

## Rarefaction waves in discrete kinetic theory

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### 1. Introduction

We consider the initial value problem for the discrete Boltzmann equation in one space dimension:

$$(1.1) \quad c_i \left( \frac{\partial F_i}{\partial t} + v_i \frac{\partial F_i}{\partial x} \right) = Q_i[F], \quad i \in \Lambda,$$

$$(1.2) \quad F_i(0, x) = F_{i0}(x), \quad i \in \Lambda.$$

The discrete Boltzmann equation (1.1) describes the evolution of a gas of particles allowed to move with a finite number of admissible velocities, the  $x$ -components of which are denoted by  $v_i, i \in \Lambda$ , where  $\Lambda = \{1, 2, \dots, m\}$ . Each  $F_i(t, x)$  denotes the mass density of gas particles corresponding to  $v_i$ ,  $Q_i[F]$  is the term related to binary and higher order multiple collisions (we do not exclude the multiple collision case), and  $c_i$  is a positive number.

We want to study large-time behaviors of solutions to the problem (1.1), (1.2) when the initial function  $F_0(x) = (F_{i0}(x))_{i \in \Lambda}$  satisfies

$$(1.3) \quad F_0(x) \rightarrow F^\pm \quad \text{as } x \rightarrow \pm\infty,$$

where  $F^\pm = (F_i^\pm)_{i \in \Lambda}$  are constant Maxwellians so that  $Q_i[F^\pm] = 0$  for  $i \in \Lambda$ . Our main interest is in the study of wave phenomena arising in the solutions of (1.1), (1.2) as  $t \rightarrow +\infty$ . In this respect, it was observed in [2] that if  $F^+ = F^-$ , then the solution of (1.1), (1.2) behaves, asymptotically as  $t \rightarrow +\infty$ , like a local Maxwellian which is characterized by the superposition of nonlinear diffusion waves given in terms of the self-similar solution of the Burgers equation. On the other hand, when  $F^+ \neq F^-$ , it is expected that the solution of (1.1), (1.2) approaches, in general, the superposition of rarefaction waves and/or (smooth) shock waves as  $t \rightarrow +\infty$ , and this combination of waves is determined by the relative position of the end states  $F^\pm$  in the space of all Maxwellians (see [3]). However, the analysis of wave phenomena in this general case is so complicated and nothing is known except for the Broadwell

model, the simplest model written in the form of (1.1): For the Broadwell model, asymptotics toward rarefaction waves and (smooth) shock waves are discussed in [8] and [7,1], respectively.

The aim of this paper is to discuss the asymptotic stability of rarefaction waves for a certain class of models described by (1.1). Our main theorem states that under suitable assumptions the solution of (1.1), (1.2) behaves, asymptotically as  $t \rightarrow +\infty$ , like a local Maxwellian which is characterized by the superposition of centered rarefaction waves for the Euler equation associated with (1.1). This is a generalization of the stability result in [8] for the Broadwell model. The detailed proof of the main theorem will be given in a joint paper with Bellomo [6].

## 2. Assumptions and examples

We denote by  $\mathcal{M}_0$  the space of collision invariants for (1.1).  $\mathcal{M}_0$  is a linear subspace of  $\mathbb{R}^m$ ,  $m$  being the cardinality of  $\Lambda$ , and consists of vectors  $\phi = (\phi_i)_{i \in \Lambda}$  satisfying

$$(2.1) \quad \sum_{i \in \Lambda} \phi_i Q_i[F] = 0 \quad \text{for any } F = (F_i)_{i \in \Lambda} \in \mathbb{R}^m.$$

Throughout the paper we make the following assumption that characterizes the models under consideration.

**Assumption 2.1.** (i)  $\dim \mathcal{M}_0 = 2$ , and  $\mathcal{M}_0$  is spanned by  $(1)_{i \in \Lambda}$  and  $(v_i)_{i \in \Lambda}$ , where  $(1)_{i \in \Lambda}$  means the vector with components all equal to one.  
(ii)  $(v_i^2)_{i \in \Lambda}$  is not an element of  $\mathcal{M}_0$ .

Note that Assumption 2.1 implies the set  $\{v_i\}_{i \in \Lambda}$  of all  $x$ -components of admissible velocities contains at least three different values. The collision invariants  $(1)_{i \in \Lambda}$  and  $(v_i)_{i \in \Lambda}$  correspond to the conservation of mass and momentum (in the  $x$ -direction), respectively, and we here neglect the conservation of energy. In other words, we assumed that the collision invariant corresponding to the energy conservation is linearly dependent to  $(1)_{i \in \Lambda}$ . This is the case where the admissible velocities have the same modulus.

