数理解析研究所講究録
第 864 巻 1994 年 154-163

A GENERALIZED SECRETARY PROBLEM WITH UNCERTAIN EMPLOYMENT

愛知大学　玉置光司(Mitsushi Tamaki)

1. Introduction

We shall consider and solve a modification of the classical secretary problem with uncertain employment. The problem is described as follows : n applicants appear one by one in random order with all n! permutations equally likely. We are able, at any time, to rank the applicants that have so far appeared. As each applicant appears we must decide whether or not to make an offer to that applicant with the objective of maximizing the probability of selecting the best(most preferred)applicant. It is assumed that each applicant only accepts an offer of employment with constant probability p and that an applicant to whom an offer is not made cannot be recalled later (see Smith[1] and Tamaki[2]).

The problem we consider here allows the applicants to refuse an offer depending on the rank of the applicant. Let $p_i(q_i = 1-p_i)$ be the acceptance probability (rejection probability) of the i-th ranked applicant $1 \le i \le n$. We treat the best choice problem in Section 2 and the Gusein-Zade problem in Section 3.

2. Best Choice Problem

Our objective is to maximize the probability of choosing the best applicant. Imagine a situation where r-1 applicants have so far appeared, and offer(s) were made to k of them but rejected $0 \le k \le r-1 \le n$. Then, if $k \ge 1$, this state is described as $(r-1\,;\,i_1,i_2,\dots,i_k)$. where the information pattern $(i_1,i_2,\dots,i_k)$ represents the relative ranks , among the first r-1 applicants, of those who have rejected offers arranged in ascending order, i.e., $1 \le i_1 < i_2 < \dots < i_k \le r-1$. Furthermore denote by $(r\,;\,i \mid i_1,i_2,\dots,i_k)$, $1 \le i \le r$ the state where, after leaving state $(r-1\,;\,i_1,i_2,\dots,i_k)$, we have just observed the r-th applicant to be relatively i-th best. When k=0,i.e., no offer has been made so far, the information pattern is denoted by ϕ and the corresponding states will be denoted by $(r-1\,;\,\phi)$ and $(r\,;\,i \mid \phi)$. In this section, our trial is said to be a success if the chosen applicant is the overall best.

Let $v_{r-1}(i_1,i_2,\dots,i_k)$ be the probability of success assuming that we proceed optimally after leaving state $(r-1\,;\,i_1,i_2,\dots,i_k)$. Also let $s_r(i \mid i_1,i_2,\dots,i_k)((c_r(i \mid i_1,i_2,\dots,i_k))$ be the corresponding probability when we make(when we do not make) an offer to the r-th applicant in state $(r\,;\,i \mid i_1,i_2,\dots,i_k)$ and proceed optimally thereafter. Corresponding to states $(r-1\,;\,\phi)$ and $(r\,;\,i \mid \phi)$, $v_{r-1}(\phi)$ and $s_r(i \mid \phi)$, $c_r(i \mid \phi)$ can be respectively defined in a similar way. Let $p_r(i \mid i_1,i_2,\dots,i_k)$ be the transition probability from state $(r-1\,;\,i_1,i_2,\dots,i_k)$ into state $(r\,;\,i \mid i_1,i_2,\dots,i_k)$. Also let $p_r(i \mid \phi)$ be the transition probability from state $(r-1\,;\,\phi)$ into state $(r\,;\,i \mid \phi)$. Then we have from the principle of optimality,

$$v_{r-1}(i_1,\dots,i_k) = \sum_{i=1}^{r} p_r(i \mid i_1,\dots,i_k).\max\{s_r(i \mid i_1,\dots,i_k),\ c_r(i \mid i_1,\dots,i_k)\} \quad (1 \le k \le r-1<n) \quad (2.1)$$

$$v_{r-1}(\phi) = \sum_{i=1}^{r} p_r(i \mid \phi).\max\{s_r(i \mid \phi),\ c_r(i \mid \phi)\} \quad (0 \le r-1<n) \quad (2.2)$$

with the boundary conditions $v_n(i_1,\dots,i_k)\equiv 0$ and $v_n(\phi)=0$.
Obviously optimal success probability will be calculated as $v_0(\phi)$. It is easy to see

$$p_r(i\mid\phi) = 1/r, \qquad 1\le i\le r\ . \tag{2.3}$$

from the assumption that the arrival orders of the applicants are equally likely . However, $p_r(i\mid i_1,i_2,\dots,i_k)$ is not equal to 1/r in general, because the information pattern $(i_1,i_2,\dots,i_k)$ observed so far has influence on estimating the future arrival of the remaining applicants.

