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Kyoto University
On a Local Energy Decay of Solutions

of a Dissipative Wave Equation

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§1. Introduction.

This study is concerned with a local energy decay property of solutions to the following initial boundary value problem of the dissipative wave equation:

\[
\begin{aligned}
(D) \begin{cases}
    u_{tt} + u_t - \Delta u = 0 & \text{in } \Omega \text{ and } t > 0, \\
    u = 0 & \text{on } \Gamma \text{ and } t > 0, \\
    u(0, x) = u_0(x), \ u_t(0, x) = u_1(x) & \text{in } \Omega,
\end{cases}
\end{aligned}
\]

where \(\Omega\) is an exterior domain in an \(n\)-dimensional Euclidean space \(\mathbb{R}^n\), whose boundary \(\Gamma\) is a \(C^\infty\) and compact hypersurface. Below, \(r_0 > 0\) is a fixed constant such that \(\Omega^c \subset B_{r_0} = \{x \in \mathbb{R}^n \mid |x| < R\}\). (\(\Omega^c\) is the complement of \(\Omega\).)

In the wave equation case, the local energy decays exponentially fast if \(n\) is odd and polynomially fast if \(n\) is even, when \(\Omega\) is at least non-trapping (cf. [9], [10], [11], [16]). In fact, from a physical point of view the energy propagates along the wave fronts, so that the motion stops after time passes unless the wave front is trapped in a bounded set.

In the dissipative wave case, the energy also propagates along the wave front. Moreover, the trapped energy also decreases in virtue of the dissipative term \(u_t\), so that we can expect to get the local energy decay result for any domains. In fact, in 1983 Shibata [14] proved the following theorem.

**Theorem 1.1.** Assume that \(n \geq 3\). Let \(R > r_0\) and let \(u(t, x)\) be a smooth solution of \((D)\) such that \(\text{supp}u(0, x), \text{supp}u_t(0, x) \subset \Omega_R = \{x \in \Omega \mid |x| < R\}\). Then, there exists
a constant $C > 0$ depending on $n$ and $R$ such that

$$
\int_{\Omega_{R}} \{ |u_t(t, x)|^2 + \sum_{|\alpha| \leq 1} |\partial_x u(t, x)|^2 \} dx
\leq C(1 + t)^{-n} \left\{ \sum_{|\alpha| \leq 3} \int_{\Omega} |\partial_x^{\alpha} u_{t}(0, x)|^2 dx + \sum_{|\alpha| \leq 4} \int_{\Omega} |\partial_x^{\alpha} u(0, x)|^2 dx \right\},
$$

where $\partial_x^{\alpha} v = \partial^{|\alpha|} v / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The purpose of this study is to show the decay rate of the local energy of even the weak solutions of (D) is also $n/2$ when $n \geq 2$, that is, we shall prove the following theorem.

**Theorem 1.2.** Assume that $n \geq 2$. Let $R > r_0$ and $u_0 \in H_0^{1, R}(\Omega)$ and $u_1 \in L_{R}^{2}(\Omega)$, where

$$
L_{R}^{2}(\Omega) = \{ f \in L^{2}(\Omega) \mid \text{supp} f \subset \Omega_{R} \},
$$

$$
H_{0, R}^{1}(\Omega) = \{ f \in H^{1}(\Omega) \mid \text{supp} f \subset \Omega_{R}, \ f = 0 \text{ on } \Gamma \}.
$$

Let $u(t, x)$ be a weak solution of (D). Then, there exists a constant $C$ depending on $n$ and $R$ such that

$$
\int_{\Omega_{R}} \{ |u_t(t, x)|^2 + \sum_{|\alpha| \leq 1} |\partial_x^{\alpha} u(t, x)|^2 \} dx
\leq C(1 + t)^{-n} \left\{ \int_{\Omega} |u_1(x)|^2 dx + \sum_{|\alpha| \leq 1} \int_{\Omega} |\partial_x^{\alpha} u_0(x)|^2 dx \right\}.
$$

Compared with Theorem 1.1, in Theorem 1.2 we remove the smoothness assumption on solutions of (D) and we consider the case that $n = 2$ as well as the case that $n \geq 3$.