The simplest model satisfying Assumption 2.1 is the Broadwell model which is written in the form

$$(2.2) \quad \begin{cases} \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial x} = F_2^2 - F_1 F_3, \\ 4 \frac{\partial F_2}{\partial t} = 2(F_1 F_3 - F_2^2), \\ \frac{\partial F_3}{\partial t} - \frac{\partial F_3}{\partial x} = F_2^2 - F_1 F_3. \end{cases}$$

For this model,  $\mathcal{M}_0$  is a two-dimensional subspace spanned by  $(1, 1, 1)$  and  $(1, 0, -1)$ .

Another interesting model satisfying Assumption 2.1 is the following one-dimensional version of the 6-velocity coplanar model with both binary and ternary collisions.

$$(2.3) \quad \left\{ \begin{array}{l} \frac{\partial F_1}{\partial t} + 2\frac{\partial F_1}{\partial x} = a_1(F_2F_3 - F_1F_4) + a_2(F_2^2F_4 - F_1F_3^2), \\ 2\left(\frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial x}\right) = a_1(F_1F_4 - F_2F_3) + 2a_2(F_1F_3^2 - F_2^2F_4), \\ 2\left(\frac{\partial F_3}{\partial t} - \frac{\partial F_3}{\partial x}\right) = a_1(F_1F_4 - F_2F_3) + 2a_2(F_2^2F_4 - F_1F_3^2), \\ \frac{\partial F_4}{\partial t} + 2\frac{\partial F_4}{\partial x} = a_1(F_2F_3 - F_1F_4) + a_2(F_1F_3^2 - F_2^2F_4), \end{array} \right.$$

where  $a_1$  and  $a_2$  are positive constants. The subspace  $\mathcal{M}_0$  of this model is two-dimensional and is spanned by  $(1, 1, 1, 1)$  and  $(2, 1, -1, -2)$ .

### 3. Maxwellians and the Euler equation

Consider a vector  $F = (F_i)_{i \in \Lambda}$  with  $F_i > 0$  for  $i \in \Lambda$ . Such a vector  $F$  is called Maxwellian if  $Q_i[F] = 0$  for  $i \in \Lambda$ . It is known that any Maxwellian  $F$  can be characterized by the corresponding fluid-dynamic variables. Under Assumption 2.1, there are two independent fluid-dynamic variables which are, for example, the mass density  $\rho$  and the mean velocity  $u$ :

$$(3.1) \quad \rho = \sum_{i \in \Lambda} c_i F_i, \quad \rho u = \sum_{i \in \Lambda} c_i v_i F_i,$$

and any Maxwellian  $F$  can be parametrized by  $(\rho, u)$  in the form

$$(3.2) \quad F = \rho M(u).$$

Here  $M(u) = (M_i(u))_{i \in \Lambda}$  is given implicitly by the formula

$$(3.3) \quad M_i(u) = \frac{e^{\beta v_i}}{G(\beta)}, \quad i \in \Lambda, \quad u = \frac{G'(\beta)}{G(\beta)},$$

in which  $G(\beta) = \sum_{i \in \Lambda} c_i e^{\beta v_i}$ , and  $G'(\beta)$  denotes the derivative of  $G(\beta)$  with respect to  $\beta \in \mathbb{R}$ .

Now, let  $F = (F_i)_{i \in \Lambda}$  be a solution of (1.1) such that  $F_i > 0$  for  $i \in \Lambda$ , and let  $(\rho, u)$  be the corresponding fluid-dynamic variables defined by (3.1). From Assumption 2.1 we have two independent conservation laws which are of the form

$$(3.4) \quad \begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + S_x = 0, \end{cases}$$

where  $S = \sum_{i \in \Lambda} c_i v_i^2 F_i$ . Obviously, (3.4) is not a closed system of  $(\rho, u)$ . To get a closed system, we apply the Chapman-Enskog expansion

$$F = F^{(0)} + F^{(1)} + \dots$$

It is known that the first order term  $F^{(0)}$  in the expansion is the Maxwellian which is characterized by the fluid-dynamic variables  $(\rho, u)$  of the original  $F$ , that is,  $F^{(0)} = \rho M(u)$ . If we neglect the higher order terms in the expansion and use the approximation  $F = F^{(0)}$ , then (3.4) becomes a closed system of the form

$$(3.5) \quad \begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho \sigma(u))_x = 0, \end{cases}$$

where

$$(3.6) \quad \sigma(u) = \sum_{i \in \Lambda} c_i v_i^2 M_i(u).$$

This closed system is called the Euler equation associated with (1.1). It was observed in [5,4] that our Euler equation (3.5) forms a strictly hyperbolic system with genuinely nonlinear characteristic fields. This is the same property as the classical Euler equation in fluid mechanics has.