To derive $p_r(i\mid i_1,i_2,\dots,i_k)$, some notations must be introduced. For convenience of exposition, we denote by C_i^r, $1\le i\le r$, $1\le r\le n$ the i-th best among the first r applicants(in particular, C_i^n represents the applicant of absolute rank i).

Let A(r,i;n) be a random variable representing the absolute rank of the applicant C_i^r, i.e., A(r,i;n)=j if C_i^r is C_j^n. Then it is easy to see that the joint probability $P(A(r,i_1:n)=j_1, A(r,i_2;n)=j_2,\dots,A(r,i_k;n)=j_k)$, which is denoted simply by $p(j_1,j_2,\dots j_k\,;n\mid i_1,i_2,\dots,i_k\,;r)$, is given by

$$p(j_1,j_2,\dots j_k\,;n\mid i_1,i_2,\dots,i_k\,;r) =$$

$$\frac{\binom{j_1-1}{i_1-1}\binom{j_2-j_1-1}{i_2-i_1-1}\cdots\binom{j_k-j_{k-1}-1}{i_k-i_{k-1}-1}\binom{n-j_k}{r-i_k}}{\binom{n}{r}} \tag{2.4}$$

for $(j_1,j_2,\dots j_k)\in W_r(i_1,i_2,\dots,i_k)$,

where $W_r(i_1,i_2,\dots,i_k)$ stands for the set of possible values $(j_1,j_2,\dots j_k)$ for given values $(i_1,i_2,\dots,i_k)$, i.e., $W_r(i_1,i_2,\dots,i_k) = \{(j_1,j_2,\dots j_k) : j_1<j_2<\dots<j_k,\ i_s\le j_s\le n-r+i_s,\ 1\le s\le k\}$.

Some properties of $p(j_1,j_2,\dots j_k\,;n\mid i_1,i_2,\dots,i_k\,;r)$ are listed in the following lemma.

LEMMA 2.1

(i) $p(j_1,j_2,\dots j_k\,;n\mid i_1+1,i_2+1,\dots,i_k+1\,;r)$

$$= \left(\frac{r}{n-r+1}\right)\left(\frac{j_1-i_1}{i_1}\right) p(j_1,j_2,\dots j_k\,;n\mid i_1,i_2,\dots,i_k\,;r-1)$$

$p(j_1,\dots j_k\,;n\mid i_1,\dots,i_{s-1},i_s+1,\dots,i_k+1\,;r)$

$$= \left(\frac{r}{n-r+1}\right)\left(\frac{(j_s-i_s)-(j_{s-1}-i_{s-1})}{i_s-i_{s-1}}\right) p(j_1,j_2,\dots j_k\,;n\mid i_1,i_2,\dots,i_k\,;r-1)$$

(2≤s≤k)

$p(j_1,j_2,\dots j_k\,;n\mid i_1,i_2,\dots,i_k\,;r)$

$$= \left(\frac{r}{n-r+1}\right)\left(\frac{(n-r+1)-(j_k-i_k)}{r-i_k}\right) p(j_1,j_2,\dots j_k\,;n\mid i_1,i_2,\dots,i_k\,;r-1)$$

(ii) $p(j_1,j_2,\dots j_k\,;n\mid i_1,i_2,\dots,i_k\,;r)$

$$= p(j_1,j_2,\dots j_{s-1}\,;j_s-1\mid i_1,i_2,\dots,i_{s-1}\,;i_s-1)\; p(j_s,j_{s+1},\dots j_k\,;n\mid i_s,i_{s+1},\dots,i_k\,;r)$$

(2≤s≤k)

(iii) $p(j_1,j_2,\dots j_k\,;n\mid i_1+1,i_2+1,\dots,i_k+1\,;r)$

$$= \left(\frac{r}{n}\right)\left(\frac{j_1-1}{i_1}\right)p(j_1-1,j_2-1,\dots,j_k-1;n-1 \mid i_1,i_2,\dots,i_k;r-1)$$

(iv) $p(j_1,\dots,j_k;n \mid i_1,\dots,i_k;r-1)$

$$= \left(\frac{i_1}{r}\right) p(j_1,\dots,j_k;n \mid i_1+1,\dots,i_k+1;r)$$
$$+\sum_{s=2}^{k} \left(\frac{i_s-i_{s-1}}{r}\right) p(j_1,\dots,j_k;n \mid i_1,\dots,i_{s-1},i_s+1,\dots,i_k+1;r)$$
$$+\left(\frac{r-i_k}{r}\right)p(j_1,\dots,j_k;n \mid i_1,\dots,i_k;r)$$

PROOF. Straightforward from (2.4).

