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Asymptotic Stability of Traveling Waves with shock profile for Non-convex Viscous Scalar Conservation Laws

Akitaka Matsumura(松村　昭孝)[1] and Kenji Nishihara(西原　健二)[2]
[1] Department of Mathematics, Osaka University
[2] School of Political Science and Economics, Waseda University

1 Introduction

We consider the Cauchy problem for scalar viscous conservation laws:

$$u_t + f(u)_x = \mu u_{xx}, \quad x \in \mathbf{R}, \mathbf{t} > \mathbf{0} \tag{1.1}$$

$$u(0,x) = u_0(x), \quad x \in \mathbf{R}, \tag{1.2}$$

where μ is a positive constant and the initial data $u_0(x)$ is asymptotically constant as $x \to \pm\infty$:

$$u_0(x) \to u_\pm \quad \text{as} \quad x \to \pm\infty. \tag{1.3}$$

We note that $f \in C^2$ is not assumed to be necessarily convex.

Asymptotic behavior of the solution of (1.1),(1.2) closely corresponds to that of the solution of corresponding Riemann problem. In this note, let Eq.(1.1) admit traveling wave solutions with shock profile such that

$$u = U(x - st) \equiv U(\xi), \quad U(\xi) \to u_\pm \quad \text{as} \quad \xi \to \pm\infty, \tag{1.4}$$

where the constants $u_\pm$ and s (shock speed) satisfy the Rankine-Hugoniot condition

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0 \tag{1.5}$$

and the generalized shock condition(Oleinik's shock condition)

$$h(u) \equiv -s(u - u_\pm) + f(u) - f(u_\pm) \begin{cases} < 0 & \text{if} \quad u_+ < u < u_- \\ > 0 & \text{if} \quad u_- < u < u_+. \end{cases} \tag{1.6}$$

It is noted that the condition (1.6) implies

$$f'(u_+) \le s \le f'(u_-). \tag{1.7}$$

and that, especially when $f'' > 0$, the condition (1.6) is equivalent to

$$f'(u_+) < s < f'(u_-),$$

which is well-known as Lax's shock condition (Lax [5]).

Substituting $U(\xi)$ into (1.1) we have

$$\mu U_{\xi\xi} = -sU_\xi + f(U)_\xi = h'(U)U_\xi. \tag{1.9}$$

Integrating (1.9) over $(\pm\infty, \xi)$ and noting the Rankine-Hugoniot condition (1.5) we also have

$$\mu U_\xi = -s(U - u_\pm) + f(U) - f(u_\pm) = h(U). \tag{1.10}$$

Lemma 1 *Assume (1.5), (1.6) and*

$$|h(U)| \sim |U - u_\pm|^{1+k_\pm}, \quad U \to u_\pm \tag{1.11}$$

with $k_\pm \geq 0$. Then there exists a traveling wave solution $U(\xi)$ of (1.1) with $U(\pm\infty) = u_\pm$, unique up to a shift, which is determined by the ordinary differential equation (1.9) or (1.10). Moreover, it holds as $\xi \to \pm\infty$

$$|U(\xi) - u_\pm| \sim \exp(-c_\pm|\xi|) \quad \textit{if} \quad f'(u_+) < s < f'(u_-) \tag{1.12}$$

for some positive constants $c_\pm$ and

$$|U(\xi) - u_+| \sim |\xi|^{-1/k_\pm} \quad \textit{if} \quad s = f'(u_\pm) \tag{1.13}$$

Remark. Since $h'(u_\pm) = -s + f'(u_\pm)$, the condition $h'(u_\pm) = 0$ is corresponding to the equality in (1.7) and $k_\pm = 0$ in (1.11), which is called as degenerate shock condition. While $h'(u_\pm) \neq 0$ corresponds to (1.8) and $k_\pm > 0$ in (1.11), which is the non-degenerate shock. We note the behavior of U as $\xi \to \infty$ is likely (1.12) or (1.13) depending on the non-degenerate or degenerate shock, respectively.

To investigate the stability of traveling wave solution U, we assume $u_0 - U$ is integrable and determine a unique shift of U as

$$\int_{-\infty}^{\infty} (u_0(x) - U(x))dx = 0. \tag{1.14}$$

Hence

$$\psi_0(x) = \int_{-\infty}^{x} (u_0(y) - U(y))dy. \tag{1.15}$$

is well-defined. Under these considerations we obtain three theorems. To state them, we first mention several notations.

Notations. We denote several positive constants depending on $a, b, \cdots$ by $C_{a,b,\cdots}$ or only by C without confusions. We also denote $f(x) \sim g(x)$ as $x \to a$ when $C^{-1}g < f < Cg$ in a neighborhood of a, though we have already used it. For funcion spaces, L^2 denotes the space of square integrable functions on $\mathbf{R}$ with the norm

$$\| f \| = \left(\int_{\mathbf{R}} |f(x)|^2 dx\right)^{1/2}.$$

Here and after the integrand $\mathbf{R}$ is abreviated. $H^l(l \geq 0)$ denotes the usual l-th order Sobolev space with the norm

$$\| f \|_l = (\sum_{j=0}^{l} \| \partial_x^j f \|^2)^{1/2}.$$

For the weight function w, L^2_w denotes the space of measurable functions f satisfying $\sqrt{w} f \in L^2$ with the norm