We give some details about this property. Let us rewrite (3.5) as

$$(3.7) \quad \begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ (\sigma(u) - u^2)/\rho & \sigma'(u) - u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0.$$

We can verify that the coefficient matrix in (3.7) has real and distinct eigenvalues  $\lambda_j, j = 1, 2$ , which depend only on  $u$  (independent of  $\rho$ ). More precisely, we see that

$$(3.8) \quad \lambda_1(u) < u < \lambda_2(u),$$

which implies the strict hyperbolicity of (3.5). The right eigenvectors  $r_j(\rho, u)$  corresponding to  $\lambda_j(u)$  are

$$(3.9) \quad r_j(\rho, u) = \begin{pmatrix} \rho \\ \lambda_j(u) - u \end{pmatrix}, \quad j = 1, 2,$$

for which we can check that  $\nabla \lambda_j \cdot r_j \neq 0, j = 1, 2$ , where the gradient  $\nabla$  is with respect to  $(\rho, u)$ . This last property means the genuine nonlinearity of both characteristic fields.

#### 4. Rarefaction waves for the Euler equation

Let  $F^\pm$  be the Maxwellians appearing in (1.3) and let  $(\rho_\pm, u_\pm)$  be the corresponding fluid-dynamic variables, that is,

$$(4.1) \quad F^\pm = \rho_\pm M(u_\pm).$$

We consider the initial value problem for the Euler equation (3.5) with the step initial data

$$(4.2) \quad (\rho, u)(0, x) = (\rho_\pm, u_\pm), \quad x \gtrless 0.$$

This kind of initial value problem, called the Riemann problem, admits a weak solution which is, in general, expressed as the superposition of centered rarefaction waves and/or centered (discontinuous) shock waves, and this combination of waves is completely determined by the relative position of the end states  $(\rho_\pm, u_\pm)$  in the state space. This weak solution depends on  $(t, x)$  only through the variable  $\xi = x/t$  so that we may write it by  $(\rho^R, u^R)(x/t)$ .

In this paper we discuss the case where  $(\rho^R, u^R)(x/t)$  is the superposition of centered rarefaction waves only. This is the case where the end states  $(\rho_\pm, u_\pm)$  satisfy

$$(4.3) \quad (\rho_+, u_+) \in RR(\rho_-, u_-).$$

Here  $RR(\rho_-, u_-)$  is the sector between the two rarefaction curves  $R_j(\rho_-, u_-)$ ,  $j = 1, 2$ , in the state space.  $R_j(\rho_-, u_-)$  is the integral curve of the right eigenvector  $r_j(\rho, u)$  in (3.9), that passes through the point  $(\rho_-, u_-)$  and has the property that  $\lambda_j(u) \geq \lambda_j(u_-)$  for  $(\rho, u) \in R_j(\rho_-, u_-)$ . This curve is represented in the form

$$(4.4) \quad R_j(\rho_-, u_-) = \{(\rho, u); \rho/\rho_- = f_j(u; u_-), u \geq u_-\},$$

where

$$(4.5) \quad f_j(u; u_-) = \exp\left(\int_{u_-}^u \frac{ds}{\lambda_j(s) - s}\right), \quad j = 1, 2.$$

Therefore, the region  $RR(\rho_-, u_-)$  is given explicitly as

$$(4.6) \quad RR(\rho_-, u_-) = \{(\rho, u); f_1(u; u_-) \leq \rho/\rho_- \leq f_2(u; u_-), u \geq u_-\}.$$

Now we suppose that (4.3) holds, and we give a precise expression of the weak solution  $(\rho^R, u^R)(x/t)$  to the Riemann problem (3.5), (4.2). We see that if (4.3) holds, then there exists uniquely an intermediate state  $(\rho_*, u_*)$  such that

$$(4.7) \quad (\rho_*, u_*) \in R_1(\rho_-, u_-) \quad \text{and} \quad (\rho_+, u_+) \in R_2(\rho_*, u_*).$$

