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On damped or strongly damped hyperbolic system

By

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1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^k$ with Lipschitz boundary $\partial\Omega$. For a map $u : \Omega \times (0,\infty) \to \mathbf{R}^\ell$, we consider the hyperbolic system

$$(1.1)\quad \begin{aligned} &a_{ij}(x)D_t^2u^i(x,t) - D_\beta\left(b_{ij}^{\alpha\beta}(x)D_\alpha u^i(x,t)\right) + c_{ij}(x)\|u(x,t)\|_c^{m-2}u^i(x,t) \\ &\quad + a_{ij}(x)D_tu^i(x,t) = 0 \quad \text{in } \Omega\times(0,\infty), \quad j=1,\cdots,\ell, \end{aligned}$$

or

$$(1.2)\quad \begin{aligned} &a_{ij}(x)D_t^2u^i(x,t) - D_\beta\left(b_{ij}^{\alpha\beta}(x)D_\alpha u^i(x,t)\right) + c_{ij}(x)\|u(x,t)\|_c^{m-2}u^i(x,t) \\ &\quad - D_tD_\beta\left(f_{ij}^{\alpha\beta}(x)D_\alpha u^i(x,t)\right) = 0 \quad \text{in } \Omega\times(0,\infty), \quad j=1,\cdots,\ell. \end{aligned}$$

Here D_t and D_α mean the partial derivatives with respect to variable t and x^α, *i.e.*,

$$D_t = \partial/\partial t, \quad D_\alpha = \partial/\partial x^\alpha.$$

The c-norm $\|\cdot\|_c$ of u is the square root of the quadratic form $c_{ij}u^iu^j$. Similar notations $\|u\|_a$, $\|Du\|_b$ and $\|Du\|_f$ will appear later, and their meaning are

$$\|u\|_a = \left(a_{ij}u^iu^j\right)^{\frac{1}{2}}, \quad \|Du\|_b = \left(b_{ij}^{\alpha\beta}D_\alpha u^iD_\beta u^j\right)^{\frac{1}{2}}, \quad \|Du\|_f = \left(f_{ij}^{\alpha\beta}D_\alpha u^iD_\beta u^j\right)^{\frac{1}{2}}.$$

And $m > 1$ is a constant.

Here and in the following, summation over repeated indices is understood, the greek indices run from 1 to k, and the latin ones from 1 to ℓ. We assume that the coefficients $a_{ij}(x)$, $b_{ij}^{\alpha\beta}(x)$ and $c_{ij}(x)$ are bounded functions defined on Ω and satisfy the coercive condition

$$(1.3)\qquad \begin{cases} a_{ij}(x)\xi^i\xi^j \geq \lambda_0|\xi|^2 & \text{for all } \ \xi \in \mathbf{R}^\ell, \\ b_{ij}^{\alpha\beta}(x)\eta_\alpha^i\eta_\beta^j \geq \lambda_1|\eta|^2 & \text{for all } \ \eta \in \mathbf{R}^{k\ell}, \\ c_{ij}(x)\xi^i\xi^j \geq \lambda_2|\xi|^2 & \text{for all } \ \xi \in \mathbf{R}^\ell, \\ f_{ij}^{\alpha\beta}(x)\eta_\alpha^i\eta_\beta^j \geq \lambda_3|\eta|^2 & \text{for all } \ \eta \in \mathbf{R}^{k\ell} \end{cases}$$

for some positive constants λ_0, λ_1, λ_2 and λ_3, and the symmetry

$$(1.4)\qquad a_{ij}(x) = a_{ji}(x), \quad b_{ij}^{\alpha\beta}(x) = b_{ji}^{\beta\alpha}(x), \quad c_{ij}(x) = c_{ji}(x), \quad f_{ij}^{\alpha\beta}(x) = f_{ji}^{\beta\alpha}(x).$$

We call (1.1) the *damped hyperbolic system*, or the hyperbolic system with a *damping term* $a_{ij}(x)D_tu^i(x,t)$. And the second system is called the *strongly damped hyperbolic system*, or the hyperbolic system with a *strongly damping term* $-D_tD_\beta\left(f_{ij}^{\alpha\beta}(x)D_\alpha u^i(x,t)\right)$. The strongly damping term is also called the *viscosity term*. These system appear in some models of continuum mechanics. For the historical remark we can refer [2] and references cited therein.