$$|f|_w = (\int w(x)|f(x)|^2 dx)^{1/2}.$$

When $w(x) = \langle x \rangle^\alpha = (1+x^2)^{\alpha/2}$, we write $L^2_w = L^2_\alpha$ and $|\cdot|_w = |\cdot|_\alpha$ without confusions. Moreover when w is replaced by $\langle x \rangle^\alpha w$, we denote that space by $L^2_{\alpha,w}$ with the norm

$$|f|_{\alpha,w} = (\int \langle x \rangle^\alpha w(x)|f(x)|^2 dx)^{1/2}.$$

We also use $\langle x \rangle_+ = \begin{cases} \sqrt{1+x^2} & x \geq 0 \\ 1 & x < 0 \end{cases}$ or $\langle x \rangle_- = \begin{cases} \sqrt{1+x^2} & x < 0 \\ 1 & x \geq 0 \end{cases}$ as the weight function. When $C^{-1} \leq w(x) \leq C$, we note that $L^2 = H^0 = L^2_0 = L^2_w$ with $\|\cdot\| = \|\cdot\|_0 = |\cdot|_0 \sim |\cdot|_w$ and that $L^2_{\alpha,w} = L^2_\alpha$ with $|\cdot|_{\alpha,w} \sim |\cdot|_\alpha$.

Theorem 1 *(Stability) Assume (1.5), (1.6) and (1.11) and let U be a traveling wave solution uniquely determined by (1.14). Then the followings hold.*
(i) When $f'(u_+) < s < f'(u_-)$, suppose $u_0 - U$ is integrable and $\psi_0 \in H^2$. Then there exists a positive constant ε_1 such that if $\| \psi_0 \|_2 < \varepsilon_1$, then the Cauchy problem (1.1), (1.2) has a unique global solution $u(t,x)$ satisfying

$$u - U \in C^0([0,\infty); H^1) \cap L^2(0,\infty; H^2)$$

and moreover

$$\sup_{\mathbf{R}} |u(t,x) - U(x-st)| \to 0 \quad as \quad t \to \infty. \tag{1.16}$$

(ii) When $s = f'(u_+) < f'(u_-)$, there exists a positive constant ε_1 such that if $\| \psi_0 \|_2 + |\psi_0|_{\langle \xi \rangle_+} < \varepsilon_1$, then the Cauchy problem (1.1), (1.2) has a unique global solution $u(t,x)$ satisfying

$$u - U \in C^0([0,\infty); H^1) \cap L^2([0,\infty); H^2 \cap L^2_{\langle \xi \rangle_+})$$

and moreover

$$\sup_{\mathbf{R}} |u(t,x) - U(x-st)| \to 0 \quad as \quad t \to \infty. \tag{1.17}$$

(iii) When $f'(u_+) < s = f'(u_-)$ or $s = f'(u_+) = f'(u_-)$, then $L^2_{\langle \xi \rangle_+}$ in (ii) should be replaced by $L^2_{\langle \xi \rangle_-}$ or $L^2_{\langle \xi \rangle} = L^2_1$, respectively.

Remark 1 *When $s = f'(u_+)$ or $f'(u_-)$(degenerate shock), we need a weight of order $\langle \xi \rangle = \sqrt{1+\xi^2}$ as $\xi \to +\infty$ or $-\infty$ for a stability theorem in our method.*

Theorem 2 *(Rate of asymptotic speed for $f'(u_+) < s < f'(u_-)$) Let u be a solution obtained in Theorem 1(i) and let ψ_0 lie in L^2_α for some $\alpha > 0$. If α is an integer, then it holds*

$$\sup_{\mathbf{R}} |u(t,x) - U(x-st)| \leq C(1+t)^{-\alpha/2}(\| u_0 - U \|_1 + |\psi_0|_\alpha), \tag{1.18}$$

while if α is not an integer, then

$$\sup_{\mathbf{R}} |u(t,x) - U(x-st)| \leq C_\varepsilon (1+t)^{-\alpha/2+\varepsilon} (\| u_0 - U \|_1 + |\psi_0|_\alpha) \tag{1.19}$$

for any constant $\varepsilon > 0$ and some constant C_ε such that $C_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Next we state the result for $f'(u_+) = s < f'(u_-)$. When $f'(u_+) < s = f'(u_-)$ or $s = f'(u_+) = f'(u_-)$, the similar result is obtained as in Theorem 1(iii).

Theorem 3 *(Rate of asymptotic speed for $f'(u_+) = s < f'(u_-)$) Let u be a solution obtained in Theorem 1(ii) and $f''(u_+) = \cdots = f^{(n)}(u_+) = 0$ and $f^{(n+1)}(u_+) \neq 0$ for $n \geq 1$. Then if $\psi_0 \in L^2_{\alpha,\langle\xi\rangle_+}$ $(0 < \alpha < 2/n)$, it holds for any $\varepsilon > 0$*

$$\sup_{\mathbf{R}} |u(t,x) - U(x-st)| \leq C_\varepsilon (1+t)^{-\alpha/4+\varepsilon} (\| u_0 - U \|_1 + |\psi_0|_{\alpha,\langle\xi\rangle_+}). \tag{1.20}$$