Define, for $1\le i_1<i_2<\dots<i_k\le r$, $1\le r\le n$

$$a_r(i_1,i_2,\dots,i_k) = E\left[\prod_{t=1}^{k} q_{A(r,i_t;n)}\right], \tag{2.5}$$

where E denotes an operator of taking expectation. $a_r(i_1,i_2,\dots,i_k)$ implies the probability that all k offers will be rejected , provided that these offers are given to $C^r_{i_1},C^r_{i_2},\dots,C^r_{i_k}$(we write $a_r(i_1,i_2,\dots,i_k;n)$ to make explicit the dependence on n, total number of applicants that appears, if necessary). We can write (2.5) as

$$a_r(i_1,i_2,\dots,i_k) = \Sigma\Sigma\dots\Sigma\left(\prod_{t=1}^{k} q_{j_t}\right)p(j_1,j_2,\dots,j_k;n \mid i_1,i_2,\dots,i_k;r)$$

where summations with respect to $(j_1,j_2,\dots,j_k)$ are taken over $W_r(i_1,i_2,\dots,i_k)$.

We now have the following lemma.

LEMMA 2.2

$$p_r(i \mid i_1,i_2,\dots,i_k) = \begin{cases} \left(\frac{1}{r}\right)\left[\frac{a_r(i_1+1,i_2+1,\dots,i_k+1)}{a_{r-1}(i_1,i_2,\dots,i_k)}\right], & 1\le i\le i_1 \\ \left(\frac{1}{r}\right)\left[\frac{a_r(i_1,\dots,i_{s-1},i_s+1,\dots,i_k+1)}{a_{r-1}(i_1,i_2,\dots,i_k)}\right], & i_{s-1}<i\le i_s(2\le s\le k) \\ \left(\frac{1}{r}\right)\left[\frac{a_r(i_1,i_2,\dots,i_k)}{a_{r-1}(i_1,i_2,\dots,i_k)}\right], & i_k<i\le r \end{cases}$$

PROOF. Let $\tilde{p}(j_1,j_2,\dots,j_k;n \mid i_1,i_2,\dots,i_k;r-1)$ be the conditional joint probability that $C^{r-1}_{i_t}$ is in effect $C^n_{j_t}$, $1\le t\le k$ provided that offers given to $C^{r-1}_{i_1},\dots,C^{r-1}_{i_k}$ are all rejected. Then by the Bayes formula

$$\tilde{p}(j_1,j_2,\dots,j_k;n \mid i_1,i_2,\dots,i_k;r-1) = \frac{(q_{j_1}q_{j_2}\cdots q_{j_k})p(j_1,j_2,\dots,j_k;n \mid i_1,i_2,\dots,i_k;r-1)}{a_{r-1}(i_1,i_2,\dots,i_k)} \tag{2.6}$$

Let R be the relative rank of the r-th applicant. We easily see that the conditional probability distribution of R, given that $C^{r-1}_{i_t}$ is $C^n_{j_t}$ for $1\le t\le k$, is given by

$$P(R=i \mid j_1,j_2,\dots,j_k) = \begin{cases} \left(\frac{1}{i_1}\right)\left(\frac{j_1-i_1}{n-r+1}\right) & 1\le i\le i_1 \\ \left(\frac{1}{i_s-i_{s-1}}\right)\left(\frac{(j_s-i_s)-(j_{s-1}-i_{s-1})}{n-r+1}\right) & i_{s-1}<i\le i_s\ (2\le s\le k) \\ \left(\frac{1}{r-i_k}\right)\left(\frac{(n-r+1)-(j_k-i_k)}{n-r+1}\right) & i_k<i\le r \end{cases} \tag{2.7}$$

•Thus the result follows from (2.6), (2.7) and Lemma 2.1(i), since $p_r(i \mid i_1,i_2,\dots,i_k)$ is calculated through

$$p_r(i \mid i_1,i_2,\dots,i_k)= \Sigma\Sigma\dots\Sigma P(R=i \mid j_1,j_2,\dots j_k)\cdot\tilde{p}(j_1,j_2,\dots j_k;n \mid i_1,i_2,\dots,i_k;r-1)\,,$$

where summations are taken over $W_{r-1}(i_1,i_2,\dots,i_k)$.

