数理解析研究所講究録
第 867 巻 1994 年 1-15

FLAG TRANSITIVE GEOMETRIES BELONGING TO SOME TITS – BUEKENHOUT DIAGRAMS

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1. DEFINITIONS AND NOTATION

Diagrams geometries were introduced by Buekenhout about 1975, starting from the Tits buildings and the Fisher 3–transpositions. As a result of this, many mathematicians have taken interest in diagram geometries theory, which has turned out to be a very fruitful field.

Associating a diagram to a class of geometries allows us to have in a synthetic way some essential informations about the incidence structure of the geometries belonging to the diagram.

Moreover diagrams give an unified view of geometries even when they look very different one from another.

A particular diagram gains interest when several examples of geometries belonging to it are known. In this case it is natural to look for other examples or for a better classification of all geometries belonging to that diagram.

The aim of the present paper is to give a review of the classification of geometries belonging to some diagrams of remarkable interest. In view of this we need to premise some basic notation and definitions.

Defined a geometry of rank 1 as a set of size greater than 2, we call *geometry of rank* $n \geq 2$ an n–partite graph $\Gamma = (V, *)$ such that, for every vertex x, the neighbourhood Γ_x of x is a geometry of rank $n-1$. We call

elements: the vertices of Γ,
incidence: the adjacency relation (denoted by $*$) in Γ,
flags: the cliques of Γ,
chambers: the maximal flags,
rank of a flag F: $|F|$
corank of a flag F: $n - |F|$.

Given a bijection between the partition classes of Γ and the set $I = \{0, 1, \cdots, n-1\}$, we call *type* of an elements x, denoting it by $t(x)$, the element of I associated to the partition class which x belongs to. If $F = \{x_1, \cdots, x_r\}$ is a flag, we call $t(F) = \{t(x_1), \cdots, t(x_r)\}$ the *type* of F. $I - t(F)$ is called the *cotype* of F.

The neighbourhood Γ_x of an element x, equipped with the type partition inherited from Γ is called the *residue* of x. We define the *residue of a flag F* as $\Gamma_F = \bigcap_{x \in F} \Gamma_x$. The cotype of F is the *type* of Γ_F . Given a flag F and a subset J of I, the *J–shadow* $\sigma_J(F)$ is the set of flags of type J incident with F.

A geometry of rank 2 is said to be a *generalized digon* if it is a complete bi–partite graph.

Given a set $\mathcal{D} = \{\mathcal{D}_{ij}\}_{i,j \in I, i \neq j}$ where $\mathcal{D}_{ij}$ is some class of rank 2 geometries, we say that a geometry Γ of rank n belongs to the *Tits–Buekenhout diagram* $\mathcal{D}$ if all residues of Γ of type $\{i, j\}$ belong to $\mathcal{D}_{ij}$ $(i, j \in I, i \neq j)$.

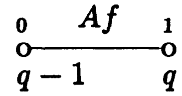
We associate to a diagram $\mathcal{D}$ as above the graph $\overline{\mathcal{D}}$ in which the vertices are the elements of I and two distinct vertices i and j are adjacent if and only if $\mathcal{D}_{ij}$ is different from the class of generalized digons. If $\overline{\mathcal{D}}$ is a string we call $\mathcal{D}$ a *string diagram.*

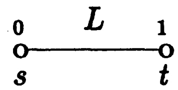
If Γ is a geometry of rank n belonging to a string diagram, the elements of Γ of type $0, 1, 2$ and $n-1$ are usually called *points, lines, planes* and *hyperplanes*, respectively. We will use phrases as "the point x is on the plane y", "the line z passes through the point x" etc. to mean that $x * y, x * z$ etc.

A geometry is said to be *flat* if every point is incident with all hyperplanes.

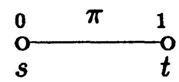
We recall some standard notation for some classes of geometries of rank 2. The parameters below the nodes $0, 1$ denote the number decreased by one of elements of type 0 (respectively 1) incident with an element of type 1 (respectively 0). In particular the following diagrams denote:

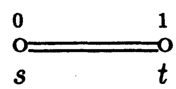

projective planes of order q


affine planes of order q

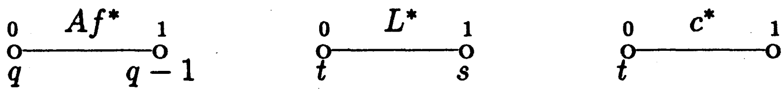

linear geometries


circular geometries


partial planes

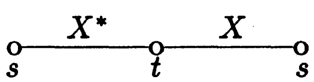

generalized quadrangles of order (s, t)

Finally

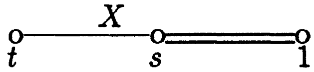


denote dual affine planes, dual linear geometries, dual circular geometries, respectively.