We now mention the background of our theorems. Pioneering work in this field was given by Il'in and Oleinik [1] in 1960. They showed the exponential stability of the traveling wave solutions when $f'' > 0$ and so $f'(u_+) < s < f'(u_-)$, together with the stability of rarefaction waves. Kawashima and Matsumura [3] have obtained the stability of algebraic order, $\sup_{\mathbf{R}} |u - U| \leq Ct^{-[\alpha]/2}$ if $\psi_0 \in L^2_\alpha$. Recently, in the absence of $f'' > 0$ the stability problems have been investigated by Kawashima and Matsumura [4], Jones, Gardner and Kapitula[2], Mei [6]. When f has only one inflection point, the stability theorem has been obtained by Kawashima and Matsumura [4] including the system case and the rate of asymptotic speed by Mei [6], both of which are due to the weighted energy method. Mei [6] also has obtained the stability theorem in the degenerate case $s = f'(u_\pm)$ for the first time. For general function $f \in C^2$ and $f'(u_+) < s < f'(u_-)$ (non-degenerate shock case), Jones et al. [2] have obtained the stability and the rate of asymptotics, $\sup_{\mathbf{R}} |u - U| \leq C(1+t)^{-[\alpha]/4}$ if $\psi_0 \in L^2_\alpha$, which is based on spectral analysis. Our theorems 1 and 2 cover these stability results and improve the rate of asymptotics in non-degenerate shock case. Further, our rate seems to be almost optimal from the view point of the optimality in Nishihara [7], in which he has showed that, when $f = u^2/2$, $\sup_{\mathbf{R}} |u - U| \leq Ct^{-\alpha/2}$ if $|\psi_0(x)| \leq C|x|^{-\alpha/2}$ and this estimate is optimal in general. In the degenerate shock case, we have obtained the rate in Theorem 3 for the first time. However, it seems to be less sufficient and more contributions may be expected.

2 Reformulation of the problem

Letting $U(\xi)$ be the traveling wave solution in Theorem 1, we put

$$u(t,x) = U(\xi) + \psi_\xi(t,\xi), \quad \xi = x - st. \tag{2.1}$$

Then the problem (1.1), (1.2) is reduced to

$$\psi_t - s\psi_\xi + f(U + \psi_\xi) - f(U) = \mu\psi_{\xi\xi} \tag{2.2}$$

$$\psi(0,\xi) = \psi_0(\xi) \equiv \int_{-\infty}^{\xi} (u_0 - U)(\eta)d\eta. \tag{2.3}$$

Eq.(2.2) is rewritten as

$$\psi_t + h'(U)\psi_\xi - \mu\psi_{\xi\xi} = F, \tag{2.4}$$

$$F \equiv -\{f(U+\psi_\xi) - f(U) - f'(U)\psi_\xi\}. \tag{2.5}$$

Now we select the weight as

$$w = w(U) = |\frac{(U-u_+)(U-u_-)}{h(U)}|. \tag{2.6}$$

Since $w(U) \sim const.$ in the case $f'(u_+) < s < f'(u_-)$, $L^2_{\alpha,w(U)} = L^2_\alpha$. While if $s = f'(u_+) < f'(u_-)$, then $w(U) \sim |U-u_+|^{-k_+}$ as $U \to u_+$ and $w(U(\xi)) \sim \langle\xi\rangle$ as $\xi \to +\infty$, and hence $L^2_{w(U)} = L^2_{\langle\xi\rangle_+}$. Also, $L^2_{w(U)} = L^2_{\langle\xi\rangle_-}$ if $f'(u_+) < f'(u_-) = s$ and $L^2_{w(U)} = L^2_{\langle\xi\rangle} = L^2_1$ if $f'(u_+) = f'(u_-) = s$. Noting these we define the solution space of (2.2) and (2.3)

$$X(0,T) = \{\psi \in C^0([0,T]; H^2 \cap L^2_{w(U)}), \psi_\xi \in L^2(0,T; H^2 \cap L^2_{w(U)})\}$$

with $0 < T \le +\infty$. Then the problem (2.2), (2.3) can be solved globally in time as follows.

Theorem 2.1 *Suppose $\psi_0 \in H^2 \cap L^2_{w(U)}$. Then there exists a positive constant ε_2 such that if $\| \psi_0 \|_2 + |\psi_0|_{w(U)} < \varepsilon_2$, the problem (2.2), (2.3) has a unique global solution $\psi \in X(0,\infty)$ satisfying*

$$\| \psi(t) \|_2^2 + |\psi(t)|^2_{w(U)} + \int_0^t \| \psi_\xi(\tau) \|_2^2 + |\psi_\xi(\tau)|^2_{w(U)} d\tau \le C(\| \psi_0 \|_2^2 + |\psi_0|^2_{w(U)}) \tag{2.7}$$

for any $t \ge 0$. Moreover, ψ_ξ tends to 0 in the maximum norm as $t \to \infty$, that is,

$$\sup_{\mathbf{R}} |\psi_\xi(t,\xi)| \to 0 \quad as \quad t \to \infty.$$

For the decay rate we have the followings.

Theorem 2.2 *(Non-degenerate shock case) Suppose $f'(u_+) < s < f'(u_-)$. Then the solution $\psi(t)$ obtained in Theorem 2.1 satisfies*

$$(1+t)^\gamma \| \psi(t) \|_2^2 + \int_0^t (1+\tau)^\gamma \| \psi_\xi(\tau) \|_2^2 d\tau \le C(|\psi_0|^2_\alpha + \| \psi_0 \|_2^2) \tag{2.8}$$

for any γ such that $0 \le \gamma \le \alpha$ if α is an integer and that $0 \le \gamma < \alpha$ if α is not an integer.

