NAVIER-STOKES EQUATIONS WITH DISTRIBUTIONS AS INITIAL DATA

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§1 Introduction.

Let $\Omega$ be an exterior domain in $\mathbb{R}^n(n \geq 3)$, i.e., a domain having a compact complement $\mathbb{R}^n \setminus \Omega$, and assume that the boundary $\partial \Omega$ is of class $C^{2+\mu}(0 < \mu < 1)$. The motion of the incompressible fluid occupying $\Omega$ is governed by the Navier-Stokes equations:

\[
\begin{cases}
- \Delta w + w \cdot \nabla w + \nabla \pi = \text{div} \, F & \text{in } \Omega, \\
\text{div} \, w = 0 & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega, \\
w(x) \to 0 & \text{as } |x| \to \infty,
\end{cases}
\]

where $w = w(x) = (w^1(x), \cdots, w^n(x))$ and $\pi = \pi(x)$ denote the velocity vector and the pressure of the fluid at point $x \in \Omega$, respectively, while $F = F(x) = \{F_{ij}(x)\}_{i,j=1,\cdots,n}$ is the given $n \times n$ matrices with $\text{div} \, F$ the external force. In the previous paper [14], the first author and Ogawa showed the stability in $L^n$ of solutions $w$ in the class

\[(CL) \quad w \in L^n(\Omega) \quad \text{and} \quad \nabla w \in L^{n/2}(\Omega).\]

In case $n \geq 4$ we can show the existence and uniqueness for solutions $w$ of (S) with (CL). In the three dimensional case, however, the solution in the class (CL) yields that the net force exerted to the body is equal to zero:

\[
\int_{\partial \Omega} (T(w, \pi) + F) \cdot \nu dS = 0,
\]

where $T(w, \pi) = \{\partial w^i/\partial x^j + \partial w^j/\partial x^i - \delta_{ij} \pi\}_{i,j=1,\cdots,n}$ and $\nu$ denote the stress strain and the unit outer normal to $\partial \Omega$, respectively (see Kozono-Sohr [16]). Introducing another class

\[(CL') \quad \sup_{x \in \Omega} |x||w(x)| + \sup_{x \in \Omega} |x|^2|\nabla w(x)| \equiv C_w < \infty\]
Borchers-Miyakawa [3] constructed the solution with (CL') and showed that if $C_w$ is small, then $w$ is stable under the initial disturbance in weak- $L^n$ space $L^{n,\infty}(\Omega)$.

The purpose of this note is to find a larger class of stable flows than (CL'). Indeed, we shall show that stationary flows in the class

$$(CL'') \quad w \in L^{n,\infty}(\Omega)$$

are stable under such perturbation as Borchers-Miyakawa's [3]. As a result, we shall obtain the same class of stable solutions and initial disturbances. More precisely, if $w$ is perturbed by $a$, then the perturbed flow $v(x, t)$ is governed by the following non-stationary Navier-Stokes equations:

$$(N-S) \quad \begin{cases} \frac{\partial v}{\partial t} - \Delta v + v \cdot \nabla v + \nabla q = f \quad \text{in } \Omega, t > 0, \\
\text{div } v = 0 \quad \text{in } \Omega, t > 0, \\
v = 0 \quad \text{on } \partial \Omega, t > 0, \quad v(x, t) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \\
v(x, 0) = w(x) + a(x) \quad \text{for } x \in \Omega. \end{cases}$$

In this note we shall show: if the stationary flow $w$ and the initial disturbance $a$ are both small enough in $L^{n,\infty}(\Omega)$, then there is a unique global strong solution $v$ of (N-S) such that the integrals

$$\int_{\Omega} |v(x, t) - w(x)|^r dx \quad \text{for } n < r < \infty$$

converges to zero with definite decay rates as $t \rightarrow \infty$. Let $w$ and $v$ be solutions of (S) and (N-S), respectively. Then the pair of functions $u \equiv v - w, p \equiv q - \pi$ satisfies

$$(N-S') \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u + w \cdot \nabla u + u \cdot \nabla w + u \cdot \nabla u + \nabla p = 0 \quad \text{in } \Omega, t > 0, \\
\text{div } u = 0 \quad \text{in } \Omega, t > 0, \\
u = 0 \quad \text{on } \partial \Omega, t > 0, \quad u(x, t) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \\
u|_{t=0} = a. \end{cases}$$