Some properties of $a_r(i_1,i_2,\dots,i_k)$ are listed in the following lemma.

LEMMA 2.3

(i) For $1<r\le n$,

$$\begin{aligned} a_{r-1}(i_1,i_2,\dots,i_k) &= \left(\frac{i_1}{r}\right) a_r(i_1+1,i_2+1,\dots,i_k+1) \\ &+\sum_{s=2}^{k}\left(\frac{i_s-i_{s-1}}{r}\right)a_r(i_1,\dots,i_{s-1},i_s+1,\dots,i_k+1) \\ &+\left(\frac{r-i_k}{r}\right)a_r(i_1,i_2,\dots,i_k) \end{aligned}$$

(ii)

$$\begin{aligned} &a_r(i_1,i_2,\dots,i_k;n) \\ &=\Sigma\Sigma\dots\Sigma a_{i_s-1}(i_1,\dots,i_{s-1};j_s-1)\Big(\prod_{t=s}^{k} q_{j_t}\Big)p(j_s,\dots j_k;n \mid i_s,\dots,i_k;r)\,, \end{aligned}$$

where summations with respect to $(j_s,\dots j_k)$ are taken over $W_r(i_s,\dots,i_k)$, $(2\le s\le k)$.

(iii) Assume that $\{q_j\}$ is non-increasing in j. Then $a_r(i_1,i_2,\dots,i_k)$ is non-decreasing in r and non-increasing in i_s.

PROOF. (i) is immediate from Lemma 2.2, since $\sum_{i=1}^{r} p_r(i \mid i_1,i_2,\dots,i_k)$ must be unity.
(ii) is straightforward from Lemma 2.1(ii).
(iii) can be shown by induction on r.

In our problem, the j-th best applicant is assumed to reject an offer with probability q_j. So we denote this problem by $\begin{bmatrix} 1 & 2 & \dots & n \\ q_1 & q_2 & & q_n \end{bmatrix}$. Consider a modified problem $\begin{bmatrix} 1 & 2 & \dots & n-1 \\ q_2 & q_3 & & q_n \end{bmatrix}$, where total number of applicants is n-1 and the j-th best applicant rejects with probability q_{j+1}, $1\le j\le n-1$ and let $b_r(i_1,i_2,\dots,i_k)$, $1\le r\le n-1$ denote the probability that all k offers will be rejected when these offers are given to $C^r_{i_1},C^r_{i_2},\dots,C^r_{i_k}$. More specifically

$$b_r(i_1,i_2,\dots,i_k) = \Sigma\Sigma\dots\Sigma\Big(\prod_{t=1}^{k} q_{j_t+1}\Big)p(j_1,j_2,\dots j_k;n-1 \mid i_1,i_2,\dots,i_k;r) \tag{2.8}$$

Note that, since $b_r(i_1,i_2,\ldots,i_k)$ corresponds to $a_r(i_1,i_2,\ldots,i_k)$ in the original problem, Lemma 2.3(i) holds with $a_r(i_1,i_2,\ldots,i_k)$ replaced by $b_r(i_1,i_2,\ldots,i_k)$ for $2\le r\le n-1$ and $b_r(i_1,i_2,\ldots,i_k)$ can be solved recursively starting with the boundary condition $b_{n-1}(i_1,i_2,\ldots,i_k) = \prod_{t=1}^{k} q_{i_t+1}$

We can now express $s_r(i\mid i_1,\ldots,i_k)$ and $c_r(i\mid i_1,\ldots,i_k)$ in terms of $v_r(i_1,\ldots,i_k)$, $a_r(i_1,i_2,\ldots,i_k)$ and $b_r(i_1,i_2,\ldots,i_k)$.