For the Cauchy problem of the dissipative wave equation (i.e. $\Omega = \mathbb{R}^n$), A. Matsumura [8] studied the decay rate of solutions in 1976. His argument was based on the concrete representation of solutions by using the Fourier transform. When $\Omega$ is bounded, it is well-known that the energy of solutions decays exponentially fast. In fact, this fact is easily proved by the multiplications of the equation with $u_t$ and $u$.
and by use of Poincaré's inequality. Since $\Omega$ is unbounded in our case, we cannot use Poincaré's inequality. And also, because of the boundary, we can not use the Fourier transform. Our method is based on a spectral analysis to the corresponding stationary problem.

§2. A construction of $C_0$ semigroup solving (D).

Putting $u_t = v$, let us rewrite the problem (D) in the following form:

$$\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix}.$$

To consider $A$ to be dissipative, we introduce a space $H_D(\Omega)$. For any open set $\mathcal{O} \subset \mathbb{R}^n$, $C_0^\infty(\Omega)$ denotes the space of all $C^\infty$ functions on $\mathbb{R}^n$ whose support is compact and lies in $\mathcal{O}$ (in particular, such functions vanish near the boundary of $\mathcal{O}$), $L^2(\mathcal{O})$ a usual $L^2$ space on $\mathcal{O}$ with norm $\| \cdot \|_\mathcal{O}$ innerproduct $( \cdot , \cdot )_\mathcal{O}$ and $H^s(\mathcal{O})$ a usual Sobolev space of order $s$ on $\mathcal{O}$ with norm $\| \cdot \|_{s, \mathcal{O}}$. $\| \cdot \|_{k, \Omega}$ will be denoted simply by $\| \cdot \|_k$. Likewise for $\| \cdot \|_\Omega$ and $( \cdot , \cdot )_\Omega$. Then, we put

$$H_D(\Omega) = \{ u \in H_{loc}^1(\Omega) | \nabla u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}) \in L^2(\Omega), \ u = 0 \ \text{on} \ \Gamma, \ \exists \ \{ u_n \} \subset C_0^\infty(\Omega) \ \text{s.t.} \ \| \nabla (u_n - u) \| \rightarrow 0 \ \text{as} \ n \rightarrow \infty \} ,$$

where $H_{loc}^1(\Omega) = \{ u \in D'(\Omega) | u \in H^1(\Omega_R) \ \forall R > r_0 \}$. $H_D(\Omega)$ has the following properties.

**Theorem 2.1.** If $u \in H_D(\Omega)$, then $u$ satisfies the following inequalities:

$$\| u \|_{0, \Omega_R} \leq C(R) \| \nabla u \|_{0, \Omega_R} ,$$

$$\int_{\Omega} \frac{|u(x)|^2}{d(x)^2} dx \leq C \| \nabla u \|^2 .$$

Moreover, $H_D(\Omega)$ is a Hilbert space equipped with an inner product $(u, v)_D = (\nabla u, \nabla v)$. 
Then, an underlying space for $A$ is

$$\mathcal{H} = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \mid u \in H_D(\Omega), v \in L^2(\Omega) \right\}. $$

From Theorem 2.1 we know that $\mathcal{H}$ is a Hilbert space equipped with the inner product

$$\left( \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} w \\ z \end{bmatrix} \right)_\mathcal{H} = (u, w)_D + (v, z).$$

The domain of $A$ is

$$D(A) = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H} \mid A \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H} \right\}$$

$$= \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H} \mid v \in H_D(\Omega), \Delta u \in L^2(\Omega) \right\}. $$

Then, $A$ has the following properties.

**Proposition 2.2.**

1. $A$ is a closed operator.
2. $A$ is a dissipative operator.
3. $\mathcal{R}(I - A) = \mathcal{H}.$
4. $D(A)$ is dense in $\mathcal{H}.$

Lumer and Phillips theorem [13, Chapter 1, Theorem 4.3] implies that $A$ generates a $C^0$ semigroup $\{T(t)\}$ on $\mathcal{H}.$

**§3. A proof of Theorem 1.2.**

Our purpose in this section is to prove the following result, which implies our main theorem.