Given a geometry Γ of rank 3, we call *Grassmannian of* Γ *with respect to the lines* the geometry Γ' in which *points* are the lines of Γ, *lines* are (point, plane) flags of Γ and *planes* are the points and the planes of Γ. If Γ has diagram as follows



where X is any diagram of rank 2 and X^* is the dual of X, then Γ' has diagram as follows



For the general definitions of Grassmannian see [27; 5].

We denote the group of type–preserving automorphisms of a geometry Γ by $Aut(\Gamma)$. If H is a subgroup of $Aut(\Gamma)$, we call *quotient of* Γ *over* H the geometry in which the elements are the orbits of H with the incidence inherited by Γ; we denote it by Γ/H.

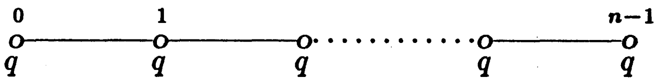
The reader is referred to [27] for more details on geometries and diagrams.

2. INTRODUCTION

We are interested in the following kind of problem: given a diagram, classify all geometries belonging to it. This problem has been completely solved for some diagrams. The following theorems, rephrasing two results of Tits and Jonsson respectively [30] [20], give an example in this sense.

THEOREM 2.1 *All geometries belonging to the following diagram*

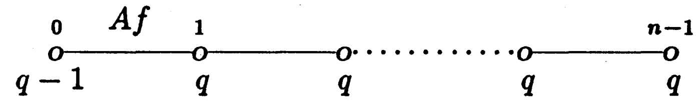
(A_n)



are projective spaces of dimension n *and order* q.

THEOREM 2.2 *All geometries belonging to the following diagram*

$(Af.A_{n-1})$



are affine spaces of dimension n *and order* q.

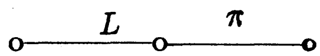
In general it is too difficult to achieve a complete classification of all geometries belonging to a given diagram, without assuming any additional conditions of geometric or algebraic type. An example of geometric condition is the "Intersection Property" (IP for short):

(IP) Given two flags F, G, for any $J \subseteq I$ there exists a flag H such that
$$\sigma_J(F) \cap \sigma_J(G) = \sigma_J(H).$$

Another usual geometric condition which is weaker than IP is

(LL) Given two distinct lines, there is at most one point incident with both of them.

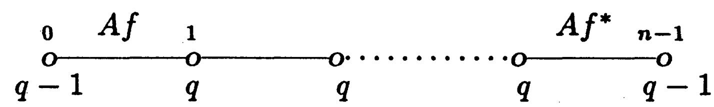
Even if LL is weaker than IP in general, nevertheless for some diagrams LL is equivalent to IP. For istance this happens for the following diagram [27; th. 7.5]



An example of classification of geometries satisfying IP is given by the following result due to Lefevre–Percsy and Van Nypelseer [21]:

THEOREM 2.3 *All geometries belonging to the diagram*

$(Af.A_{n-2}.Af^*)$



and satisfying IP are obtained from the projective space $PG(n,q)$ *by deleting the residues of a point and a hyperplane.*

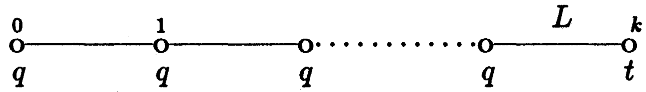
The most usual condition of algebraic type is

(FT) The automorphism group of the geometry is flag–transitive.

Whe might ask from a classification theorem for some diagram that it states that the geometries belonging to the given diagram are obtained by some general construction (such as cancellations, truncations, quotients etc.) except some sporadic cases. In fact this often happens as we can see in the following examples.

EXAMPLE 2.4 The geometries belonging to the diagram $Af.A_{n-2}$, i. e. the affine spaces (see theorem 2.2), are obtained by removing a hyperplane from a projective space.

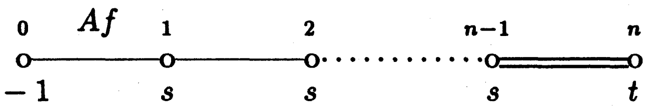
EXAMPLE 2.5 The geometries belonging to the diagram



are obtained from a projective space $PG(n,q)$, $n \geq k$, by truncating all types $\geq k$ [3].

EXAMPLE 2.6 All geometries belonging to the diagram

$(Af.C_n)$



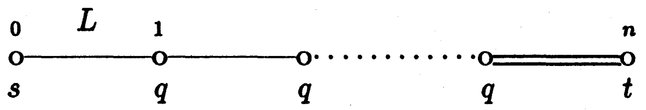
and satisfying IP are affine polar spaces (namely obtained from a polar space by removing a geometric hyperplane) or quotients of affine polar spaces [8],[9].

