A NOTE ON MILNOR AND THURSTON'S MONOTONICITY THEOREM

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The aim of this note is to give a simplified proof of the so-called Milnor and Thurston's monotonicity theorem. We begin with stating the theorem.

Let us consider a family of maps $Q_a(x) = a - x^2$ from $\mathbb{R}$ to itself. The kneading sequence for $Q_a$ is an infinite sequence $K(a) = (e_1, e_2, \cdots)$ of three symbols $L$, $C$ and $R$ defined as

$$e_i = \begin{cases} L, & \text{if } Q_a^i(0) < 0; \\ C, & \text{if } Q_a^i(0) = 0; \\ R, & \text{if } Q_a^i(0) > 0, \end{cases}$$

On the set $\{L, C, R\}^\mathbb{N}$, the so-called signed lexicographical order $\prec$ is defined as follows: for three sequences with the same first $n$-entries, $I_L = (e_1, e_2, \cdots, e_n, L, \cdots)$, $I_C = (e_1, e_2, \cdots, e_n, C, \cdots)$, $I_R = (e_1, e_2, \cdots, e_n, R, \cdots)$, we decide $I_L \prec I_C \prec I_R$ if the number of the symbol $R$ in $\{e_1, e_2, \cdots, e_n\}$ is even, and $I_R \prec I_C \prec I_L$ otherwise. Milnor and Thurston's monotonicity theorem is

**Theorem.** *The correspondence $a \mapsto K(a)$ is monotone increasing.*

This surprisingly strong theorem was conjectured by Milnor and Thurston, and proved firstly by Duady, Hubberd and Sullivan. The proof we give here is a modification of the proof in [2].

The theorem follows from

**Proposition.** *If $K(a_0) = (e_1, e_2, \cdots, e_n, C, \cdots)$ and $e_i \neq C$ for $1 \leq i \leq n$, then

\[
\frac{\partial_a(Q_a^{n+1}(0))|_{a=a_0}}{DQ_a^n(Q_{a_0}(0))} > 0
\]

where $DQ_a^n$ denote the derivative of the $n$-times iteration of the map $Q_{a_0}$ and $\partial_a(Q_a^{n+1}(0))$ denote the derivative of $Q_a^{n+1}(0)$ as a function of the parameter $a$.

In fact, suppose that $a_0$ satisfies the assumption of the proposition and that the number of $R$ in $(e_1, e_2, \cdots, e_n)$ is even (resp. odd). Then the denominator in the left hand side of (1) is positive (resp. negative) and, from the proposition, so is the numerator. This implies that $e_{n+1}$ varies as $L \rightarrow C \rightarrow R$ (resp. $R \rightarrow C \rightarrow L$) when
the parameter $a$ pass $a_0$ from the left to the right. Now consider the truncated
kneading sequence $K^{(n)}(a) = (e_1, e_2, \ldots, e_n)$ for each $n$. For each parameter at
which $K^{(n)}(a)$ changes, we can find the situation in the proposition. Thus the above
observation shows that $K^{(n)}(a)$ depends on $a$ monotonously. Letting $n \to \infty$, we
get the theorem.

Let us denote $w_i = Q^n_a(0)$ for $i = 1, 2, \ldots, n$ and put $\omega = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$.
Consider the so-called Thurston map:

$$T(z_1, z_2, \ldots, z_n) = (\sigma_1 \sqrt{z_1 - z_2}, \sigma_2 \sqrt{z_1 - z_3}, \ldots, \sigma_{n-1} \sqrt{z_1 - z_n}, \sigma_n \sqrt{z_1})$$

where $\sigma_i$ is the sign of $w_i$. Then $T(\omega) = \omega$ and $T$ is defined on a neighborhood of
$\omega$. By easy calculations, we obtain

Lemma 1. $\frac{\partial_a(Q^{n+1}_a(0))|_{a=a_0}}{DQ^a_a(Q^{a_0}(0))} = \det(I_n - D\omega T)$ where $I_n$ denotes the $n \times n$ unit
matrix and $D\omega T$ the derivative of $T$ at $\omega$.

So we reduce the proposition to

Lemma 2. No eigenvalue of $D\omega T$ is contained in $[1, \infty)$.

Let $X = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid 0 < |z_i| < 3, \text{ and } z_i \neq z_j \text{ if } i \neq j \}$ and

$$Y_{\varepsilon} = \{(z_1, z_2, \ldots, z_n) \in X \mid |z_i| > 10^i \varepsilon \text{ and } |z_i - z_j| > 10^{\min(i, j)} \varepsilon \text{ if } i \neq j \}$$

for $\varepsilon > 0$. Then the (multi-valued) complex extension of $T$,

$$T_C(z_1, z_2, \ldots, z_n) = (\sqrt{z_1 - z_2}, \sqrt{z_1 - z_3}, \ldots, \sqrt{z_1}) : \mathbb{C}^n \to \mathbb{C}^n,$$

maps $X$ into itself in the sense that, for every $x \in X$ and every branch of $T_C$, the
image belongs to $X$. Moreover, if $\varepsilon$ is sufficiently small, $T_C$ maps $Y_{\varepsilon}$ into a compact
subset of $Y_{\varepsilon}$ in this sense. Take $\varepsilon$ so small that $\omega \in Y_{\varepsilon}$. Let $M_\mu : \mathbb{C} \to \mathbb{C}$ be a map
defined by

$$M_\mu(z_1, \ldots, z_n) = (w_1 + \mu(z_1 - w_1), \ldots, w_n + \mu(z_n - w_n)) : \mathbb{C}^n \to \mathbb{C}^n.$$

We choose $\mu > 1$ so close to 1 that the composition $S := M_\mu \circ T_C$ maps $Y_{\varepsilon}$ into
itself. Let $\pi : Y_{\varepsilon} \to Y_{\varepsilon}$ be the universal covering and let $\tilde{\omega} \in \tilde{Y}_{\varepsilon}$ be a point such that
$\pi(\tilde{\omega}) = \omega$. Then there is a (single valued) lift $\tilde{S} : \tilde{Y}_{\varepsilon} \to \tilde{Y}_{\varepsilon}$ of $S$ such that $\tilde{S}(\tilde{\omega}) = \tilde{\omega}$.

Now consider the Kobayashi metric $| \cdot |_K$ on $\tilde{Y}_{\varepsilon}$, which is defined as

$$|v|_K = [\sup\{r \geq 0 \mid \text{there is a holomorphic map } \phi : D_r \to \tilde{Y}_{\varepsilon} \text{ s.t. } d\phi(e) = v.\}]^{-1}$$

for any tangent vector $v$ where $D_r = \{z \in \mathbb{C} \mid |z| < r\}$ and $e$ is the unit vector at
$0 \in D_r$. (See [1] for generalities.) Then it is easy to check that $| \cdot |_K$ is equivalent to the
Euclidean metric at each point. From the definition, we have $|d\Phi(v)|_K \leq |v|_K$
for any holomorphic map $\Phi : \tilde{Y}_{\varepsilon} \to \tilde{Y}_{\varepsilon}$ and any tangent vector $v$. So it follows that
the spectral radius of $D\omega \tilde{S}$ is not bigger than 1. Since $D\omega \tilde{S} = \mu \cdot D\omega T$ and $\mu > 1$,
the spectral radius of $D\omega \tilde{S}$ is smaller than 1. We have proved lemma 2 and so the
main theorem.

References