LEMMA 2.4

(i) For $1<r\le n$

$$s_r(i\mid i_1,\ldots,i_k) = \begin{cases} p_1(\frac{r}{n})\left[\dfrac{b_{r-1}(i_1,i_2,\ldots,i_k)}{a_r(i_1+1,i_2+1,\ldots,i_k+1)}\right] + v_r(1,i_1+1,\ldots,i_k+1)\left[\dfrac{a_r(1,i_1+1,\ldots,i_k+1)}{a_r(i_1+1,i_2+1,\ldots,i_k+1)}\right], & i=1 \\ v_r(i,i_1+1,\ldots,i_k+1)\left[\dfrac{a_r(i,i_1+1,\ldots,i_k+1)}{a_r(i_1+1,i_2+1,\ldots,i_k+1)}\right], & 1<i\le i_1 \\ v_r(i_1,\ldots,i_{s-1},i,i_s+1,\ldots,i_k+1)\left[\dfrac{a_r(i_1,\ldots,i_{s-1},i,i_s+1,\ldots,i_k+1)}{a_r(i_1,\ldots,i_{s-1},i_s+1,\ldots,i_k+1)}\right], & i_{s-1}<i\le i_s \\ & (2\le s\le k) \\ v_r(i_1,\ldots,i_k,i)\left[\dfrac{a_r(i_1,\ldots,i_k,i)}{a_r(i_1,\ldots,i_k)}\right], & i_k<i\le r \end{cases}$$

$$c_r(i\mid i_1,\ldots,i_k) = \begin{cases} v_r(i_1+1,\ldots,i_k+1), & 1\le i\le i_1 \\ v_r(i_1,\ldots,i_{s-1},i_s+1,\ldots,i_k+1), & i_{s-1}<i\le i_s \\ & (2\le s\le k) \\ v_r(i_1,\ldots,i_k), & i_k<i\le r \end{cases}$$

(ii) For $1\le r\le n$,

$$s_r(i\mid \phi) = \begin{cases} p_1(\frac{r}{n}) + a_r(1)v_r(1), & i=1 \\ a_r(i)v_r(i), & 1<i\le r \end{cases}$$

$c_r(i \mid \phi) = v_r(\phi)$, $1 \le i \le r$

PROOF. We'll only derive $s_r(1 \mid i_1, \ldots, i_k)$, since others can be obtained in a similar way. Suppose that we are in state $(r;1 \mid i_1, \ldots, i_k)$, the forcasting probability that $C^r_{i_t+1}$ is $C^n_{j_t}$ for $1 \le t \le k$ is given by $\tilde{p}(j_1, j_2, \ldots, j_k; n \mid i_1+1, i_2+1, \ldots, i_k+1; r)$ defined in (2.6). On the other hand, given that $C^r_{i_t+1}$ is $C^n_{j_t}$, making an offer to C^r_1 leads to a success with probability

$$p_1 p(1;j_1-1 \mid 1;i_1) + a_{i_1}(1;j_1-1) v_r(1, i_1+1, \ldots, i_k+1) .$$

The first term corresponds to acceptance of the offer and the second term corresponds to rejection and subsequent continuation in an optimal manner. Thus we have

$$s_r(1 \mid i_1, \ldots, i_k) = \Sigma\Sigma \ldots \Sigma[p_1 p(1;j_1-1 \mid 1;i_1) + a_{i_1}(1;j_1-1) v_r(1, i_1+1, \ldots, i_k+1)] \times \tilde{p}(j_1, j_2, \ldots, j_k; n \mid i_1+1, i_2+1, \ldots, i_k+1; r) , \qquad (2.9)$$

where summations with respect to $(j_1, j_2, \ldots, j_k)$ are taken over $W_r(i_1+1, \ldots, i_k+1)$. From Lemma 2.1(iii), the first term in the RHS of (2.9) can be reduced to

$$\frac{p_1}{a_r(i_1+1, \ldots, i_k+1)} \Sigma\Sigma \ldots \Sigma\left(\prod_{t=1}^{k} q_{j_t}\right)\left(\frac{i_1}{j_1-1}\right) \cdot p(j_1, \ldots, j_k; n \mid i_1+1, \ldots, i_k+1; r)$$

$$= p_1\left(\frac{r}{n}\right) \cdot \frac{1}{a_r(i_1+1, \ldots, i_k+1)} \Sigma\Sigma \ldots \Sigma\left(\prod_{t=1}^{k} q_{j_t}\right) \cdot p(j_1-1, \ldots, j_k-1; n-1 \mid i_1, \ldots, i_k; r-1)$$

$$= p_1\left(\frac{r}{n}\right) \cdot \frac{b_{r-1}(i_1, \ldots, i_k)}{a_r(i_1+1, \ldots, i_k+1)} . \qquad (2.10)$$

