1. **Prime graphs.** Let $G$ be a finite group and $\Gamma(G)$ be the prime graph of $G$. This is the graph such that the vertex-set $V(\Gamma(G)) = \pi(G)$, the set of prime divisors of $|G|$ and two distinct primes $p$ and $r$ are joined by an edge if and only if there exists an element of order $pr$ in $G$. The concept of prime graph arose from cohomological questions associated with integral representation of finite groups (See Gruenberg[4],[5], Gruenberg-Roggenkamp[6],[7]). Let $n(\Gamma(G))$ be the number of connected components of $\Gamma(G)$ and $d_G(p,r)$ the length of the shortest path between $p$ and $r$. If there is no path between $p$ and $r$, then $d_G(p,r)$ is defined to be infinite.

**Theorem 1** ([10],[13],[14]).

\[
n(\Gamma(G)) = \begin{cases} 
1, \\
2, \\
3, \\
4, \\
5, \\
6
\end{cases}
\]

**Theorem 2** ([11]).

\[
d_G(p,r) = \begin{cases} 
1, \\
2, \\
3, \\
4, \\
\infty
\end{cases}
\]

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Remark 1. Theorems 1 and 2 hold for any finite group $G$. The proofs depend upon the classification of finite simple groups. Theorem 1 is the solution of Gruenberg-Kegel’s conjecture. We classify not only the number of connected components but also the components themselves. The significance of Theorem 1 can be found in [5],[8],[9],[12] and [15].

Remark 2. If $G$ is solvable or simple, then $d_G(p, r) = 1, 2, 3$ or $d_G(p, r) = \infty$. For the sporadic simple group $G$, $d_G(p, r) = 3$ if and only if $G = F_1$ and $p = 29$, $r = 47$ or $G = M_{23}$ and $p = 3$, $r = 7$. Unfortunately we have no application of Theorem 2. We are trying to find applications of Theorem 2.

2. Related topics. Let $\chi$ be a character (resp. $p$-Brauer character) of $G$ and $L$ be the set of values of $\chi$ on nonidentity elements (resp. nonidentity $p$-regular elements) of $G$. We say that $\chi$ is sharp (resp. $p$-Brauer sharp) if $f_L(\chi(1)) = |G|$ (resp. $f_L(\chi(1)) = |G|_{p'}$) where $f_L(x)$ is the monic polynomial of least degree whose set of roots is $L$. We note that $|G|$ (resp. $|G|_{p'}$) always divides $f_L(\chi(1))$ by Blichfeldt’s theorem (See [1]). Recently Alvis and Nozawa [1] classified the groups with sharp character $\chi$ such that $\chi$ takes an irrational value and $(\chi, 1_G) = 1$. Therefore we can assume that $L$ is contained in $\mathbb{Z}$. Let $L = \{l_1, l_2, ..., l_t\}$. The ($p$-Brauer) sharp character $\chi$ is said to be $t$-connected if and only if $L \subseteq \mathbb{Z} - \{\chi(1) - 1, \chi(1) + 1\}$ and $(\chi(1) - l_i, \chi(1) - l_j) = 1$ for $i \neq j$.

Theorem 3 ([3],[8]). The following two conditions are equivalent.

1. $G$ has a 2-connected ($p$-Brauer) sharp character.

2. $\Gamma(G) - \{p\}$ is disconnected.

Remark 3. $\Gamma(G) - \{p\}$ is a subgraph of $\Gamma(G)$ such that the vertex-set is $V(\Gamma(G)) - \{p\}$. If $p$ does not divide $|G|$, then $\Gamma(G) - \{p\} = \Gamma(G)$ and the result is for ordinary (generalized)
characters.

Remark 4. In [1] the authors assume that $\chi$ is the character of its representation. However in [3] and [8] $\chi$ may not have its representation.

Let $\mathfrak{R}(G) = \{ n \in \mathbb{Z} | G has a conjugacy class C with |C| = n \}$. Thompson made the following conjecture.

**Thompson's conjecture.** Let $G$ be a finite group and $M$ a non abelian simple group. If $\mathfrak{R}(G) = \mathfrak{R}(M)$ and $Z(G) = 1$, then $G$ is isomorphic with $M$.

**Theorem 4 ([2]).** Thompson's conjecture holds for a finite simple group $M$ with $n(\Gamma(M)) > 1$.

The proof heavily depends upon the classification of the connected components of prime graphs of finite simple groups in Theorem 1.

**REFERENCES**


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