{\epsilon}^\delta = \begin{cases} 
\alpha, & \text{if } x \in \Omega \setminus \Omega_{0}, \\
\hat{w}_{\epsilon}^\delta, & \text{if } x \in \Omega_{\delta},
\end{cases}
$$

where $\Omega_{0} = \bigcup_{0 < t < \epsilon*} \pi_{t}(\partial\Omega)$ and where $\hat{w}_{\epsilon}^\delta$ is a function constructed in Step 2. In $\Omega_{\delta} \equiv \Omega_{0} \setminus \Omega_{\delta}$, we construct $w_{\epsilon}^\delta$ by combining between $g(x)$ and $w_{\epsilon}^\delta(\pi_{\delta}(x))$ i.e. for $x \in \Omega_{0} \setminus \Omega_{\delta}$,

$$
(3.19) \quad w_{\epsilon}^\delta(x) = \frac{d_{\partial\Omega}(x)}{\delta}w_{\epsilon}^\delta|_{(\partial\Omega)_{\delta}}(\pi_{\delta}(x_{i})) + \left(1 - \frac{d_{\partial\Omega}(x)}{\delta}\right)g(\pi_{\delta}(x_{i}))
$$

Here $\pi_{\delta}(x)$ and $\pi_{\partial\Omega}(x)$ are functions appearing in Lemma 3–1. Then we can see easily $w_{\epsilon}^\delta \in W^{1,2}(\Omega)$ and $w_{\epsilon}^\delta(x) = g(x)$ for all $x \in \partial\Omega$.

In order to estimate the gradient of $w_{\epsilon}^\delta$, we fix $\epsilon, \delta$, and fix $\Omega_{\eta,i}^\delta(x_{i}), i = 1, 2$ and $\omega_{\eta}$. Then we set

$$
\Omega_{1}^\delta = \{x \in \Omega^\delta: \pi_{\delta} \circ \pi_{d_{\partial\Omega}}^{-1}(x) \in \bigcup_{1 \leq i \leq p} \partial\Omega_{\eta,1}^\delta(x_{i})\},
$$

$$
\Omega_{2}^\delta = \{x \in \Omega^\delta: \pi_{\delta} \circ \pi_{d_{\partial\Omega}}^{-1}(x) \in \bigcup_{1 \leq i \leq p} \partial\Omega_{\eta,2}^\delta(x_{i})\},
$$

$$
\omega_{\eta}^\delta = \bigcup_{0 < t < \epsilon} \pi_{t}(\omega_{\eta}),
$$

and have $\Omega^\delta = \Omega_{1}^\delta \cup \Omega_{2}^\delta \cup \omega_{\eta}^\delta$. Here $\Omega_{\eta,i}^\delta(x_{i}), i = 1, 2$ is a domain appearing in Step 1. Furthermore for simplicity, we set

$$
\hat{g}(x) = g(\pi_{d_{\partial\Omega}}^{-1}(x)) \quad \text{and} \quad \hat{w}_{\epsilon}^\delta(x) = w_{\epsilon}^\delta|_{(\partial\Omega)_{\delta}}(\pi_{\delta} \circ \pi_{d_{\partial\Omega}}^{-1}(x))
$$

for $x \in \Omega^\delta$. Then from Lemma 3–1 we can see that there exists a constant $C$ such that $|\nabla \hat{g}(x)| \leq C$ for almost all $x \in \Omega^\delta$. 


Now in the domains $\Omega_{\eta,1}^\delta$, $\Omega_{\eta,2}^\delta$, and $\Omega^\delta$, we will estimate the gradient of $w_\epsilon^\delta$, and obtain the inequality (2.1). If $x \in \omega^\delta_\eta$, then from the construction of $w_\epsilon$ in Step 2 we see $w_\epsilon^{\delta,\eta} \equiv \alpha$ in a neighborhood of $x$, and so for almost all $x \in \omega^\delta_\eta$ we have

$$|
abla w_\epsilon^{\delta,\eta}| \leq C(1 + 1/\delta).$$

So we obtain

$$\int_{\omega^\delta_\eta} [\epsilon|\nabla w_\epsilon^{\delta,\eta}|^2 + \frac{1}{\epsilon} W(x, w_\epsilon^{\delta,\eta})] dx \leq C \left( \frac{\epsilon}{\delta^2} + \epsilon + \frac{1}{\epsilon} \right) \delta \mathcal{H}_{N-1}(\omega).$$