The second term can be written as, from Lemma 2.3(ii),

$$\frac{v_r(1, i_1+1, \ldots, i_k+1)}{a_r(i_1+1, \ldots, i_k+1)} \Sigma\Sigma \ldots \Sigma a_{i_1}(1;j_1-1)\left(\prod_{t=1}^{k} q_{j_t}\right) \cdot p(j_1, \ldots, j_k; n \mid i_1+1, \ldots, i_k+1; r)$$

$$= v_r(1, i_1+1, \ldots, i_k+1) \frac{a_r(1, i_1+1, \ldots, i_k+1)}{a_r(i_1+1, \ldots, i_k+1)} . \qquad (2.11)$$

Substituting (2.10) and (2.11) into (2.9) yields the desired result.
$c_r(i \mid i_1, i_2, \ldots, i_k)$, $i_{s-1} < i \le i_s$, is immediate since , if we do not make an offer, the information pattern is changed by incrementing i_t by one for $t \ge s$ (when $i \le i_1$, each component of the information pattern increases by one, and when $i > i_k$, no change occurs). (ii) is easy to see and hence omitted.

Define

$$V_r(i_1, \ldots, i_k) = a_r(i_1, \ldots, i_k) v_r(i_1, \ldots, i_k) , \quad 1 \le r \le n$$

$$V_r(\phi) = v_r(\phi) , \quad 0 \le r \le n$$

and apply Lemmas 2.2 and 2.4 to (2.1) and (2.2). Then we have the following lemma.

LEMMA 2.5
Given additional information, $V_r(i_1,\dots,i_k)$ does not increase. To be more precise, for information pattern $(i_1,\dots,i_k)$ with $i_s-i_{s-1}>1$ for some $s(2\le s\le k)$,

$$V_r(i_1,\dots,i_{s-1},i_s,\dots,i_k) \ge V_r(i_1,\dots,i_{s-1},i,i_s,\dots,i_k)\} \qquad (2.12)$$
$$,i_{s-1}<i<i_s.$$

When $i<i_1$ or $i>i_k$, $V_r(i_1,\dots,i_k) \ge V_r(i,i_1,\dots,i_k)$ or $V_r(i_1,\dots,i_k) \ge V_r(i_1,\dots,i_k,i)$ holds respectively. Moreover $V_r(\phi) \ge V_r(i)$ for $1\le i\le r$.
Thus

$$\begin{aligned}
&V_{r-1}(i_1,\dots,i_k)\\
&\quad = \frac{1}{r}\max\{p_1(\tfrac{r}{n})b_{r-1}(i_1,\dots,i_k) + V_r(1,i_1+1,\dots,i_k+1),\ V_r(i_1+1,\dots,i_k+1)\}\\
&\quad + (\frac{i_1-1}{r})V_r(i_1+1,\dots,i_k+1)\\
&\quad + \sum_{t=2}^{k}(\frac{i_t-i_{t-1}}{r})V_r(i_1,\dots,i_{t-1},i_t+1,\dots,i_k+1)\\
&\quad + (\frac{r-i_k}{r})V_r(i_1,\dots,i_k)
\end{aligned} \qquad (2.13)$$
$$(1<r\le n\ ;\ V_n(i_1,\dots,i_k)\equiv 0)$$

$$V_{r-1}(\phi) = \frac{1}{r}\max\{p_1(\frac{r}{n}) + V_r(1),\ V_r(\phi)\} + (\frac{r-1}{r})V_r(\phi) \qquad (2.14)$$
$$(1\le r\le n\ ;\ V_n(\phi)=0).$$

PROOF. We show (2.12) by induction on r. For r=n-1, (2.12) is evident since

$$V_{n-1}(i_1,\dots,i_k) = (\frac{p_1}{n})b_{n-1}(i_1,\dots,i_k) = (\frac{p_1}{n})\prod_{t=1}^{k} q_{i_t+1} \qquad (2.15)$$

Assume that (2.12) holds. Then (2.13) holds and yields immediately

$$V_{r-1}(i_1,\dots,i_{s-1},i_s,\dots,i_k) - V_{r-1}(i_1,\dots,i_{s-1},i,i_s,\dots,i_k) \ge 0$$

from the induction hypothesis and the fact that $b_{r-1}(i_1,\dots,i_k)$, by definition, does not increase with additional information.

The following lemma is concerned with the asymptotic result.