**Theorem 3.1.**

$$\| \varphi_R T(t) x \|_\mathcal{H} \leq C(1 + t)^{-n/2} \| x \|_\mathcal{H},$$

for $x \in \mathcal{H}_{1,R}$, where $C = C(R).$

**Sketch of proof.**
Since $A$ is dissipative, $T(t)$ is a $C_0$ semigroup of contractions, so that

$$\|T(t)\| \leq 1 \quad \forall t \geq 0.$$  \hspace{1cm} (3.1)

Let $\alpha$ be a positive number. In view of (3.1), we have the following expression:

$$T(t)x = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\alpha-i\omega}^{\alpha+i\omega} e^{\lambda t}(\lambda I - A)^{-1}x d\lambda$$  \hspace{1cm} (3.2)

for $x \in D(A^2)$.

(cf. [12, p.295] or [13, Chapter 1, Corollary 7.5]). By a lemma due to F. Huang in [4, §1, Lemma 1] (also see [7]), we have the following lemma.

**Lemma 3.2.** For any $\alpha > 0$ and $x \in \mathcal{H}$, put

$$g(\omega) = \|(\alpha + i\omega)I - A)^{-1}x\|_{\mathcal{H}}.$$  

Then $g(\omega) \in L^2(\mathbb{R})$ and

$$\lim_{|\omega| \to \infty} g(\omega) = 0,$$

$$\int_{-\infty}^{\infty} g(\omega)^2 d\omega \leq \frac{\pi}{\alpha} \|x\|_{\mathcal{H}}^2.$$  

In view of Lemma 3.2, the high frequency part decays sufficiently fast, so that we have to investigate the low frequency part. Now we shall introduce some functional spaces. Let $E$ be a Banach space with norm $| \cdot |_E$, $N \geq 0$ an integer and $k = N + \sigma$ with $0 < \sigma \leq 1$. Put

$$C^k(\mathbb{R}^1; E) = \{u \in C^{N-1}(\mathbb{R}^1; E) \cap C^{\infty}(\mathbb{R}^1 - \{0\}; E); \ll u \gg k, E < \infty\},$$

where

$$\ll u \gg k, E = \sum_{j=0}^{N} \int_{\mathbb{R}} |\left(\frac{d}{d\tau}\right)^j u(\tau)|_E d\tau$$

$$+ \sup_{h \neq 0} |h|^{-\sigma} \int_{\mathbb{R}} |\left(\frac{d}{d\tau}\right)^N u(\tau + h) - \left(\frac{d}{d\tau}\right)^Nu(\tau)|_E d\tau$$  \hspace{1cm} if $0 < \sigma < 1,$

$$\ll u \gg k, E = \sum_{j=0}^{N} \int_{\mathbb{R}} |\left(\frac{d}{d\tau}\right)^j u(\tau)|_E d\tau$$
\[ + \sup_{h \neq 0} |h|^{-1} \int_{\mathbb{R}} \left| \left( \frac{d}{d\tau} \right)^N u(\tau + 2h) - 2 \left( \frac{d}{d\tau} \right)^N u(\tau + h) + \left( \frac{d}{d\tau} \right)^N u(\tau) \right|_E d\tau, \]

if \( \sigma = 1 \). Here, \( \left( \frac{d}{d\tau} \right)^0 = 1 \).

The following lemma is concerned with the properties of the Fourier transformation of functions belonging to \( C^k(\mathbb{R}^1, E) \), which was proved in [14, Part 1, Theorem 3.7].

**Lemma 3.3.** Let \( E \) be a Banach space with norm \( |\cdot|_E \). Let \( N \geq 0 \) be an integer and \( \sigma \) a positive number \( \leq 1 \). Assume that \( f \in C^{N+\sigma}(\mathbb{R}^1; E) \). Put

\[ F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\tau) \exp(\sqrt{-1}\tau t) d\tau. \]

Then,

\[ |F(t)|_E \leq C (1 + |t|)^{-(N+\sigma)} \ll f \gg N+\sigma, E. \]

Here and hereafter, we put \( \mathcal{H}_R = \{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H} | \text{supp} u, \text{supp} v \subset \Omega_R \} \). \( \varphi_R(x) \) always refers to a function in \( C_0^\infty(\mathbb{R}^n) \) such that \( \varphi_R(x) = 1 \) if \( |x| \leq R \) and \( = 0 \) if \( |x| \geq R + 1 \).