Theorem 2.3 *(Degenerate shock case) Suppose $s = f'(u_+) < f'(u_-)$ and $f''(u_+) = \cdots = f^{(n)}(u_+) = 0$ and $f^{(n+1)}(u_+) \neq 0$ for $n \ge 1$. If $0 < \alpha < 2/n$, then the solution $\psi(t,x)$ obtained in Theorem 2.1 satisfies*

$$(1+t)^\gamma \| \psi(t) \|_2^2 + \int_0^t (1+\tau)^\gamma \| \psi_\xi(\tau) \|_2^2 d\tau \le C(\| \psi_0 \|_2^2 + |\psi_0|^2_{\alpha,w(U)}) \tag{2.9}$$

for γ such that $0 \le \gamma < \alpha/2$.

All assertions (i)-(iii) in Theorem 1 are direct consequences of Theorem 2.1. Theorem 2 and Theorem 3 are, respectively, consquences of Theorem 2.2 and Theorem 2.3. Theorems 2.1-2.3 are all proved by the weighted energy method combining the local existence with a priori estimates. These are on the same line in Kawashima and Matsumura [3] etc.

Proposition 2.1 *(Local existence) For any $\varepsilon_0 > 0$, there exists a positive constant T_0 depending on ε_0 such that if $\psi_0 \in H^2 \cap L^2_{w(U)}$ and $\| \psi_0 \|_2 \leq \varepsilon_0$, then the problem (2.2), (2.3) has a unique solution $\psi \in X(0, T_0)$ satisfying $\| \psi(t) \|_2 < 2\varepsilon_0$ for $0 \leq t \leq T_0$.*

Proposition 2.2 *(A priori estimate) Let ψ be a solution in $X(0,T)$ for a positive constant T. Then there exists a positive constant ε_3 such that if $\sup_{0 \leq t \leq T} \| \psi(t) \|_2 < \varepsilon_3$, then $\psi(t)$ satisfies (2.7) for $0 \leq t \leq T$.*

Proposition 2.1 can be proved in the standard way. Proposition 2.2 will be proved in the next section. For the proofs of Theorems 2.2 and 2.3 more estimates are necessary.

In later sections we only show the case $u_+ < u_-$ and $h(U) \leq 0$ for $U \in [u_+, u_-]$. The other case is shown in the same way.

3 Basic estimate and stability theorem

Assuming $u_+ < u_-$ and $h(U) < 0$ for $U \in (u_+, u_-)$, we first derive the basic estimate in our all proofs.

Lemma 3.1 *Let $\psi(t) \in X(0,T)$ be a solution of (2.2), (2.3). Then it holds*

$$\frac{1}{2}|\psi(t)|^2_{w(U)} + \int_0^t (\| \sqrt{-U_\xi}\psi(\tau) \|^2 + \mu|\psi_\xi(\tau)|^2_{w(U)})d\tau \leq \frac{1}{2}|\psi_0|^2_{w(U)} + \int_0^t \int w(U)\psi F \, dx d\tau. \quad (3.1)$$

Proof. Multiplying (2.4) by $w(U(\xi))\psi(t,\xi)$ we have

$$(\frac{1}{2}w(U)\psi^2)_t + (\frac{1}{2}(wh)'(U)\psi^2 - \mu w(U)\psi_\xi\psi)_\xi + \mu w(U)\psi_\xi^2 - \frac{1}{2}(wh)''(U)U_\xi\psi^2 = w(U)\psi F. \quad (3.2)$$

Here we have used $\mu U_\xi = h(U)$. Since we have taken the weight w as (2.6), we obtain (3.1) by integrating (3.2) over $(0,t) \times \mathbf{R}$ and noting $U_\xi < 0$. Q.E.D.

We now put

$$N(t) = \sup_{0 \leq \tau \leq t} \| \psi(\tau) \|_2,$$

and assume $N(t) \leq \varepsilon_0$. Since $|\psi| \leq N(t)$, $|F| \leq C\psi_\xi^2$. Hence, if $N(t) < \varepsilon_3$ for sufficiently small $\varepsilon_3 > 0$, then we have

$$|\psi(t)|^2_{w(U)} + \int_0^t |\psi_\xi(\tau)|^2_{w(U)} d\tau \leq C|\psi_0|^2_{w(U)}. \quad (3.3)$$

Moreover, we apply ∂_ξ to (2.4), multiply it by $\partial_\xi\psi$ and $\partial_\xi^3\psi$ and integrate the resulting equations over $(0,t) \times \mathbf{R}$. Noting $|F_\xi| \leq o(1)|\psi_\xi| + C|\psi_\xi\psi_{\xi\xi}|$ as $\sup_{\mathbf{R}} |\psi_\xi| \to 0$ we can get the next lemma. We omit the details.

Lemma 3.2 *There is a positive constant $\varepsilon_4 (\leq \varepsilon_0)$ such that if $N(t) \leq \varepsilon_4$, the estimate holds:*

$$\| \psi_\xi(t) \|_1^2 + \int_0^t \| \psi_{\xi\xi}(\tau) \|_1^2 \, d\tau \leq C(|\psi_0|^2_{w(U)} + \| \psi_{0\xi} \|_1^2).$$

Combining Lemma 3.2 with (3.3) gives the proof of Proposition 2.2 and so Theorem 2.1.