The focus of the present paper is a survey on the classification of geometries belonging to some diagrams and satisfying FT.

3. DIAGRAM $L.C_n$

The classification of flag–transitive geometries belonging to the following diagram

$(L.C_n)$ $(n \geq 2)$



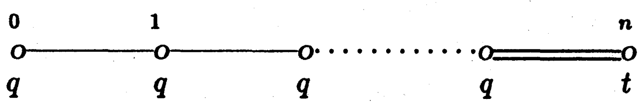
has kept busy many people. Aschbacher, Buekenhout, Cuypers, Del Fra, Ghinelli, Hubaut, Meixner, Neumaier, Pasini, Weiss and Yoshiara have contributed to this (see [1], [2], [6], [8], [9], [12], [13], [17], [22], [24], [25], [26], [28], [31], [32]).

We distinguish two cases according to $n > 2$ or $n = 2$.

THEOREM 3.1 *A flag–transitive $L.C_n$ geometry Γ with $n > 2$ belongs to one of the following diagrams:*

i) The diagram C_{n+1}

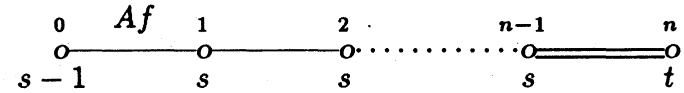
(C_{n+1})



In this case Γ is a classical polar space;

ii) *The diagram $Af.C_n$*

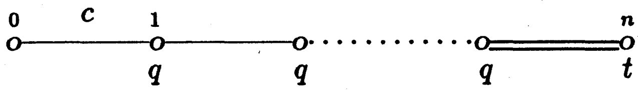
$(Af.C_n)$



In this case Γ is a standard quotient Π/H of an affine polar space Π ("standard" means that two points belonging to the same orbit of H are at distance 3 in Π);

iii) *The diagram $c.C_n$*

$(c.C_n)$



and in this case Γ is

a) *either a standard quotient of an affine polar space of rank $n+1$ (namely of polar rank n) with $q = 2$ (in this case the labels Af and c are the same),*

b) *or the so-called Neumaier geometry with $n = 3, q = 2, t = 2$ for the group A_8,*

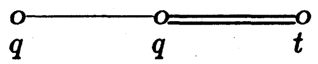
c) *or a (unique) geometry with $n = 3, q = 4, t = 2$ for the group Fi_{22}.*

The classification in the case $n = 2$ is not yet complete. In fact the following theorem uses an additional condition on the point-residues.

THEOREM 3.2 *A flag-transitive $L.C_2$ geometry Γ with classical generalized quadrangles as point-residues belongs to one of the following diagrams:*

i) *The diagram C_3*

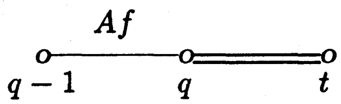
(C_3)



In this case Γ is either a classical polar space or the A_7-geometry;

ii) *The diagram $Af.C_2$*

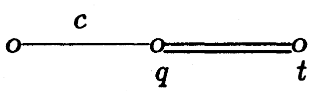
$(Af.C_2)$



In this case Γ is a standard quotient of an affine polar space of rank 3 (polar rank 2);

iii) *The diagram $c.C_2$*

$(c.C_2)$



In this case Γ is

a) *either a standard quotient of an affine polar space of rank 3 with $q = 2$,*

or one of six sporadic examples with

b) $q = 2, t = 3$ *for* $U_5(2)$

c) $q = 3, t = 9$ *for* McL

d) $q = 4, t = 2$ *for* $O_6^-(3)$

e) $q = 4, t = 2$ *for* $3.O_6^-(3)$

f) $q = 9, t = 3$ *for* Suz

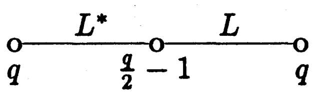
g) $q = 9, t = 3$ *for* $Aut(HS)$.

The above–mentioned groups are the minimal flag–transitive automorphism groups of these examples.

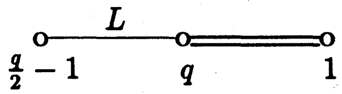
We again remark that the previous result holds if the point–residues are classical generalized quadrangles. In fact there are some flag–transitive examples of $L.C_2$ geometries with non classical point–residues.

EXAMPLE 3.3 [19] Given the hyperbolic quadric $Q_3^+(q)$, q even, let us consider the following geometry Γ:
the *points* of Γ are the points of $PG(3,q) \setminus Q_3^+(q)$,
the *lines* of Γ are the lines of $PG(3,q)$ external to $Q_3^+(q)$,
the *planes* of Γ are the planes of $PG(3,q)$ secant for $Q_3^+(q)$.
The incidence relation is the natural one except the point–plane case in which we claim that a point p is incident with a plane π if and only if $p \in \pi$ and $p^\perp \neq \pi$.