We impose the initial and boundary conditions

$$(1.5)\qquad u(x,0) = u_0(x), \quad D_tu(x,0) = v_0(x) \quad \text{in } \ \Omega,$$

$$(1.6)\qquad u(x,t) = w(x) \quad \text{on } \ \partial\Omega \times (0,\infty),$$

where $u_0(x)$, $v_0(x)$ and $w(x)$ are given maps satisfying the compatibility condition $u_0(x) = w(x)$ on $\partial\Omega$.

Our aim is two-folds. The first one is to construct global weak solutions by the method of time-discretization. And the second one is to show their decay properly as $t \to \infty$ in case of $w \equiv 0$, *i.e.*, homogeneous Dirichlet's boundary condition.

First we give the notion of weak solution. Let $\gamma_{\partial\Omega}$ and $\gamma_{t=0}$ denote the trace operators to $\partial\Omega$ and $\Omega \times \{0\}$ respectively.

Definition 1.1. For u_0, $w \in H^{1,2}(\Omega) \cap L^m(\Omega)$ and $v_0 \in L^2(\Omega)$ satisfying $\gamma_{\partial\Omega}u_0 = \gamma_{\partial\Omega}w$, a map $u : \Omega \times [0,T) \to \mathbf{R}^\ell$ is called a *weak solution* of (1.1) on $\Omega \times [0,T)$ with the initial and boundary conditions (1.5) – (1.6), if the following conditions are satisfied:
(i) $u \in L^\infty(0,T; H^{1,2}(\Omega) \cap L^m(\Omega))$ with $D_tu \in L^\infty(0,T; L^2(\Omega))$.
(ii) $\gamma_{t=0}u(x,t) = u_0(x)$ and $\gamma_{\partial\Omega}u(x,t) = \gamma_{\partial\Omega}w(x)$ for $0 < t < T$.

(iii) For any $\psi(x,t) \in C_0^1([0,T); C_0(\Omega)) \cap C([0,T); C^1(\Omega))$,

$$\int_0^T \int_\Omega \Big(-a_{ij}(x) D_t u^i(x,t) D_t \psi^j(x,t) + b_{ij}^{\alpha\beta}(x) D_\alpha u^i(x,t) D_\beta \psi^j(x,t)$$

$$+ c_{ij}(x) \|u(x,t)\|_c^{m-2} u^i(x,t) \psi^j(x,t) + a_{ij}(x) D_t u^i(x,t) \psi^j(x,t) \Big) \, dxdt$$

$$= \int_\Omega a_{ij}(x) v_0^i(x) \psi^j(x,0) dx.$$

Definition 1.2. For u_0, $w \in H^{1,2}(\Omega) \cap L^m(\Omega)$ and $v_0 \in H_0^{1,2}(\Omega)$ satisfying $\gamma_{\partial\Omega} u_0 = \gamma_{\partial\Omega} w$, a map $u : \Omega \times [0,T) \to \mathbf{R}^\ell$ is called a *weak solution* of (1.2) on $\Omega \times [0,T)$ with the initial and boundary conditions (1.5) – (1.6), if the following conditions are satisfied:
(i) $u \in L^\infty(0,T; H^{1,2}(\Omega) \cap L^m(\Omega))$ with $D_t u \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^{1,2}(\Omega))$.
(ii) $\gamma_{t=0} u(x,t) = u_0(x)$ and $\gamma_{\partial\Omega} u(x,t) = \gamma_{\partial\Omega} w(x)$ for $0 < t < T$.
(iii) For any $\psi(x,t) \in C_0^1([0,T); C_0(\Omega)) \cap C([0,T); C^1(\Omega))$,

$$\int_0^T \int_\Omega \Big(-a_{ij}(x) D_t u^i(x,t) D_t \psi^j(x,t) + b_{ij}^{\alpha\beta}(x) D_\alpha u^i(x,t) D_\beta \psi^j(x,t)$$

$$+ c_{ij}(x) \|u(x,t)\|_c^{m-2} u^i(x,t) \psi^j(x,t) + f_{ij}^{\alpha\beta}(x) D_t D_\alpha u^i(x,t) D_\beta \psi^j(x,t) \Big) \, dxdt$$

$$= \int_\Omega a_{ij}(x) v_0^i(x) \psi^j(x,0) dx.$$

Definition 1.3. We say u is a *global weak solution* if $u|_{\Omega \times [0,T)}$ is a weak solution on $\Omega \times [0,T)$ for any $T > 0$.

