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Information Structures and Perfect Information in Simple Exchange Games

名古屋商科大　坂口 実 (Minoru SAKAGUCHI)

Abstract This paper is concerned with the evaluation in two and three person simple exchange games (SEG), in which hands are delivered to the players face-down and/or face-up. It discloses how changes in information about the hands delivered to either players affects the value of the simple exchange game. By comparing solutions to the games under various information structures, it is shown that some relations between different information structures (IS) are intuitively plausible in three-person SEGs and some others are not so in two-person SEGs. It is surprising that some three-person SEGs under very simple ISs have very much complicated solutions. Finally we discuss the expected value of perfect information (EVPI) in SEG under any IS in terms of bargaining cost transferred between the two or three players and an arbitrator.

1. Introduction and Summary.

The two-person simple exchange game (SEG) is described as follows: Each of two players (I and II) draws a number (x and y, respectively) according to a uniform distribution on $[0, 1]$. After observing his number each player can then choose either to keep (K) his number or to exchange (E) it for the other player's number. If only one player chooses E, an exchange of the numbers occurs with probability p. A player's payoff is the number he holds after the players have made their choices and a possible exchange has occured. The outcome of a "trade" (i.e., choice-pair) is therefore

$$\begin{array}{ccc} & K & E \\ \begin{array}{c} K \\ E \end{array} & \left[\begin{array}{c} (x, \ y) \\ (py+\bar{p}x, \ px+\bar{p}y) \end{array} \right. & \left. \begin{array}{c} (py+\bar{p}x, \ px+\bar{p}y) \\ (y, \ x) \end{array} \right] \end{array}$$

where $\bar{p}=1-p$. If $p=0$ [1] then the game is an AND [OR] game in which both players [at least one player] must choose E in order for an exchange to occur.

We comsider the zero-sum game version with the payoff matrix:

$$\operatorname{sgn}(x-y) \quad \text{times} \quad \begin{array}{c|cc} & K & E \\ \hline K & \multicolumn{2}{c}{\begin{bmatrix} 1 & 1-2p \\ 1-2p & -1 \end{bmatrix}} \\ E & & \end{array}$$

where sgn $t=1, 0, -1$, if $t>0, =0, <0$ (resp.).

We shall denote by $\alpha(x)$ [$\beta(y)$] the probability that player I [II] chooses E if his hand is x [y]. Under this strategy-pair $\alpha(\cdot)-\beta(\cdot)$, the expected payoff to I is

$$\text{(1. 1)} \qquad M(\alpha, \beta) \equiv \iint (\bar{\alpha}(x), \alpha(x)) \begin{bmatrix} 1 & 1-2p \\ 1-2p & -1 \end{bmatrix} \begin{bmatrix} \bar{\beta}(y) \\ \beta(y) \end{bmatrix} \operatorname{sgn}(x-y)\, dx\, dy$$

(Hereafter in this paper, if integration is taken over [0, 1], the limits will be omitted).

In the present paper we shall investigate SEGs in which one or both of players have various pattern of information-private or public-about his own and opponent's hands. We consider the information struture of our model which is described by a statement as to "who knows what".

Let $I^{ij:kl}$ be the information structure (IS) such that

$$\begin{Bmatrix} i \\ l \end{Bmatrix} = 1\,(0), \text{ if player } \begin{Bmatrix} \mathrm{I} \\ \mathrm{II} \end{Bmatrix} \text{ does (doesn't) know his hand } \begin{Bmatrix} x \\ y \end{Bmatrix};$$

$$\begin{Bmatrix} j \\ k \end{Bmatrix} = 1\,(0), \text{ if plater } \begin{Bmatrix} \mathrm{I} \\ \mathrm{II} \end{Bmatrix} \text{ does (doesn't) know his opponent's hand } \begin{Bmatrix} y \\ x \end{Bmatrix}.$$

The SEG with payoff function (1. 1) has the information structure $I^{10:01}$. There are $2^4=16$ ISs and these may be classified into the following six types: (1°) $I^{00:00}$:No player knows either hand. That is, hand to each player is dealt face-down and he is not allowed to open.