Hence our problem on the stability for (S) can now be reduced to investigation into asymptotic behaviour of the solution $u$ of (N-$S'$). In a three-dimensional exterior domain, Heywood [10,11] and Masuda [18] considered inhomogeneous boundary condition at infinity like $w(x) \rightarrow w^\infty$ as $|x| \rightarrow \infty$, where $w^\infty$ is a prescribed non-zero constant vector in $\mathbb{R}^3$. They showed the stability for such solutions in $L^2$-spaces. On account of the parabolically wake region behind obstacles, their decay rates are slower than that of our solutions. To obtain sharper decay rates in $L^r$-spaces of the solutions of (N-$S'$) with the initial data in weak- $L^n$ space, we need to establish $L^{p,\infty} - L^r$-estimates for the semigroup $e^{-tL_r}$, where $L_r$ is the operator defined by

$$L_r u \equiv A_r u + P_r(w \cdot \nabla u + u \cdot \nabla w).$$
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Here \( P_r \) is the projection operator from \( L^r(\Omega) \) onto \( L^r_\sigma(\Omega) \) and \( A_r \equiv -P_r \Delta \) denotes the Stokes operator in \( L^r_\sigma(\Omega) \).

In case \( w \equiv 0 \), we have \( L_r = A_r \) and hence our problem coincides with obtaining a global strong solution and its decay properties of the Navier-Stokes equations in exterior domains. Since the pioneer work of Kato [13] and Ukai [23], many efforts have been made to get \( L^p - L^r \)-estimates for the Stokes semigroup \( e^{-tA_r} \) in exterior domains and there are mainly two methods. One is to characterize the domain \( D(A_\alpha^p) \) of fractional powers \( A_\alpha^p(0 < \alpha < 1) \) due to Giga [7], Giga-Sohr [9] and Borchers-Miyakawa [2] and another is to obtain asymptotic expansion of the resolvent \( (A_r + \lambda)^{-1} \) near \( \lambda = 0 \) due to Iwashita [12]. In our case, since \( L_r \) is the operator with \textit{variable} coefficients, both of these methods seem to be difficult to get the same asymptotic behavior of \( e^{-tL_r} \) as that of \( e^{-tA_r} \) as \( t \to \infty \). If we restrict ourselves to the case \( n/(n-1) < r < \infty \), however, then \( L_r \) can be treated as a perturbation of \( A_r \), and for such \( r \), we can get satisfactory \( L^p, \infty - L^r \)-estimates of \( e^{-tL_r} \), which is enough to construct the global strong solution of (N-S'). Our proof needs neither estimates of the purely imaginary powers \( L^s_r(s \in \mathbb{R}) \) of \( L_r \) nor asymptotic expansion of \( (L_r + \lambda)^{-1} \) near \( \lambda = 0 \); we need only such a standard resolvent estimate of elliptic differential operators as Agmon's [1].

On account of the restriction \( n/(n-1) < r < \infty \), we cannot construct the strong solution directly in the same way as Giga-Miyakawa [8] and Kato [13]. Therefore, we need to first introduce a \textit{mild solution} which is an intermediate between weak and strong solutions (see Definition below). This procedure is due to Kozono-Ogawa [14]. Then we shall show the existence and uniqueness of the \textit{global} mild solution \( u \) of (N-S') in the class \( C((0, \infty); L^n, \infty(\Omega)) \) with decay property \( \|u(t)\|_r = O(t^{-1/2 + n/2r}) \) as \( t \to \infty \) for \( n < r < \infty \). Using a similar uniqueness criterion to Serrin [21] and Masuda [19], we may identify the mild solution with the strong solution. As a result, it will be clarified that the restriction on \( r \) causes no obstruction for our purpose.

§2 Results.