4 Decay rate for the case $f'(u_+) < s < f'(u_-)$

We proceed more a priori estimates of the solution ψ of the problem (2.2), (2.3). Since $h(U) < 0$, $U \in (u_+, u_-)$, there exists a unique number $\xi_* \in \mathbf{R}$ such that

$$U(\xi_*) = \overline{u} \equiv \frac{u_+ + u_-}{2}. \tag{4.1}$$

Putting $\langle \xi - \xi_* \rangle = \sqrt{\Gamma + (\xi - \xi_*)^2}$ and multiplying (2.2) by $2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w(U)\psi$, we get

$$\begin{aligned}
&((1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w(U)\psi^2)_t + (\cdots)_\xi + 2\mu(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w(U)\psi_\xi^2 \\
&-\gamma(1+t)^{\gamma-1}\langle \xi - \xi_* \rangle^\beta w(U)\psi^2 + (1+t)^\gamma \langle \xi - \xi_* \rangle^{\beta-1} A_\beta \psi^2 \\
&+2\mu\beta(1+t)^\gamma \langle \xi - \xi_* \rangle^{\beta-2}(\xi - \xi_*) w(U)\psi\psi_\xi \\
&= 2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w(U)\psi F,
\end{aligned} \tag{4.2}$$

where

$$A_\beta(\xi) = -\langle \xi - \xi_* \rangle U_\xi (wh)''(U) - \beta \frac{\xi - \xi_*}{\langle \xi - \xi_* \rangle}(wh)'(U) = -2\langle \xi - \xi_* \rangle U_\xi - 2\beta \frac{\xi - \xi_*}{\langle \xi - \xi_* \rangle}(U - \overline{u})$$

by virtue of (2.6).

Lemma 4.1 *Let α be a given positive number. For $\beta \in [0, \alpha]$, there is a positive number c_0 independent of β such that*

$$A_\beta(\xi) \geq c_0 \beta \quad \textit{for any} \quad \xi \in \mathbf{R}. \tag{4.3}$$

Proof. Let put $g(\xi) = -(wh)'(U(\xi)) = -2(U(\xi) - \overline{u})$, then $g(\xi_*) = 0$ by (4.1) and $g'(\xi) = -2U'(\xi) > 0$, so $g(\xi) \to u_\mp - u_\pm$ as $\xi \to \pm\infty$, respectively. Hence

$$-\frac{\xi - \xi_*}{\langle \xi - \xi_* \rangle}(wh)'(U(\xi)) \geq c(\delta), \quad |\xi - \xi_*| \geq \delta \tag{4.4}$$

for any $\delta > 0$. On the other hand,

$$-\langle \xi - \xi_* \rangle U_\xi (wh)''(U) = -2\langle \xi - \xi_* \rangle U_\xi = 2\langle \xi - \xi_* \rangle \frac{-h(U(\xi))}{\mu} \geq \frac{-h(\overline{u})}{\mu}, \quad |\xi - \xi_*| \leq \delta_0 \tag{4.5}$$

for some constant δ_0. Combinig (4.4) with (4.5) we obtain (4.3), where $c_0 = \min\{c(\delta_0), \frac{-h(\overline{u})}{\mu\alpha}\}$. Q.E.D.

Integrating (4.2) over $[0, t] \times \mathbf{R}$ and noting $C^{-1} \leq w(U) \leq C$, we have

$$\begin{aligned}
&(1+t)^\gamma |\psi(t)|_\beta^2 + \beta \textstyle\int_0^t (1+\tau)^\gamma |\psi(\tau)|_{\beta-1}^2 d\tau + \int_0^t (1+\tau)^\gamma |\psi_\xi(\tau)|_\beta^2 d\tau \\
&\leq C\{|\psi_0|_\beta^2 + \gamma \textstyle\int_0^t (1+\tau)^{\gamma-1} |\psi(\tau)|_\beta^2 d\tau \\
&\quad + \beta \textstyle\int_0^t (1+\tau)^\gamma \int \langle \xi - \xi_* \rangle^{\beta-1} |\psi\psi_\xi| d\xi d\tau + \int_0^t \int (1+\tau)^\gamma \langle \xi - \xi_* \rangle^\beta |\psi||F| d\xi d\tau\}.
\end{aligned} \tag{4.6}$$

Since

$$C\beta\langle\xi-\xi_*\rangle^{\beta-1}|\psi\psi_\xi| \le \frac{\beta}{2}\langle\xi-\xi_*\rangle^{\beta-1}\psi^2+\frac{C^2\beta}{2}\langle\xi-\xi_*\rangle^{\beta-1}\psi_\xi^2$$

and

$$\begin{aligned}
&\int \tfrac{C^2\beta}{2}\langle\xi-\xi_*\rangle^{\beta-1}\psi_\xi^2 d\xi\\
&=\int_{|\xi-\xi_*|>R}\tfrac{C^2\beta}{2(\xi-\xi_*)}\langle\xi-\xi_*\rangle^{\beta}\psi_\xi^2 d\xi+\int_{|\xi-\xi_*|\le R}\tfrac{C^2\beta}{2}\langle\xi-\xi_*\rangle^{\beta-1}\psi_\xi^2 d\xi\\
&\le \tfrac{1}{2}|\psi_\xi|_\beta^2+\beta C_R\|\psi_\xi\|^2.
\end{aligned}$$

for some fixed $R>0$, we have

$$\begin{aligned}
&(1+t)^\gamma|\psi(t)|_\beta^2+\textstyle\int_0^t\{\frac{\beta}{2}(1+\tau)^\gamma|\psi(\tau)|_{\beta-1}^2+(\frac{1}{2}-CN(\tau))(1+\tau)^\gamma|\psi_\xi(\tau)|_\beta^2\}d\tau\\
&\le C\{|\psi_0|_\beta^2+\gamma\textstyle\int_0^t(1+\tau)^{\gamma-1}|\psi(\tau)|_\beta^2 d\tau+\beta\int_0^t\|\psi_\xi\|^2 d\tau\}.
\end{aligned}$$