We discuss the damped and strongly damped hyperbolic systems in § 2 and § 3 respectively. This note is an epitome of [5, 6].

2 The damped hyperbolic system

2.1 A construction of weak solutions

Here we construct weak solutions by use of a combination of time-discretization and calculus of variations. Though our system solved in several different way, we omit the historical remark of the equations. The authors, however, think that our method is not so familiar. Hence we point out only the previous result on our method applied various equations. The method explained here was firstly introduced by Rektorys [7] in 1971. He applied it to linear parabolic equations. Independently Kikuchi [3] rediscovered this method in 1991, and he tried to apply the method to non-linear equations coming from variational problems. Actually Bethuel-Coron-Ghidaglia-Soyeur [1] constructed a weak solution of the heat flow for harmonic maps by the method. The authors also constructed weak solutions of a semi-linear hyperbolic system and the Navier-Stokes equations by the method in [9] and [4] respectively.

We firstly construct an approximate solution as follows. Let h be a positive number, which will tend to zero later. u_0 is a given initial data of u. u_1 is defined by

$$u_1(x) = u_0(x) + v(x,h),$$

where v is an $\mathbf{R}^\ell$-valued function satisfying

$$(2.7)\qquad \begin{cases} v(x,0)=0, \quad D_t v(x,0) = v_0(x) \quad \text{in } \Omega, \quad v(x,t)=0 \quad \text{on } \partial\Omega\times\mathbf{R}, \\ v \in L^\infty(\mathbf{R}; H^{1,2}(\Omega)\cap L^m(\Omega)), \\ D_t v(\cdot,t) \text{ is a weakly continuous map of } t \text{ with values in } L^2(\Omega), \\ \displaystyle\int_\Omega \left(\frac{1}{2}|D_t v^i|^2 + \frac{1}{2}\|Dv^i\|^2 + \frac{1}{m}|v^i|^m\right) dx \le \int_\Omega \frac{1}{2}|v_0^i|^2 dx. \end{cases}$$

Here $\|\cdot\|$ denotes the Euclidean norm, and $D=(D_1,\cdots,D_k)$. To get such a map v, for example, we solve the initial-boundary value problem

$$(2.8)\qquad \begin{cases} D_t^2 v^i(x,t) - \Delta v^i(x,t) + |v^i|^{m-2}v^i(x,t) = 0 & \text{on } \Omega\times\mathbf{R}, \\ v^i(x,0)=0, \quad D_t v^i(x,0) = v_0^i(x) & \text{in } \Omega, \\ v^i(x,t) = 0 & \text{on } \partial\Omega\times\mathbf{R}. \end{cases}$$

[8, Theorem 2] guarantees the existence of weak solution v satisfying (2.7).

For $n\ge 2$ we define u_n as a minimizer of the functional

$$\mathcal{F}_n(u) = \int_\Omega \left(\frac{1}{2}\frac{\|u-2u_{n-1}+u_{n-2}\|_a^2}{h^2} + \frac{1}{2}\|Du\|_b^2 + \frac{1}{m}\|u\|_c^m + \frac{1}{2}\frac{\|u-u_{n-2}\|_a^2}{2h}\right) dx$$

in the class

$$\{u\in H^{1,2}(\Omega)\cap L^m(\Omega)\ ;\ \gamma_{\partial\Omega} u = \gamma_{\partial\Omega} w\}.$$

The functional $\mathcal{F}_n(u)$ is coercive in the above class, and the standard argument of minimizing sequence can be applied.