Before stating our results, we introduce some notations and function spaces and then give our definition of mild solutions of (N-S'). Let \( C^\infty_{0, \sigma} \) denote the set of all \( C^\infty \) real vector functions \( \phi = (\phi^1, \cdots, \phi^n) \) with compact support in \( \Omega \), such that \( \Div \phi = 0 \). \( L^r_\sigma \) is the closure of \( C^\infty_{0, \sigma} \), with respect to the \( L^r \)-norm \( \| \cdot \|_r \); \( \langle \cdot, \cdot \rangle \) denotes the \( L^2 \)-inner product and the duality pairing between \( L^r \) and \( L^r' \), where \( 1/r + 1/r' = 1 \). \( L^r \) stands for the usual(vector-valued)\( L^r \)-space over \( \Omega \), \( 1 < r < \infty \). \( H^1_{0, \sigma} \) denotes the closure of \( C^\infty_{0, \sigma} \) with respect to the norm

\[ \|\phi\|_{H^1_r} = \|\phi\|_r + \|\nabla\phi\|_r, \]

where \( \nabla \phi = (\partial \phi^i/\partial x_j; i, j = 1, \cdots, n) \). When \( X \) is a Banach space, its norm is denoted by \( \| \cdot \|_X \). Then \( C^m((t_1, t_2); X) \) is a usual Banach space, where \( m = 0, 1, 2, \cdots \) and \( t_1 \) and \( t_2 \) are real numbers such that \( t_1 < t_2 \). \( BC^m((t_1, t_2); X) \) is the set of all functions \( u \in C^m((t_1, t_2); X) \) such that \( \sup_{t_1 < t < t_2} \| d^m_x u(t) \|_X < \infty \).
Let us recall the Helmholtz decomposition:
\[ L^r = L^r_{\sigma} \oplus G^r \] (direct sum), \( 1 < r < \infty \),
where \( G^r = \{ \nabla p \in L^r; p \in L^r_{loc}(\Omega) \} \).
For the proof, see Fujiwara-Morimoto[6], Miyakawa[20] and Simader-Sohr[22].
\( P_r \) denotes the projection operator from \( L^r \) onto \( L^r_{\sigma} \) along \( G^r \).
The Stokes operator \( A_r \) on \( L^r_{\sigma} \) is then defined by
\[ A_r = -P_r\Delta \]
with domain \( D(A_r) = \{ u \in H^{2,r}(\Omega); u|_{\partial\Omega} = 0 \} \cap L^r_{\sigma} \).
It is known that \( (L^r_{\sigma})^* \) (the dual space of \( L^r_{\sigma} \)) = \( L^{r'}_{\sigma} \),
where \( 1/r + 1/r' = 1 \).
Furthermore, for \( 1 < r < \infty \) and \( 1 \leq q \leq \infty \), \( L^{rq} \) denotes the Lozentz space over \( \Omega \) with norm \( ||\cdot||_{r,q} \).
Let us introduce the operator \( L_r \) in \( L^r_{\sigma} \). To this end, we make the following assumption on \( w \).

Assumption. \( w \) is a smooth solenoidal vector function on \( \overline{\Omega} \) with \( w|_{\partial\Omega} = 0 \) in the class \( w \in L^{n,\infty}_{\sigma} \).

For the existence of such solutions \( w \) of (S), see Finn[4] and Fujita[5]. Under this assumption, we define the operator \( B_r \) on \( L^r_{\sigma} \) by
\[ B_r u \equiv P_r(w \cdot \nabla u + u \cdot \nabla w) \]
with domain \( D(B_r) = H^{1,r}_{0,\sigma} \).
\( L_r \) is now defined by
\[ D(L_r) = D(A_r) \quad \text{and} \quad L_r \equiv A_r + B_r. \]
Since \( \text{div} \ w = 0 \) in \( \Omega \), we can easily verify that the operator \( L' \) defined by
\[ L'_r u = A_r u - P_r(w \cdot \nabla u + \sum_{j=1}^{n} w^j \nabla u^j), \quad D(L'_r) = D(A_r) \]
is the adjoint operator of \( L_r \) on \( L^r_{\sigma} \). It should be noted that the operator \( L' \) contains no derivative \( \partial w/\partial x^j (j = 1, \cdots, n) \) of \( w \) in its coefficients.