Moreover, we put

\[ \mathcal{H}_{loc} = \{ \begin{bmatrix} u \\ v \end{bmatrix} | u \in H^1(\Omega_R), v \in L^2(\Omega_R) \text{ \forall } R \geq r_0 \} , \]

\[ \mathcal{H}_{comp} = \bigcup_{R \geq r_0} \mathcal{H}_R, \]

and \( \mathcal{L}(B_1, B_2) \) denotes the set of all bounded linear operators from \( B_1 \) into \( B_2 \) and \( \text{Anal}(I, B) \) the set of all \( B \)-valued analytic functions in \( I \). In view of Lemma 3.3, if we prove the following fact, the proof of Theorem 3.1 is complete.

(F) Put \( Q_d = \{ \lambda \in \mathbb{C} | 0 < \Re \lambda < d, |\Im \lambda| < d \} \). Then, there exists a \( d > 0 \) and \( R(\lambda) \in \text{Anal}(Q_d; \mathcal{L}(\mathcal{H}_{comp}, \mathcal{H}_{loc})) \) such that:

(a) \( R(\lambda)x = (\lambda I - A)^{-1}x \) for \( x \in \mathcal{H}_{comp} \) and \( \lambda \in Q_d \);

(b) For any \( R \geq r_0 \) and \( \rho(s) \in C_0^\infty(\mathbb{R}) \) such that \( \rho(s) = 1 \) if \( |s| < d/2 \) and \( = 0 \) if \( |s| > d \), there exist \( M_1 > 0 \) depending on \( R, \rho \) and \( d \) such that

\[ \ll \rho(\cdot)(\varphi_R R(\alpha + i\cdot)x, y)_{\mathcal{H}} \gg n/2, \mathbb{R} \leq M_1 ||x||_{\mathcal{H}} ||y||_{\mathcal{H}}, \]
for any \( x \in \mathcal{H}_R, y \in \mathcal{H} \) and \( 0 < \alpha < d \).

We shall conclude this report by giving a brief proof of (F).

**Proof of (F).**

When \( n \geq 3 \), (F) was proved by Shibata [14, Part 1], so that we shall consider the case that \( n = 2 \). Corresponding stationary problem is

\[
(3.3) \quad (\lambda^2 + \lambda - \Delta)u = f \quad \text{and} \quad u = 0 \quad \text{on} \quad \Gamma.
\]

If \( |\lambda| \) is small, then in stead of (3.3), it is sufficient to consider the following problem :

\[
(A_{\lambda}) \quad (\lambda - \Delta)u = f \quad \text{in} \quad \Omega \subset \mathbb{R}^2 \quad \text{and} \quad u = 0 \quad \text{on} \quad \Gamma,
\]

where \( \lambda \in S_{r,\epsilon} = \{ \lambda \in \mathbb{C} \setminus \{0\} \mid |\lambda| < r, \ |\arg\lambda| < \pi - \epsilon \}, \ 0 < r < 1 \) and \( 0 < \epsilon < \pi/2 \), because \( \lambda^2 + \lambda \) is equivalent to \( \lambda \) for small \( |\lambda| \). In view of Lemma 3.4 of [14], in order to prove (F), it is sufficient to prove the following propositions.

**Proposition 3.4.** For \( \lambda \in S_{r,\epsilon} \) and \( r_0 \leq R < \infty \), there exists \( A(\lambda) : L_R^2(\Omega) \to H_{loc}^1(\Omega) \) satisfying that

\[
(\lambda - \Delta)A(\lambda)f = f \quad \text{in} \quad \Omega \quad \text{and} \quad A(\lambda)f = 0 \quad \text{on} \quad \Gamma,
\]

for \( f \in L_R^2(\Omega) \). Moreover, it satisfies that

\[
\|\varphi_R A(\lambda)f\|_1 \leq C\|f\| \quad \text{as} \quad \lambda \in S_{r,\epsilon}.
\]

**Proposition 3.5.** For \( \lambda \) and \( R \) as mentioned above, following estimates hold;

\[
\|\varphi_R \frac{d}{d\lambda} A(\lambda)f\|_1 \leq \frac{C(R)}{|\lambda||\log\lambda|^2}\|f\| \leq \frac{C(R)}{|\Im\lambda|\lambda|^2}\|f\|,
\]

\[
\|\varphi_R \frac{d^2}{d\lambda^2} A(\lambda)f\|_1 \leq \frac{C(R)}{|\lambda|^2|\log\lambda|^2}\|f\| \leq \frac{C(R)}{|\Im\lambda|^2\lambda|^2}\|f\|,
\]

\[
\|\varphi_R A(\lambda)f\|_1 \leq C\|f\| \quad \text{as} \quad \lambda \in S_{r,\epsilon}.
\]
for $f \in L^2_R(\Omega)$.