$I^{11:11}$:Both players can know hands. That is, hands are dealt face-up.

(2°) $\begin{Bmatrix} I^{10:01} \\ I^{01:10} \end{Bmatrix}$:Each player knows his $\begin{Bmatrix} \text{own} \\ \text{opponent's} \end{Bmatrix}$ hand, but not $\begin{Bmatrix} \text{the opponent's} \\ \text{his own} \end{Bmatrix}$.

In the case of poker the upper applies in High-Hand-Wins and La Relance poker. The lower applies in the so-called "Indian poker".

(3°) $I^{10:00}$ or $I^{00:01}$:One player can know his own hand, but not the opponent's, with the other player kept unaware of the hands.

$I^{01:00}$ or $I^{00:01}$:One player can know his opponent's hand but not his own, with the other player being kept unaware of the hands.

(4°) $I^{11:00}$ or $I^{00:11}$:One player knows both players' hands, whereas the other doesn't know either hand.

(5°) $\begin{Bmatrix} I^{11:01} \text{ or } I^{10:11} \\ I^{11:10} \text{ or } I^{01:11} \end{Bmatrix}$:One player knows both players' hands, whereas the other can know his $\begin{Bmatrix} \text{own} \\ \text{opponent's} \end{Bmatrix}$ hand only.

(6°) $I^{10:10}$ or $I^{01:01}$:Either one of the hands is commonly known to both players.

Types of (1°) and (2°) can be thought of as cases of symmetric information, and types of (3°)〜(6°) as those of asymmetric information. Type (6°) is regarded as the case of shared information, in the sense that information gathered by one player is also shared by the other.

We shall also discussw three-person SEG, in Section 3 of this paper, as a natural extension of the two-person game model. At the beginning of a play three players I, II and III receive hands x, y, z, respectively, chosen at random and independently from a unit distribution in [0, 1]. Then each player, looking at his hand privately, choose either one of K and E. Choices by the players should be made simultaneously and independently.

If only one player chooses E, an exchange of his hand with each one of the players who chose K occur with probability p/2, and no exchange among the three players occur with probability $\bar{p}$. If two players choose E and the other chooses K, an exchange hands within the players who chose E occurs. If all players choose E then the two possible exchange among the all three players occur with probability 1/2 each. After choices are made and the correaponding "trade" occurs, the hands, they get finally, are compared and a player with the highest hand wins. The winner gets the reward of two units and each of the two loses one unit. More precisely, payoffs are given by:

$$\begin{array}{cc} & \overbrace{\qquad\qquad\text{K}\qquad\qquad\qquad\qquad\qquad\qquad\text{E}\qquad\qquad}^{(\text{III})} \\ (\text{II})\begin{cases}\text{K}\\ \text{E}\end{cases} & \begin{bmatrix} (v(x),\ v(y),\ v(z)) & \left(\left(1-\frac{p}{2}\right)v(x)+\frac{p}{2}v(z),\ \left(1-\frac{p}{2}\right)v(y)+\frac{p}{2}v(z),\ \left(1-\frac{3p}{2}\right)v(z)\right) \\ \left(\left(1-\frac{p}{2}\right)v(x)+\frac{p}{2}v(y),\ \left(1-\frac{3p}{2}\right)v(y),\ \left(1-\frac{p}{2}\right)v(z)+\frac{p}{2}v(y)\right) & (v(x),\ v(z),\ v(y)) \end{bmatrix} \end{array}$$