Proposition 2.1. *$\mathcal{F}_n(u)$ has a minimizer, which we denote by u_n. It satisfies the Euler-Lagrange equation*

$$\begin{aligned}
0 &= \frac{d}{d\varepsilon}\mathcal{F}_n(u+\varepsilon\varphi)\bigg|_{\varepsilon=0} \\
&= \int_\Omega \left\{ \frac{1}{h^2}a_{ij}(x)(u^i - 2u^i_{n-1} + u^i_{n-2})\varphi^j + b^{\alpha\beta}_{ij}(x)D_\alpha u^i D_\beta \varphi^j + c_{ij}(x)\|u\|_c^{m-2}u^i\varphi^j \right. \\
&\qquad \left. + \frac{1}{2h}a_{ij}(x)(u^i - u^i_{n-2})\varphi^j \right\} dx \qquad \text{for all } \varphi \in H_0^{1,2}(\Omega)\cap L^m(\Omega).
\end{aligned} \tag{2.9}$$

Thus $\{u_n\}$ is well-defined inductively. Now, using $\{u_n\}$, we define two maps u_h and $\bar{u}_h$ by

$$\begin{cases}
\bar{u}_h(x,t) = \begin{cases} u_0(x) & \text{for} \quad t = 0, \\ u_n(x) & \text{for} \quad (n-1)h < t \le nh, \quad n \ge 1, \end{cases} \\
u_h(x,t) = \begin{cases} u_0(x) + v(x,t) & \text{for} \quad -1 \le t \le h, \\ \dfrac{t-(n-1)h}{h}u_n(x) + \dfrac{nh-t}{h}u_{n-1}(x) & \text{for} \quad (n-1)h < t \le nh, \quad n \ge 2. \end{cases}
\end{cases}$$

They approximate a weak solution of (1.1).

Proposition 2.2. *For small $h \in (0,1)$ it holds that*

$$\begin{cases}
\{\bar{u}_h\},\ \{u_h\} & \text{are bounded set in } L^{m'}(\Omega\times(0,T)), \text{ where } m' = \max\{2,m\}, \\
\{D_t u_h\} & \text{is a bounded set in } L^2(\Omega\times(0,T))\cap L^\infty(0,T;L^2(\Omega)), \\
\{D_\alpha \bar{u}_h\},\ \{D_\alpha u_h\} & \text{are bounded set in } L^2(\Omega\times(0,T)),
\end{cases}$$

and

$$\int_0^T\int_\Omega |\bar{u}_h - u_h|^2\,dxdt = O(h^2T).$$

Sketch of Proof. Since u_n and u_{n-2} coincide on $\partial\Omega$, $u_n - u_{n-2}$ $(n \ge 2)$ is an admissible test function for (2.9). Substituting it for ϕ in (2.9), we get the assertion. □

It follows from Propositions 2.1 and 2.2 that

$$\int_0^T \int_\Omega \Big\{ \frac{1}{h} a_{ij}(x) \left(D_t u_h^i(x,t) - D_t u_h^i(x,t-h) \right) \varphi^j(x)$$

$$+ b_{ij}^{\alpha\beta}(x) D_\alpha \bar{u}_h^i(x,t) D_\beta \varphi^j(x) + c_{ij}(x) \|\bar{u}_h(x,t)\|_c^{m-2} \bar{u}_h^i(x,t) \varphi^j(x)$$

$$+ \frac{1}{2} a_{ij}(x) \left(D_t u_h^i(x,t) + D_t u_h^i(x,t-h) \right) \varphi^j(x) \Big\} \eta(t) dx dt$$

$$= o(1) \quad \text{as} \quad h \downarrow 0$$

for any $T > 0$ and $\eta \in C_0^\infty[0,T)$.

The weak(-star) compactness argument and the diagonal argument give the fact that $\bar{u}_h$ and u_h converge to a global weak solution u along a suitable subsequence of $h \downarrow 0$. Thus we get the following result.

Theorem 2.1. *Let $m > 1$. For any u_0, $w \in H^{1,2}(\Omega) \cap L^m(\Omega)$ and $v_0 \in L^2(\Omega)$ satisfying $\gamma_{\partial\Omega} u_0 = \gamma_{\partial\Omega} w$, there exists at least one global weak solution u to* (1.1), (1.5) and (1.6).

2.2 Decay of our weak solutions

In this subsection we assume $w \equiv 0$ and $m \geq 2$.