Our definition of mild solutions of (N-S') is as follows:

Definition. Let \( a \in L^{n,\infty}_{\sigma} \) and let \( w \) satisfy the Assumption. Suppose that \( n < r < \infty \). A measurable function \( u \) defined on \( \Omega \times (0, \infty) \) is called a mild solution of (N-S') in \( L^r_{\sigma} \) if
\begin{enumerate}
  \item \( u \in BC((0, \infty); L^{n,\infty}_{\sigma}) \) and \( t^{(1-n/r)/2}u(\cdot) \in BC((0, \infty); L^r_{\sigma}) \);
  \item \( (u(t), \phi) = (e^{-tL}a, \phi) + \int_{0}^{t} (u(s) \cdot \nabla e^{-(t-s)L'} \phi, u(s))ds \)
\end{enumerate}
for all \( \phi \in C_{0,\sigma}^{\infty} \) and all \( 0 < t < \infty \).

Our results now read:
Theorem 1. (1) (existence) Let $a \in L^n_{\sigma}^{n, \infty}$ and let $w$ satisfy the Assumption. Then for every $n < r < \infty$, there is a positive number $\lambda = \lambda(n, r)$ such that if

$$\|a\|_{n, \infty} \leq \lambda, \quad \|w\|_{n, \infty} \leq \lambda,$$

there exists a mild solution $u$ of $(N-S')$ in $L^r_{\sigma}$ such that

$$u(t) \to a \text{ weakly } \ast \text{ in } L^{n, \infty}_{\sigma} \text{ as } t \downarrow +0.$$ 

(2) (uniqueness) There is a constant $k = k(n, r)$ such that any mild solution $u$ of $(N-S')$ in $L^r_{\sigma}$ with

$$\limsup_{t \to +0} t^{\frac{n}{2} \left(\frac{1}{n} - \frac{1}{r}\right)} \|u(t)\|_r \leq k$$

is unique.

Concerning the regularity of the solution, we have

Theorem 2. The mild solution $u$ given in Theorem 1 is actually a strong solution in the following sense:

1. $u \in C^1((0, \infty); L^r_{\sigma})$;
2. $u(t) \in D(L_r)$ for $t \in (0, \infty)$ and $L_r u \in C((0, \infty); L^r_{\sigma})$;
3. $u$ satisfies

$$\frac{du}{dt} + L_r u + P_r (u \cdot \nabla u) = 0, \quad t > 0 \text{ in } L^r_{\sigma}.$$

Remarks. (1) The above theorems show that the space $L^n_{\sigma}^{n, \infty}$ is the class of stable stationary flows and that it is the same class as that of initial disturbances. Borchers-Miyakawa [3] obtained, among others, similar results to ours including the uniform $L^\infty$ estimate in time. They make, however, such a stronger assumption as

$$\sup_{x \in \Omega} |x||w(x)| + \sup_{x \in \Omega} |x|^2 |\nabla w(x)|$$

is small enough. On the other hand, our theorems assert that the assumption on the spacial decay of $\nabla w(x)$ as $|x| \to \infty$ is not necessary. Moreover, the class of the space $L^n_{\sigma}^{n, \infty}$ is larger than that of functions $w$ such that $\sup_{x \in \Omega} |x||w(x)| < \infty$.

(2) Since the semigroup $\{e^{-tL}\}_{t \geq 0}$ is not strongly continuous in $L^n_{\sigma}^{n, \infty}$, we cannot assure whether our solution $u$ satisfies

$$\lim_{t \to +0} t^{\frac{n}{2} \left(\frac{1}{n} - \frac{1}{r}\right)} \|u(t)\|_r = 0.$$ 

(3) When $\Omega = \mathbb{R}^n (n \geq 3)$, without assuming any regularity on the stationary flow $w$, Kozono-Yamazaki [17] obtained a similar strong solution with a uniform decay estimate.
REFERENCES


