A NOTE ON MILNOR AND THURSTON'S MONOTONICITY THEOREM

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The aim of this note is to give a simplified proof of the so-called Milnor and Thurston's monotonicity theorem. We begin with stating the theorem.

Let us consider a family of maps $Q_a(x) = a - x^2$ from $\mathbb{R}$ to itself. The kneading sequence for $Q_a$ is an infinite sequence $K(a) = (e_1, e_2, \cdots)$ of three symbols $L, C$ and $R$ defined as

$$e_i = \begin{cases} L, & \text{if } Q_a^i(0) < 0; \\ C, & \text{if } Q_a^i(0) = 0; \\ R, & \text{if } Q_a^i(0) > 0, \end{cases}$$

On the set $\{L, C, R\}^\mathbb{N}$, the so-called signed lexicographical order $\prec$ is defined as follows: for three sequences with the same first $n$-entries, $I_L = (e_1, e_2, \cdots, e_n, L, \cdots)$, $I_C = (e_1, e_2, \cdots, e_n, C, \cdots)$, $I_R = (e_1, e_2, \cdots, e_n, R, \cdots)$, we decide $I_L \prec I_C \prec I_R$ if the number of the symbol $R$ in $(e_1, e_2, \cdots, e_n)$ is even, and $I_R \prec I_C \prec I_L$ otherwise. Milnor and Thurston's monotonicity theorem is

**Theorem.** The correspondence $a \mapsto K(a)$ is monotone increasing.

This surprisingly strong theorem was conjectured by Milnor and Thurston, and proved firstly by Duady, Hubbard and Sullivan. The proof we give here is a modification of the proof in [2].

The theorem follows from

**Proposition.** If $K(a_0) = (e_1, e_2, \cdots, e_n, C, \cdots)$ and $e_i \neq C$ for $1 \leq i \leq n$, then

$$(1) \quad \frac{\partial_a(Q_a^{n+1}(0))|_{a=a_0}}{DQ_a^n(Q_{a_0}(0))} > 0$$

where $DQ_a^n$ denote the derivative of the $n$-times iteration of the map $Q_{a_0}$ and $\partial_a(Q_a^{n+1}(0))$ denote the derivative of $Q_a^{n+1}(0)$ as a function of the parameter $a$.

In fact, suppose that $a_0$ satisfies the assumption of the proposition and that the number of $R$ in $(e_1, e_2, \cdots, e_n)$ is even (resp. odd). Then the denominator in the left hand side of (1) is positive (resp. negative) and, from the proposition, so is the numerator. This implies that $e_{n+1}$ varies as $L \rightarrow C \rightarrow R$ (resp. $R \rightarrow C \rightarrow L$) when
the parameter $a$ pass $a_0$ from the left to the right. Now consider the truncated
kneading sequence $K^{(n)}(a) = (e_1, e_2, \cdots, e_n)$ for each $n$. For each parameter at
which $K^{(n)}(a)$ changes, we can find the situation in the proposition. Thus the above
observation shows that $K^{(n)}(a)$ depends on $a$ monotonously. Letting $n \to \infty$, we
get the theorem.

Let us denote $w_i = Q^i_a(0)$ for $i = 1, 2, \cdots, n$ and put $\omega = (w_1, w_2, \cdots, w_n) \in \mathbb{R}^n$.
Consider the so-called Thurston map:
\[
T(z_1, z_2, \cdots, z_n) = (\sigma_1 \sqrt{z_1 - z_2}, \sigma_2 \sqrt{z_2 - z_3}, \cdots, \sigma_{n-1} \sqrt{z_{n-1} - z_n}, \sigma_n \sqrt{z_1})
\]
where $\sigma_i$ is the sign of $w_i$. Then $T(\omega) = \omega$ and $T$ is defined on a neighborhood of
$\omega$. By easy calculations, we obtain

Lemma 1. \[
\frac{\partial_a Q^n_a(1)}{DQ^n_a(1)} \Bigl|_{a=a_0} = \det(I_n - D\omega T) \text{ where } I_n \text{ denotes the } n \times n \text{ unit }
\text{matrix and } D\omega T \text{ the derivative of } T \text{ at } \omega.
\]

So we reduce the proposition to

Lemma 2. No eigenvalue of $D\omega T$ is contained in $[1, \infty)$.

Let $X = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n \mid 0 < |z_i| < 3, \text{ and } z_i \neq z_j \text{ if } i \neq j \}$ and
\[
Y_\epsilon = \{(z_1, z_2, \cdots, z_n) \in X \mid |z_i| > 10^i \epsilon \text{ and } |z_i - z_j| > 10^{\min(i,j)} \epsilon \text{ if } i \neq j \}
\]
for $\epsilon > 0$. Then the (multi-valued) complex extension of $T$,
\[
T_C(z_1, z_2, \cdots, z_n) = (\sqrt{z_1 - z_2}, \sqrt{z_2 - z_3}, \cdots, \sqrt{z_1}) : \mathbb{C}^n \to \mathbb{C}^n,
\]
maps $X$ into itself in the sense that, for every $z \in X$ and every branch of $T_C$, the
image belongs to $X$. Moreover, if $\epsilon$ is sufficiently small, $T_C$ maps $Y_\epsilon$ into a compact
subset of $Y_\epsilon$ in this sense. Take $\epsilon$ so small that $\omega \in Y_\epsilon$. Let $M_\mu : \mathbb{C} \to \mathbb{C}$
be a map defined by
\[
M_\mu(z_1, \cdots, z_n) = (w_1 + \mu(z_1 - w_1), \cdots, w_n + \mu(z_n - w_n)) : \mathbb{C}^n \to \mathbb{C}^n.
\]

We choose $\mu > 1$ so close to 1 that the composition $S := M_\mu \circ T_C$ maps $Y_\epsilon$ into
itself. Let $\pi : \tilde{Y}_\epsilon \to Y_\epsilon$ be the universal covering and let $\tilde{\omega} \in \tilde{Y}_\epsilon$ be a point such that
$\pi(\tilde{\omega}) = \omega$. Then there is a (single valued) lift $\tilde{S} : \tilde{Y}_\epsilon \to \tilde{Y}_\epsilon$ of $S$ such that
$\tilde{S}(\tilde{\omega}) = \tilde{\omega}$.

Now consider the Kobayashi metric $| \cdot |_K$ on $\tilde{Y}_\epsilon$, which is defined as
\[
|v|_K = \sup\{r \geq 0 \mid \text{ there is a holomorphic map } \phi : D_r \to \tilde{Y}_\epsilon \text{ s.t. } d\phi(e) = v, \}^{-1}
\]
for any tangent vector $v$ where $D_r = \{z \in \mathbb{C} \mid |z| < r\}$ and $e$ is the unit vector at
$0 \in D_r$. (See [1] for generalities.) Then it is easy to check that $| \cdot |_K$ is equivalent to
the Euclidean metric at each point. From the definition, we have $|d\Phi(v)|_K \leq |v|_K$
for any holomorphic map $\Phi : \tilde{Y}_\epsilon \to \tilde{Y}_\epsilon$ and any tangent vector $v$. So it follows that
the spectral radius of $D\Phi(\tilde{S})$ is not bigger than 1. Since $D\Phi(\tilde{S}) = \mu \cdot D\omega T$ and $\mu > 1$,
the spectral radius of $D\Phi(\tilde{S})$ is smaller than 1. We have proved lemma 2 and so the
main theorem.

REFERENCES