Since we are posing the homogeneous boundary condition, u_n is an admissible test function for (2.9). Therefore we can see that

$$\int_\Omega \frac{1}{h^2} a_{ij} (u_n^i - u_{n-1}^i)(u_{n-1}^j - u_{n-2}^j) dx$$

$$= \int_\Omega \Big\{ \frac{1}{h} \left(a_{ij} u_n^i \frac{u_n^j - u_{n-1}^j}{h} - a_{ij} u_{n-1}^i \frac{u_{n-1}^j - u_{n-2}^j}{h} \right) + a_{ij} u_n^i \frac{u_n^j - u_{n-1}^j}{h}$$

$$+ \|Du_n\|_b^2 + \|u_n\|_c^m - \frac{1}{2h} a_{ij} (u_n^i - 2u_{n-1}^i + u_{n-2}^i) u_n^j \Big\} dx.$$

Next we test (2.9) by $\varphi = u_n - u_{n-1}$ to get

$$0 = \int_\Omega \Big[\frac{1}{h^2} a_{ij} \left\{(u_n^i - u_{n-1}^i) - (u_{n-1}^i - u_{n-2}^i)\right\}(u_n^j - u_{n-1}^j)$$

$$+ b_{ij}^{\alpha\beta} D_\alpha u_n^i (D_\beta u_n^j - D_\beta u_{n-1}^j) + c_{ij} \|u_n\|_c^{m-2} u_n^i (u_n^j - u_{n-1}^j)$$

$$+ \frac{1}{2h} a_{ij}(u_n^i - u_{n-2}^i)(u_n^j - u_{n-1}^j)\Big]\, dx$$

$$= \int_\Omega \Big[\frac{1}{h^2}\left\{\|u_n - u_{n-1}\|_a^2 - a_{ij}(u_{n-1}^i - u_{n-2}^i)(u_n^j - u_{n-1}^j)\right\}$$

$$+ \|Du_n\|_b^2 - b_{ij}^{\alpha\beta} D_\alpha u_n^i D_\beta u_{n-1}^j + \|u_n\|_c^m - \|u_n\|_c^{m-2} c_{ij} u_n^i u_{n-1}^j$$

$$+ \frac{1}{2h}\|u_n - u_{n-1}\|_a^2 + \frac{1}{2h} a_{ij}(u_n^i - u_{n-1}^i)(u_{n-1}^j - u_{n-2}^j)\Big]\, dx.$$

Combining these relations, and estimating non-coercive terms by use of Young's inequality, we get

Proposition 2.3. *It holds that*

$$\frac{\Psi_h(t) - \Psi_h(t-h)}{h} + \Psi_h(t) \le hK_1,$$

where

$$\Psi_h(t) = \int_\Omega \left(\frac{1}{2}\|D_t u_h\|_a^2 + \frac{1}{2} a_{ij} \bar{u}_h^i D_t u_h^j + \frac{1}{2}\|D\bar{u}_h\|_b^2 + \frac{1}{m}\|\bar{u}_h\|_c^m\right) dx,$$

and K_1 is a constant depending on the initial data but not on h. And therefore we have

$$\Psi_h(t) \le \left(\frac{1}{1+h}\right)^n \Psi_h(+0) + hK_1,$$

where the relation between t and n is given by

$$n = \lceil t/h \rceil,$$

$\lceil\ \rceil$ *denotes the ceiling, i.e., $\lceil x \rceil$ is the smallest integer greater than or equal to x.*

Passing to $h \downarrow 0$, we have

$$D_t \int_\Omega \|u(x,t)\|_a^2 dx + C_1 \int_\Omega \|u(x,t)\|_a^2 dx \le K_2 e^{-t}$$

for almost every $t \in (0,\infty)$. Since u is a weak solution, it belongs to $C([0,T]; L^2(\Omega))$ and $D_t u$ to $L^\infty(0,T; L^2(\Omega))$. Hence it follows from the above differential inequality that

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \le K_3 e^{-C_2 t}.$$

Using Ψ_h again, we have

$$\int_\Omega \left(\frac{1}{4}\|D_t u_h\|_a^2 + \frac{1}{2}\|D\bar{u}_h\|_b^2 + \frac{1}{m}\|\bar{u}_h\|_c^m\right) dx \le \left(\frac{1}{1+h}\right)^n K_4 + hK_1 + C_3\|\bar{u}_h(\cdot,t)\|^2_{L^2(\Omega)}.$$

Passing to $h \downarrow 0$ again, we obtain

Theorem 2.2. *Our weak solution satisfies*

$$\|u(\cdot,t)\|^2_{H^{1,2}(\Omega)} + \|u(\cdot,t)\|^m_{L^m(\Omega)} + \|D_t u(\cdot,t)\|^2_{L^2(\Omega)} \le Ke^{-Ct},$$

provided $w \equiv 0$ *and* $m \le 2$.