LEMMA 2.6
Let n and r tend to infinity with r/n=x, then $V_r(i_1,\dots,i_k)$ approaches $V(x\mid i_1,\dots,i_k)$, which satisfies the following differential equations:

$$-\frac{d}{dx}V(x \mid i_1,\dots,i_k)$$
$$= \frac{1}{x}\max\{p_1 x b(x \mid i_1,\dots,i_k) + V(x \mid 1, i_1+1,\dots,i_k+1),\ V(x \mid i_1+1,\dots,i_k+1)\}$$
$$+ \left(\frac{i_1-1}{x}\right)V(x \mid i_1+1,\dots,i_k+1)$$
$$+ \sum_{t=2}^{k} \left(\frac{i_t-i_{t-1}}{x}\right)V(x \mid i_1,\dots,i_{t-1}, i_t+1,\dots,i_k+1)$$
$$- \left(\frac{i_k}{x}\right)V(x \mid i_1,\dots,i_k),$$

$$-\frac{d}{dx}V(x|\phi) = \frac{1}{x}\max\{p_1x+V(x|1),\ V(x|\phi)\} - \frac{1}{x}V(x|\phi),$$

where

$$b(x \mid i_1,\dots,i_k) = \sum_{j_1=i_1}^{\infty} \sum_{j_2=j_1+i_2-i_1}^{\infty} \cdots \sum_{j_k=j_{k-1}+i_k-i_{k-1}}^{\infty} \left(\prod_{s=1}^{k} q_{j_s+1}\right) \binom{j_1-1}{i_1-1} \binom{j_2-j_1-1}{i_2-i_1-1} \cdots \binom{j_k-j_{k-1}-1}{i_k-i_{k-1}-1} x^{i_k}(1-x)^{j_k-i_k}$$

PROOF. immediate

EXAMPLE 2.1

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \dots \\ q_1 & q_2 & q_3 & 0 & 0 & \dots \end{bmatrix}$$

An optimal policy is threshold type with critical number α i.e., pass over the first αn applicants and then give an offer successively to a candidate that appears until an offer is accepted or deadend comes.

α is the unique root x of the equation

$$1+2[q_2+q_3(1+q_2)](1-x)-\frac{3}{4}q_3(1+q_2)(1-x^2)= -(1+q_2)(1+\frac{1}{2}q_3)\log x$$

Moreover the optimal success probability is

$$P(S)= p_1\alpha[(1+q_2)(1+\frac{1}{2}q_3)-(q_2+q_3+q_2q_3)\alpha+\frac{1}{2}q_3(1+q_2)\alpha^2]$$

EXAMPLE 2.2

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \dots \\ q_1 & q_2 & q & q & \dots \end{bmatrix}$$

This problem is simplified by noting that

$$V_r(i_1,\dots,i_k)=\begin{cases} q^{k-1}V_r(2), & \text{if } i_1\neq 1 \\ q^{k-1}V_r(1), & \text{if } i_1=1 \end{cases}$$

An optimal policy is threshold type with critical number α, where α is the unique root x of the equation

$$(1+q)(1-q+q_2)x= (1-q)(1+q_2)x^{1-q}+2q(q_2-q)x^2$$

The optimal success probability is

$$P(S)=\frac{p_1}{q(1+q)}[(1+q_2)\alpha^{1-q}-(1+q)(1-q+q_2)\alpha+q(q_2-q)\alpha^2]$$

EXAMPLE 2.3

$$\begin{bmatrix} 1 & 2 & \cdots & m-1 & m & m+1 & \cdots \\ q_1 & 0 & \cdots & 0 & q_m & 0 & \cdots \end{bmatrix}$$

An optimal policy is not necessarily threshold type. Consider the case $\begin{bmatrix} 1 & 2 & \cdots & m-1 & m & m+1 & \cdots \\ q_1 & 0 & \cdots & 0 & q_m & 0 & \cdots \end{bmatrix}$ with m sufficiently large. Then if the first offer is rejected, we'll pass over about $e^{-1}m$ candidates and then give an offer. My conjecture is that, if q_j is non-increasing in j, then the optimal policy is threshold type.