if I chooses K; and

$$\begin{bmatrix} \left(\left(1-\frac{3p}{2}\right)v(x),\ \left(1-\frac{p}{2}\right)v(y)+\frac{p}{2}v(x),\ \left(1-\frac{p}{2}\right)v(z)+\frac{p}{2}v(x)\right) & (v(z),\ v(y),\ v(x)) \\ (v(y),\ v(x),\ v(z)) & \left(-\frac{1}{2}v(x),\ -\frac{1}{2}v(y),\ -\frac{1}{2}v(z)\right) \end{bmatrix}$$

if I chooses E. Here we have set $v(x)=u(x-y\vee z)$, $v(y)=u(y-x\vee z)$, etc, and $u(t)=2, 0, -1$; if $t>0, =0, <0$, respectively, and $y\vee z=\max(y, z)$, etc.

Note that $v(\cdot)$ is a function of x, y and z, although only one variable is mentioned explicitly, and $v(x)+v(y)+v(z)=0$ for all (x, y, z), without ties. Also note that the case where p=0 [1] is an AND [OR] game in which majority [at least one] of the players must choose E in order for an exchange to occur.

Let $\alpha(x)$, $\beta(y)$ and $\gamma(z)$ be the probability that players I, II and III choose E if their hands are x, y and z, respectively. Under this strategy-triple $\alpha(\cdot)-\beta(\cdot)-\gamma(\cdot)$ the expected payoffs to the three players are

$$(1.2)\quad M_1(\alpha,\beta,\gamma)=\iiint(\bar{\beta}(y),\beta(y))\left\{\frac{p}{2}\bar{\alpha}(x)\begin{bmatrix}0 & v(z)-v(x)\\ v(y)-v(x) & 0\end{bmatrix}\right.$$
$$\left.+\alpha(x)\begin{bmatrix}-(3p/2)v(x) & v(z)-v(x)\\ v(y)-v(x) & -(3/2)v(x)\end{bmatrix}\right\}\begin{bmatrix}\bar{\gamma}(z)\\ \gamma(z)\end{bmatrix}dx\,dy\,dz,$$

$$(1.3)\quad M_2(\alpha,\beta,\gamma)=\iiint(\bar{\beta}(y),\beta(y))\left\{\bar{\alpha}(x)\begin{bmatrix}0 & (1/2)(v(z)-v(y))\\ -(3p/2)v(y) & v(z)-v(y)\end{bmatrix}\right.$$
$$\left.+\alpha(x)\begin{bmatrix}(p/2)(v(x)-v(y)) & 0\\ v(x)-v(y) & -(3/2)v(y)\end{bmatrix}\right\}\begin{bmatrix}\bar{\gamma}(z)\\ \gamma(z)\end{bmatrix}dx\,dy\,dz$$

and $M_3(\alpha,\beta,\gamma)=-M_1(\alpha,\beta,\gamma)-M_2(\alpha,\beta,\gamma)$.

The notion of the IS in two-person games is easily extended to the one in three-person games. There are $2^9=512$ ISs. The SEG with payoff functions (1.2)-(1.3) has the information structure $I^{100:010:001}$.

In Section 2 (3) of the present paper, solutions to two-person (three-person) SEGs under various ISs are derived. By comparing the solutions under these different ISs it is shown that some relations between different ISs are intuitively plausible in three-person SEGs and some others are not so in two-person SEGs. It turns out that some three-person SEGs under very simple ISs have complicated solutions. Finally in Section 4 we discuss EVPI in two and three person SEGs in terms of bargaining cost transfered between the players and an arbitrator.

EGs were first discussed by Brams, Kilgour and Davis [1] and then by Garnaev [2] and Sakaguchi [8]. By its structure SEG is similar to poker games (see Karlin [3], Sakaguchi [4, 5, 6] and Sakaguchi and Sakai [7] and the method of deriving the solution in SEG and poker games is the same.

2. Solution of Simple Exchange Game Under Various Information Structures.

3. Solution to Three-Person Simple Exchange Games under Various Information Structures.

4. Expected Values of Perfect Information in Simple Exchange Games.

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