Thus we get the following.

Lemma 4.2 *There is a positive constant ε_5 such that if $N(T)<\varepsilon_5$, then it holds for $t\in[0,T]$*

$$\begin{aligned}
&(1+t)^\gamma|\psi(t)|_\beta^2+\textstyle\int_0^t\{\beta(1+\tau)^\gamma|\psi(\tau)|_{\beta-1}^2+(1+\tau)^\gamma|\psi_\xi(\tau)|^2\}d\tau\\
&\le C\{|\psi_0|_\beta^2+\gamma\textstyle\int_0^t(1+\tau)^{\gamma-1}|\psi(\tau)|_\beta^2 d\tau+\beta\int_0^t(1+\tau)^\gamma\|\psi_\xi(\tau)\|^2 d\tau\}.
\end{aligned}\tag{4.7}$$

Applying the induction to (4.7) we have

Lemma 4.3 *It holds for $k=0,1,\cdots,[\alpha]$*

$$(1+t)^k|\psi(t)|_{\alpha-k}^2+\int_0^t\{(\alpha-k)(1+\tau)^k|\psi(\tau)|_{\alpha-k-1}^2+(1+\tau)^k|\psi_\xi(\tau)|_{\alpha-k}^2\}d\tau\le C|\psi_0|_\alpha^2. \tag*{$(4.8)_k$}$$

Consequently, if α is an integer, then the following estimate holds for $0\le\gamma\le\alpha$

$$(1+t)^\gamma\|\psi(t)\|^2+\int_0^t(1+\tau)^\gamma\|\psi_\xi(\tau)\|^2 d\tau\le C|\psi_0|_\alpha^2. \tag{4.9}$$

Proof. First, letting $\gamma=0$ and $\beta=\alpha$ in (4.7) we have $(4.8)_0$, which shows the lemma for $\alpha<1$. Here we have used (3.3). Next we take $1\le\alpha<2$. Letting $\beta=0$ and $\gamma=1$ and letting $\beta=\alpha-1$ and $\gamma=1$ in (4.7) show $(4.8)_1$. Hence the proof for $\alpha<2$ is completed. Repeating the same procedure we can get the desired estimate $(4.8)_k$ for any $\alpha\ge 0$. Q.E.D.

Methods used in this section till now are almost same as in Kawashima and Matsumura [3]. Further we show sharper estimate. Let α be not an integer and γ be $[\alpha]<\gamma<\alpha$. Taking $\beta=0$ in (4.7) we have

$$(1+t)^\gamma|\psi(t)|_0^2+\int_0^t(1+\tau)^\gamma|\psi_\xi(\tau)|_0^2 d\tau\le C(|\psi_0|_0^2+\gamma\int_0^t(1+\tau)^{\gamma-1}|\psi(\tau)|_0^2 d\tau). \tag{4.10}$$

Using $(4.8)_k$ with $k=[\alpha]$

$$\begin{aligned}
&(1+t)^{[\alpha]}|\psi(t)|_{\alpha-[\alpha]}^2+\textstyle\int_0^t\{(\alpha-[\alpha])(1+\tau)^{[\alpha]}|\psi(\tau)|_{\alpha-[\alpha]-1}^2\\
&\quad+(1+\tau)^{[\alpha]}|\psi_\xi(\tau)|_\alpha^2\}d\tau\le C|\psi_0|_\alpha^2,
\end{aligned}\tag*{$(4.8)_{[\alpha]}$}$$

we estimate the final term in (4.10):

$$\begin{aligned}
&\textstyle\int_0^t (1+\tau)^{\gamma-1}|\psi(\tau)|_0^2 d\tau \\
&= \textstyle\int_0^t (1+\tau)^{\gamma-1} \int \langle \xi-\xi_*\rangle^{(\alpha-[\alpha])([\alpha]+1-\alpha)-(\alpha-[\alpha])([\alpha]+1-\alpha)} (\psi^2)^{([\alpha]+1-\alpha)+(\alpha-[\alpha])} d\xi d\tau \\
&\le \textstyle\int_0^t (1+\tau)^{\gamma-1} (\int \langle \xi-\xi_*\rangle^{\alpha-[\alpha]}\psi^2 d\xi)^{[\alpha]+1-\alpha} (\int \langle \xi-\xi_*\rangle^{-([\alpha]+1-\alpha)}\psi^2 d\xi)^{\alpha-[\alpha]} d\tau \\
&= \textstyle\int_0^t (1+\tau)^{-([\alpha]+1-\gamma)} ((1+\tau)^{[\alpha]}|\psi|^2_{\alpha-[\alpha]})^{[\alpha]+1-\alpha} ((1+\tau)^{[\alpha]}|\psi|^2_{\alpha-[\alpha]-1})^{\alpha-[\alpha]} d\tau \\
&\le C|\psi_0|_\alpha^{2([\alpha]+1-\alpha)} \textstyle\int_0^t (1+\tau)^{-([\alpha]+1-\gamma)} ((1+\tau)^{[\alpha]}|\psi|^2_{\alpha-[\alpha]-1})^{\alpha-[\alpha]} d\tau \\
&\le C|\psi_0|_\alpha^{2([\alpha]+1-\alpha)} (\textstyle\int_0^t (1+\tau)^{-\frac{[\alpha]+1-\gamma}{[\alpha]+1-\alpha}} d\tau)^{[\alpha]+1-\alpha} (\int_0^t (1+\tau)^{[\alpha]}|\psi|^2_{\alpha-[\alpha]-1} d\tau)^{\alpha-[\alpha]} \\
&\le C|\psi_0|_\alpha^2,
\end{aligned}$$