3. Gusein-Zade Problem

Our objective is to maximize the probability of choosing either the best or the second best. Correspoding to $a_r(i_1,i_2,\ldots,i_k)$ and $b_r(i_1,i_2,\ldots,i_k)$, define $c_r(i_1,i_2,\ldots,i_k)$ for the modified problem $\begin{bmatrix} 1 & 2 & \cdots & n-2 \\ q_3 & q_4 & \cdots & q_n \end{bmatrix}$. Then we have the following optimality equations.

$$\begin{aligned} &V_{r-1}(i_1,i_2,\ldots,i_k) \\ &\quad=\frac{1}{r}\max\{\, p_1(\tfrac{r}{n})b_{r-1}(i_1,i_2,\ldots,i_k)+p_2\frac{r(n-r)}{n(n-1)}c_{r-1}(i_1,i_2,\ldots,i_k) \\ &\qquad\qquad +V_r(1,i_1+1,\ldots,i_k+1),\ V_r(i_1+1,\ldots,i_k+1)\} \\ &\quad+\frac{1}{r}\max\{\, p_2(\frac{r(r-1)}{n(n-1)})c_{r-2}(i_1-1,i_2-1,\ldots,i_k-1) \\ &\qquad\qquad +V_r(2,i_1+1,\ldots,i_k+1),\ V_r(i_1+1,\ldots,i_k+1)\} \\ &\quad+(\frac{i_1-2}{r})V_r(i_1+1,\ldots,i_k+1) \\ &\quad+\sum_{t=2}^{k}(\frac{i_t-i_{t-1}}{r})V_r(i_1,\ldots,i_{t-1},i_t+1,\ldots,i_k+1) \\ &\quad+(\frac{r-i_k}{r})V_r(i_1,\ldots,i_k)\,, \end{aligned}$$

$$\begin{aligned} V_{r-1}(\phi)&=\frac{1}{r}\max\{\, p_1(\tfrac{r}{n})+p_2\frac{r(n-r)}{n(n-1)}+V_r(1),\ V_r(\phi)\} \\ &\quad+\frac{1}{r}\max\{\, p_2\frac{r(r-1)}{n(n-1)}+V_r(2),\ V_r(\phi)\} \\ &\quad+(1-\tfrac{2}{r})V_r(\phi). \end{aligned}$$

Letting n and r tend to infinity, we can derive the defferential equations analogous to Lemma 2.7.

EXAMPLE 3.1

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \cdots \\ q_1 & q_2 & 0 & 0 & \cdots \end{bmatrix}$$

An optimal policy is described in terms of d_1, d_2 and s_2. As for the first offer, give an offer to relatively best if he appears after d_1 and give an offer to relatively second best if he appears after d_2 ($d_1<d_2$). If the first offer was given to relatively best but rejected then we immediatelygive the second offer to the next relatively best but give the second offer to the relatively second best only when he appears after s_2.

Parameter space is partitioned into R_1 and R_2, such that

$$R_1 = \{(p_1, p_2) : 0<p_2\leq p_2^*(p_1), 0<p_1\leq 1\}$$
$$R_2 = \{(p_1, p_2) : p_2^*(p_1)<p_2\leq 1, 0<p_1\leq 1\}$$

where $p_2^*(p_1)$ is defined, for a given p_1, as a unique root p_2 of the equation

$$e^{-\delta} = \frac{2(p_1+p_2)}{2(p_1+2p_2)-(1-p_1)p_2\delta}$$

and δ ia defined as

$$\delta = \frac{q_1 p_2}{p_1 q_2+q_1 p_2}.$$

It can be shown that

$$d_1\leq s_2\leq d_2, \quad \text{if } (p_1, p_2)\in R_1$$
$$d_1\leq d_2\leq s_2, \quad \text{if } (p_1, p_2)\in R_2 .$$

For $(p_1, p_2)\in R_1$, d_1, d_2 and s_2 are defined as follows :

$$s_2 = \exp(-\delta)$$

d_2 is a unique root x of the equation

$$(p_1+p_2)+\frac{(p_1q_2+q_1p_2)x}{2}\log^2 x = (p_1+2p_2)x ,$$

d_1 is a unique root x of the equation

$$2[p_2+a(1-\delta)]x - (p_1+p_2+as_2)\log x = (p_1+p_2)(1-\log d_2) + a[(3+\delta)s_2-(1-\log d_2)d_2]$$

where $a = p_1q_2+q_1p_2$.

Moreover

$$P(S) = d_1[(p_1+p_2+as_2) - \{p_2+a(1-\delta)\}d_1]$$

References

[1] Smith, M.H. 1975. A Secretary Problem With Uncertain Employment. J. Appl. Prob. 12, 620-624.
[2] Tamaki, M. 1991. A Secretary Problem With Uncertain Employment and Best Choice of Available Candidates. Operat. Res. 39, 274-284.