Our main idea to prove Propositions 3.4 and 3.5 is to use the single layer potential and the double layer potential, and to reduce $(A_{\lambda})$ to a boundary integral equation. Put $v = (\lambda - \Delta)^{-1}f$. Then $v$ is represented by the modified Bessel function:

$$(\lambda - \Delta)^{-1}f = \int_{\mathbb{R}^2} E_\lambda(x - y)f(y)dy,$$

where $E_\lambda(x) = (2\pi)^{-1}K_0(|x|\sqrt{\lambda})$, $K_m$ ($m \in \mathbb{N} \cup\{0\}$) denotes the modified Bessel function of order $m$. So we want to solve the equation

$$(A'_{\lambda}) \quad (\lambda - \Delta)w = 0 \text{ in } \Omega \quad \text{and} \quad w = f_\lambda \text{ on } \Gamma,$$

where $f_\lambda = (\lambda - \Delta)^{-1}f|_\Gamma$. To do this, let us introduce the integral operator $B_\lambda$:

$$B_\lambda \Phi = D_\lambda \Phi - \eta E_\lambda M \Phi + \frac{2\pi \alpha}{\log \sqrt{\lambda}} E_\lambda \Phi \quad \text{for } \Phi \in C^0(\Gamma).$$

Here $\alpha, \eta > 0$, $E_\lambda$ is a single layer potential defined by

$$E_\lambda \Psi(x) = \int_{\Gamma} E_\lambda(x - y)\Psi(y)do_y$$

and $D_\lambda$ is a double layer potential defined by

$$D_\lambda \Psi(x) = \int_{\Gamma} D_\lambda(x, y)\Psi(y)do_y,$$

where

$$D_\lambda(x, y) = \nabla_x E_\lambda(x - y) \cdot N(y)$$

$$= -\frac{1}{2\pi} K_1(|x - y|\sqrt{\lambda}) \frac{\sqrt{\lambda}}{|x - y|}(x - y) \cdot N(y).$$

The projection $M : C^0(\Gamma) \rightarrow C^0(\Gamma)$ is defined by

$$\Phi \rightarrow M\Phi = \Phi - \Phi_M \quad \text{with} \quad \Phi_M = \frac{1}{|\Gamma|} \int_{\Gamma} \Phi do \text{ and } |\Gamma| = \text{meas}(\Gamma).$$
Obviously $B_{\lambda}\Phi$ satisfies that $(\lambda - \Delta)B_{\lambda}\Phi = 0$ in $\Omega$, so that we obtain the following boundary integral equation:

$$B_{\lambda}\Phi|_{\Gamma} = K_{\lambda}\Phi = (-\frac{1}{2} + D_{\lambda} - \eta E_{\lambda}M + \frac{2\pi\alpha}{\log \sqrt{\lambda}} E_{\lambda})\Phi = f_{\lambda}. \quad (3.4)$$

If $\Phi$ is a solution of $(3.4)$, $B_{\lambda}\Phi$ satisfies $(A_{\lambda}')$, and $A(\lambda)f$ is expressed by

$$(3.5) \quad A(\lambda)f = (\lambda - \Delta)^{-1}f - B_{\lambda}\Phi.$$ 

Therefore, $(A_{\lambda})$ was reduced to a boundary integral equation $(3.4)$. $K_{\lambda}$ is a Fredholm operator, so that by using the Fredholm alternative theorem, we can solve the boundary equation $(3.4)$. If we consider that $A(\lambda)$ is an operator from $L_{R}^{2}(\Omega)$ to $L_{loc}^{2}(\Omega)$, by the properties of Bessel function, we know that the expansion of $A(\lambda)$ at $\lambda \to 0$ is

$$A(\lambda) = C_{0} + C_{1}\frac{1}{\log \lambda} + \cdots.$$

Therefore, we have Propositions 3.4 and 3.5.

**References**