Remark 2.1. If we define u_n $(n \ge 2)$ as a minimizer of

$$\tilde{\mathcal{F}}_n(u) = \int_\Omega \left(\frac{1}{2}\frac{\|u - 2u_{n-1} + u_{n-2}\|_a^2}{h^2} + \frac{1}{2}\|Du\|_b^2 + \frac{1}{m}\|u\|_c^m + \frac{1}{2}\frac{\|u-u_{n-1}\|_a^2}{h}\right) dx$$

instead of $\mathcal{F}_n(u)$, then we can also construct a global weak solution form them. The authors, however, cannot clarify that this weak solution has a decay property or not.

3 The strongly damped hyperbolic system

In this section we consider the initial-boundary value problem for the strongly hyperbolic system (1.2). The method is similar as in § 2, so we state only the scheme. Let h be a positive number, which will tend to zero later. u_0 is a given initial data of u. u_1 is defined by

$$u_1(x) = u_0(x) + hv_0(x),$$

where v_0 is also given initial data of $D_t u$. For $n \ge 2$ we define u_n as a minimizer of the functional

$$\mathcal{G}_n(u) = \int_\Omega \left(\frac{1}{2}\frac{\|u - 2u_{n-1} + u_{n-2}\|_a^2}{h^2} + \frac{1}{2}\|Du\|_b^2 + \frac{1}{m}\|u\|_c^m + \frac{1}{2}\frac{\|D(u-u_{n-1})\|_f^2}{h}\right) dx$$

in the class

$$\{u \in H^{1,2}(\Omega) \cap L^m(\Omega)\ ;\ \gamma_{\partial\Omega} u = \gamma_{\partial\Omega} w\}.$$

This scheme gives us

Theorem 3.1. *Let $m > 1$. For any u_0, $w \in H^{1,2}(\Omega) \cap L^m(\Omega)$ and $v_0 \in H_0^{1,2}(\Omega)$ satisfying $\gamma_{\partial\Omega} u_0 = \gamma_{\partial\Omega} w$, there exists at least one global weak solution u to* (1.2), (1.5) and (1.6).
If $w \equiv 0$, then our weak solution satisfies

$$\|u(\cdot,t)\|^2_{H^{1,2}(\Omega)} + \|u(\cdot,t)\|^m_{L^m(\Omega)} + \|D_t u(\cdot,t)\|^2_{L^2(\Omega)} \leq K e^{-Ct}.$$

Remark 2.1. If we define u_n $(n \geq 2)$ as a minimizer of

$$\tilde{\mathcal{G}}_n(u) = \int_\Omega \left(\frac{1}{2} \frac{\|u - 2u_{n-1} + u_{n-2}\|_a^2}{h^2} + \frac{1}{2}\|Du\|_b^2 + \frac{1}{m}\|u\|_c^m + \frac{1}{2}\frac{\|D(u-u_{n-2})\|_f^2}{2h} \right) dx$$

instead of $\mathcal{G}_n(u)$, then we can also construct a global weak solution form them. The authors, however, cannot clarify that this weak solution has a decay property or not.

A technical difference between $\mathcal{F}_n(u)$ and $\tilde{\mathcal{F}}_n(u)$ (see Remark 2.1), and between $\mathcal{G}_n(u)$ and $\tilde{\mathcal{G}}_n(u)$ comes form Poincaré's inequality. The inequality can be expressed by

$$(u,u)_{L^2(\Omega)} \leq C(Du, Du)_{L^2(\Omega)}$$

for $u \in H_0^{1,2}(\Omega)$. It, however, does not hold that

$$(u_n, u_{n-1})_{L^2(\Omega)} \leq C(Du_n, Du_{n-1})_{L^2(\Omega)}.$$

We must choose scheme so that such terms does not appear in calculating a decay property.

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