For almost all $x \in \Omega_{\eta,1}^\delta(x_i)$, then we have

$$|
abla w_\epsilon^{\delta,\eta}| \leq \frac{|\nabla d_{\partial\Omega}(x)|}{\delta} |\hat{w}_\epsilon^{\delta,\eta}(x)| + \frac{d_{\partial\Omega}(x)}{\delta} |\nabla \hat{w}_\epsilon^{\delta,\eta}(x)| + \frac{|\nabla d_{\partial\Omega}(x)|}{\delta} \supset(x) + (1 - \frac{d_{\partial\Omega}(x)}{\delta}) |\nabla \supset(x)|.$$

Here from the argument in Step 1, there exists a constant $C_2$ such that $|
abla w_\epsilon^{\delta,\eta}(x)| \leq C/(\epsilon^{5/8} \eta)$ for all $x \in \Omega_{\eta,1}^\delta$. Moreover we have $|\Omega_{\eta,1}^\delta| \leq C \delta (\epsilon^{5/8} \eta^{N-1}) (\mathcal{H}_{N-1}(\partial\Omega)/\eta^{N-1}) \leq C_2 \epsilon^{5/8} \delta$. So we obtain

$$\int_{\Omega_{\eta,1}^\delta} [\epsilon|\nabla w_\epsilon^{\delta,\eta}|^2 + \frac{1}{\epsilon} W(x, w_\epsilon^{\delta,\eta})] dx \leq C \left( \epsilon \left( \frac{1}{\delta} + \frac{1}{\epsilon^{5/8} \eta} + 1 \right)^2 + \frac{1}{\epsilon} \right) \delta \epsilon^{5/8} \leq C \left( \frac{\epsilon}{\delta} + \frac{\delta}{\eta^{2} \epsilon^{1/4}} + \frac{\delta}{\epsilon} \right) \epsilon^{5/8}.$$

For any $x \in \Omega_{\eta,2}^\delta(x_i)$, from Step 1 we see $w_\epsilon^\star(x) \equiv g(x_i)$ in a neighborhood of $x$. Then from the Lipschitz continuity of $g(x)$ on $\partial\Omega$ and (3.19) we have

$$|
abla w_\epsilon^{\delta,\eta}| \leq \frac{|\nabla d_{\partial\Omega}(x)|}{\delta} |g(x_i) - \hat{g}(x)| + (1 - \frac{d_{\partial\Omega}(x)}{\delta}) |\nabla \hat{g}| \leq \frac{C}{\delta} |g(x_i) - \hat{g}(x)| + C \leq C \frac{\eta}{\delta} + C.$$

So we obtain

$$\int_{\Omega_{\eta,2}^\delta} [\epsilon|\nabla w_\epsilon^{\delta,\eta}|^2 + \frac{1}{\epsilon} W(x, w_\epsilon^{\delta,\eta})] dx \leq C \left( \frac{\eta}{\delta} + \frac{\delta}{\eta^{2} \epsilon^{1/4}} + \frac{\delta}{\epsilon} \right) \epsilon^{5/8} \mathcal{H}_{N-1}(\partial\Omega).$$

Let $\sigma(\cdot)$ be a positive function with $\sigma(0) = 0$ such that $\lim_{\epsilon \to 0} \mathcal{H}_{N-1}(\omega_{\eta(\epsilon)})/\sigma(\epsilon) = 0$ and $\lim_{\epsilon \to 0} \epsilon^{5/8}/\sigma(\epsilon) = 0$. Here we set $\delta_\epsilon = \epsilon \sigma(\epsilon)$, and define $w_\epsilon = w_\epsilon^{\delta_\epsilon}$. Then from (3.21)-(3.23) we obtain

$$\lim_{\epsilon \to 0} \int_{\Omega_{\epsilon \sigma(\epsilon)}} [\epsilon|\nabla w_\epsilon|^2 + \frac{1}{\epsilon} W(x, w_\epsilon)] dx = 0.$$
Therefore from (3.13) and (3.24) we obtain
\[
\lim_{\epsilon \to 0} \sup_{0} \int_{\Omega} [\epsilon|\nabla w_\epsilon|^2 + \frac{1}{\epsilon} W(x, w)]dx \leq 2 \int_{\partial\Omega} d(x, \alpha, g(x))d\mathcal{H}_{N-1}.
\]
Hence the proof of Proposition B for the special case \( w_0 = \alpha \) is completed. \( \blacksquare \)

Finally we remark that the proof of this section is an essential part of complete proof of Proposition B.

REFERENCES