because of $\frac{[\alpha]+1-\gamma}{[\alpha]+1-\alpha} > 1$. Thus we have the following from (4.10).

Lemma 4.4 *If α is not an integer, then it holds for any $\gamma < \alpha$*

$$(1+t)^\gamma \|\psi(t)\|^2 + \int_0^t (1+\tau)^\gamma \|\psi_\xi(\tau)\|^2 d\tau \le C|\psi_0|_\alpha^2. \tag{4.11}$$

Similar estimates to Lemma 3.2 give the same decay rate for derivatives of the solution.

5 Decay rate for the case $s = f'(u_+) < f'(u_-)$

First we show the following estimate for the solution ψ obtained in Theorem 2.1.

Lemma 5.1 *For $0 < \beta \le \alpha < 2/n (n \ge 1)$, there exists a positive constant ε_7 such that if $N(T) \le \varepsilon_7$, then the estimate*

$$\begin{aligned}
&\textstyle\int w(U)^{1+\beta}\psi(t,\xi)^2 d\xi + \int_0^t \int_{\xi>0} w(U)^{\beta-1}\psi(\tau,\xi)^2 d\xi d\tau \\
&\quad + \textstyle\int_0^t \int w(U)^{1+\beta}\psi_\xi(\tau,\xi)^2 d\xi d\tau \le C\int w(U)^{1+\beta}\psi_0(\xi)^2 d\xi
\end{aligned} \tag{5.1$_\beta$}$$

Proof. Letting $z(\xi)$ be a positive function and multiplying (2.4) by $2zw(U)\psi$, we have

$$(zw(U)\psi^2)_t + (\cdots)_\xi + 2\mu zw(U)\psi_\xi^2 - (z(hw)'(U))_\xi\psi^2 + 2\mu z_\xi w(U)\psi\psi_\xi = 2zw(U)\psi F. \tag{5.2}$$

Since

$$-(z(hw)'(U))_\xi = 2(\overline{u} - U)z_\xi - 2zU_\xi$$

and

$$|2\mu z_\xi w(U)\psi\psi_\xi| \le 2\varepsilon\mu zw(U)\psi_\xi^2 + \frac{\mu w(U) z_\xi^2}{2\varepsilon z}\psi^2$$

for $\varepsilon \in (0,1)$, Eq.(5.2) yields

$$\begin{aligned}
&(zw(U)\psi^2)_t + (\cdots)_\xi + 2(1-\varepsilon)\mu zw(U)\psi_\xi^2 + \{-2zU_\xi + (2(\overline{u}-U) - \tfrac{\mu w(U) z_\xi}{2\varepsilon z})z_\xi\}\psi^2 \\
&\le 2zw(U)|\psi F|.
\end{aligned} \tag{5.3}$$

Taking $z = w(U)^\beta$, we have

$$I \equiv (2(\overline{u} - U) - \frac{\mu w(U) z_\xi}{2\varepsilon z}) z_\xi = \frac{\beta}{\mu} w(U)^{\beta-1} w'(U) h(U)(2(\overline{u} - U) - \frac{\beta w'(U)h(U)}{2\varepsilon}).$$

If we put $\delta = U - u_+ > 0$ and $\tilde{u} = u_- - u_+$, then we have near u_+ or $\xi = +\infty$

$$\begin{aligned} h(U)w'(U) &= 2\delta - \tilde{u} - \tfrac{\delta(\delta-\tilde{u})h'(U)}{h(U)} \\ &= 2\delta - \tilde{u} - \frac{\delta(\delta-\tilde{u})(h''(u_+)\delta + \cdots + \frac{h^{(n+1)}(u_+)}{n!}\delta^n + o(\delta^n))}{\frac{h''(u_+)}{2!}\delta^2 + \cdots + \frac{h^{(n+1)}(u_+)}{(n+1)!}\delta^{n+1} + o(\delta^{n+1})} \\ &= 2\delta - \tilde{u} - (\delta - \tilde{u})((n+1) + O(\delta)) \\ &= \tilde{u}n + O(\delta) \end{aligned}$$

and hence

$$I = \frac{\beta w(U)^{\beta-1}}{\mu}(\tilde{u}n + O(\delta))(\tilde{u}(1 - \frac{\beta n}{2\varepsilon}) + O(\delta)).$$