The weak solution  $(\rho^R, u^R)(x/t)$  is then decomposed into

$$(4.8) \quad (\rho^R, u^R)(x/t) = \sum_{j=1}^2 (\rho_j^R, u_j^R)(x/t) - (\rho_*, u_*),$$

where  $(\rho_j^R, u_j^R)(x/t)$  is the centered rarefaction wave in the  $\lambda_j$ -characteristic field and connects two constant states  $(\rho_-, u_-)$  and  $(\rho_*, u_*)$  for  $j = 1$ , and  $(\rho_*, u_*)$  and  $(\rho_+, u_+)$  for  $j = 2$ . These centered rarefaction waves are the weak solutions to the Riemann problem for (3.5) with the corresponding step initial data and are given by the formulas

$$(4.9) \quad \begin{aligned} \lambda_1(u_1^R(\xi)) &= w^R(\xi; \lambda_1(u_-), \lambda_1(u_*)), (\rho_1^R, u_1^R)(\xi) \in R_1(\rho_-, u_-), \\ \lambda_2(u_2^R(\xi)) &= w^R(\xi; \lambda_2(u_*), \lambda_2(u_+)), (\rho_2^R, u_2^R)(\xi) \in R_2(\rho_*, u_*). \end{aligned}$$

Here  $w^R(x/t; w_-, w_+)$  with  $w_- < w_+$  is a weak solution to the Riemann problem for the inviscid Burgers equation  $w_t + ww_x = 0$  with the step initial function  $w(0, x) = w_{\pm}, x \gtrless 0$ , which is the simplest centered rarefaction

$$(4.10) \quad w^R(\xi; w_-, w_+) = \begin{cases} w_-, & \xi < w_-, \\ \xi, & w_- \leq \xi \leq w_+, \\ w_+, & w_+ < \xi. \end{cases}$$

The formula (4.8) together with (4.9) and (4.10) gives a precise expression of the weak solution  $(\rho^R, u^R)(x/t)$  to the Riemann problem (3.5), (4.2) under the condition (4.3).

## 5. Rarefaction waves for the discrete Boltzmann equation

Let  $(\rho^R, u^R)(x/t)$  be the superposition of two centered rarefaction waves for the Euler equation (3.5), which is constructed in the previous section. We now define

$$(5.1) \quad F^R(x/t) = \rho^R(x/t)M(u^R(x/t)).$$

This is the Maxwellian with  $(\rho^R, u^R)(x/t)$  as its fluid-dynamic variables and is called simply the rarefaction wave for the discrete Boltzmann equation (1.1). It has essentially the same structure as that of  $(\rho^R, u^R)(x/t)$  and tends to the following step function as  $t \downarrow 0$ .

$$(5.2) \quad F_0^R(x) = F^{\pm}, \quad x \gtrless 0.$$

The  $F = F^R(x/t)$  thus defined is not an exact solution of (1.1) but it can be proved to be an asymptotic solution as  $t \rightarrow +\infty$  to the initial value problem (1.1), (1.2)

when  $(\rho_{\pm}, u_{\pm})$  corresponding to  $F^{\pm}$  satisfy (4.3). This is the main result of this paper and can be stated more precisely as follows.

**Main theorem.** *Suppose that Assumption 2.1 is satisfied and that the end states  $F^{\pm}$  are Maxwellians whose fluid-dynamic parameters  $(\rho_{\pm}, u_{\pm})$  satisfy (4.3). Assume in addition that*

$$(5.3) \quad F_0 - F_0^R \in L^2(\mathbb{R}), \quad \partial_x F_0 \in L^2(\mathbb{R}).$$

*Then there is a positive constant  $\varepsilon_0$  such that if  $\|F_0 - F_0^R\| + \|\partial_x F_0\| + |F^+ - F^-| \leq \varepsilon_0$ , then the initial value problem (1.1), (1.2) has a unique global solution  $F = (F_i)_{i \in \Lambda}$  satisfying*

$$(5.4) \quad F - F_0^R \in C^0([0, \infty); L^2(\mathbb{R})), \quad \partial_x F \in C^0([0, \infty); L^2(\mathbb{R})).$$

*Moreover, the solution approaches the rarefaction wave  $F^R(x/t)$ , which is defined in (5.1), as  $t \rightarrow +\infty$ . That is,*

$$(5.5) \quad \sup_{x \in \mathbb{R}} |F(t, x) - F^R(x/t)| \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

This is a generalization of the stability result by Matsumura [8] for the Broadwell model (2.2). Our stability result shows that the solution of the discrete Boltzmann equation (1.1) can be approximated, as  $t \rightarrow +\infty$ , by the superposition of rarefaction waves of the Euler equation (3.5), and this can be considered as a mathematical justification (in some sense) of the Euler level approximation of the Chapman-Enskog expansion applied to the discrete Boltzmann equation (1.1).

The proof of the above main result requires several technicalities and will be given in a joint paper with Bellomo [6].

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