Since $\beta \le \alpha < \frac{2}{n}$, if we set $\varepsilon < 1$ as $1 - \frac{\beta n}{2\varepsilon} > 0$, then there are positive constants c_1 and R_1 such that

$$I \ge c_1 \quad \text{for} \quad \xi \ge R_1. \tag{5.4}$$

Noting $C^{-1} \le w(U) \le C, C^{-1} \le (w'h)(U) \le C$ as $\xi \to -\infty$ and using Lemma 3.1, we have

$$\int_0^t \int_{\xi \le R_1} 2I \cdot \psi^2 d\xi d\tau \le C|\psi_0|^2_{w(U)}. \tag{5.5}$$

Because of (5.4) and (5.5) the integration of (5.3) over $(0,t) \times \mathbf{R}$ gives the estimate (5.1). Q.E.D.

Again multiplying (2.2) by $2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w(U)\psi$ and integrating its equation (=(4.2)) over $(0,t) \times \mathbf{R}$, we have for $0 \le \beta \le \alpha$

$$\begin{aligned} &(1+t)^\gamma |\psi(t)|^2_{\beta,w(U)} + (1 - CN(T)) \textstyle\int_0^t (1+\tau)^\gamma |\psi_\xi(\tau)|^2_{\beta,w(U)} d\tau \\ &\quad + \beta \textstyle\int_0^t (1+\tau)^\gamma |\psi(\tau)|^2_{\beta-1} d\tau \\ &\le C\{|\psi_0|^2_{\beta,w(U)} + \gamma \textstyle\int_0^t (1+\tau)^{\gamma-1} |\psi(\tau)|^2_{\beta,w(U)} d\tau \\ &\quad + \beta \textstyle\int_0^t (1+\tau)^\gamma \int \langle \xi - \xi_* \rangle^{\beta-1} w(U) |\psi \psi_\xi| d\xi d\tau\}. \end{aligned} \tag{5.6$_{\gamma,\beta}$}$$

For $(5.6)_{\gamma,\beta}$ with $\gamma = 0$ and $\beta \le \alpha$

$$|\text{last term in}(5.6)_{0,\beta}| \le \int_0^t \int \frac{\beta}{2} \langle \xi - \xi_* \rangle^{\beta-1} \psi^2 + \frac{C^2 \beta}{2} \langle \xi - \xi_* \rangle^{\beta-1} w(U)^2 \psi_\xi^2 d\xi d\tau, \tag{5.7}$$

and

$$\begin{aligned} &\tfrac{C^2\beta}{2} \textstyle\int_0^t \int \langle \xi - \xi_* \rangle^{\beta-1} w(U)^2 \psi_\xi^2 d\xi d\tau \\ &\le C \textstyle\int_0^t \int_{\xi > R_2} w(U)^{\beta+1} \psi_\xi^2 d\xi d\tau + \tfrac{1}{2} \int_0^t \int_{\xi < -R_3} \langle \xi - \xi_* \rangle^\beta \psi_\xi^2 d\xi d\tau + C \int_0^t \int_{-R_3 \le \xi \le R_2} \psi_\xi^2 d\xi d\tau \end{aligned} \tag{5.8}$$

for some constants $R_2, R_3 > 0$, becaue $w(U(\xi)) \sim \xi$ as $\xi \to \infty$ and $w(U(\xi)) \sim$ *const.* as $\xi \to -\infty$. Applying Lemma 6.1 and Lemma 4.1 to $(5.6)_{0,\beta}$, (5.7) and (5.8), and taking $\beta = \alpha$, we get the following.

Lemma 5.2 *There is a positive constant ε_7 such that if $N(T) \leq \varepsilon_7$, then the estimate*

$$|\psi(t)|^2_{\alpha,w(U)} + \int_0^t |\psi(\tau)|^2_{\alpha-1} + |\psi_\xi(\tau)|^2_{\alpha,w(U)} d\tau \leq C|\psi_0|^2_{\alpha,w(U)} \tag{5.9}$$

holds for $\alpha < \frac{2}{n}(n \geq 1)$.

Next we consider $(5.6)_{\gamma,\beta}$ with $\gamma < \alpha/2$ and $\beta = 0$:

$$\begin{aligned} &(1+t)^\gamma |\psi(t)|^2_{0,w(U)} + (1 - N(T)) \textstyle\int_0^t (1+\tau)^\gamma |\psi_\xi(\tau)|^2_{0,w(U)} d\tau \\ &\leq C(|\psi_0|^2_{0,w(U)} + \gamma \textstyle\int_0^t (1+\tau)^{\gamma-1} |\psi(\tau)|^2_{0,w(U)} d\tau). \end{aligned} \tag{$(5.6)_{\gamma,0}$}$$

We can estimate the final term provided $\gamma < \frac{\alpha}{2}$,in a similar fashion to the proof of Lemma 4.4, deviding the integrand into $\{\xi > 0\}$ and $\{\xi < 0\}$.

Thus we have had a desired estimate.

Lemma 5.3 *For $N(T) \leq \varepsilon_7$, it holds for $\gamma < \alpha/2 < 1/n$*

$$(1+t)^\gamma |\psi(t)|^2_{w(U)} + \int_0^t (1+\tau)^\gamma |\psi_\xi(\tau)|^2_{w(U)} d\tau \leq C|\psi_0|^2_{\alpha,w(U)}.$$

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