Then Γ belongs to the following diagram

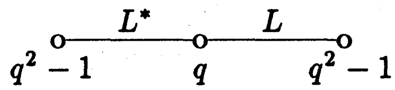


The Grassmannian of Γ with respect to the central node is a geometry of diagram

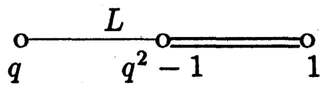


It is flag–transitive with group $O_4^+(q).2$.

EXAMPLE 3.4 Given the hermitian variety $H = H_3(q)$, consider the geometry Γ in which the *points* are those of $PG(3,q^2)$ external to H, *lines* are the lines of $PG(3,q^2)$ tangent to H and *planes* are those secant for H. The incidence relation is defined in natural way. Γ has diagram

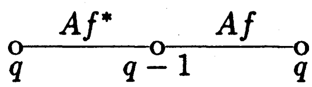


and the Grassmannian has diagram

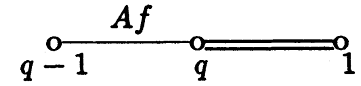


This geometry is flag–transitive too, with group $U_4(q^2).2$.

EXAMPLE 3.5 [14] Let now $W = W(q)$ be the symplectic variety in $P = PG(3,q)$, $q > 2$. Define a geometry Γ as follows. P is the point–set of Γ. The lines of Γ are the lines of P that are not lines of W. The planes of Γ are the planes of P. The incidence relation is the natural one, except that a point a is declared to be incident with a plane u only if $a^\perp \neq u$. The geometry arising in this way has diagram and parameters as follows

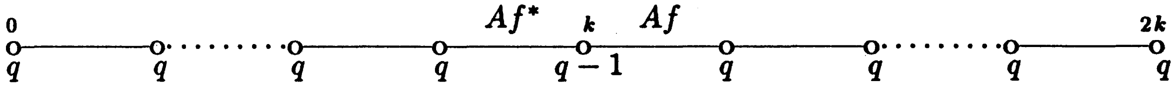


the Grassmannian has diagram

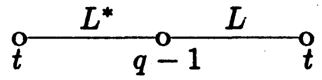


and the group $S_4(q).2$ acts flag–transitively on it.

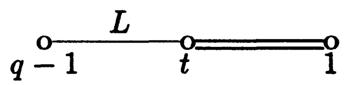
EXAMPLE 3.6 [19] In $PG(2k+1,q)$ we consider an element H of type k and delete all elements of type $< k$ meeting H and all elements of type $\geq k$ that, joined with H, do not span the whole space. We obtain a geometry of diagram



By truncating all types less than $k-1$ and greater than $k+1$, we have the following diagram



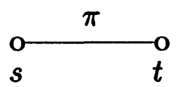
with $t = q^k + \cdots + q$. Again the Grassmannian has diagram

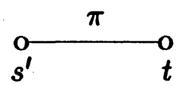


Denoted by G_H the stabilizer of H in $P\Gamma L(2k+2,q)$, this geometry is flag–transitive with $G_H.2$ as automorphism group.

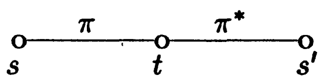
Before explaining the next example, we give a definition [7].

Let $\Pi = (\mathcal{P},\mathcal{L})$ ($\mathcal{P}$ set of points, $\mathcal{L}$ set of lines of Π) and $\Pi' = (\mathcal{P}',\mathcal{L}')$ be two partial planes with parameters (s,t) and (s',t) respectively, which are supplied with a parallelism relation. This means that there is a mapping $f : \mathcal{L} \longrightarrow I$ ($f' : \mathcal{L}' \longrightarrow I$ respectively), where $I = \{0,1,\cdots,t\}$, such that two lines mapped onto the same element of I do not meet. The (f,f') *gluing* of Π and Π' is the geometry Γ of rank 3 in which $\mathcal{P}$ is the set of points, $\mathcal{P}'$ is the set of planes and $\Lambda = \{(l,l'), l \in \mathcal{L}, l' \in \mathcal{L}' : f(l) = f'(l')\}$ is the set of lines. The incidence relation is defined as follows: all points are incident with all planes of Γ, namely Γ is flat. A point (plane) x of Γ and a line (l,l') are incident if $x \in l$ (if $x \in l'$).

Denoted by 

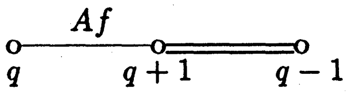
and 

the diagrams of Π and Π' respectively, the gluing Γ has diagram



We call I the *common line at infinity of* Π *and* Π'. Given an automorphism α of Π (α' of Π') we denote by α^∞ (by α'^∞) the action of α (of α') on I. We set $Aut(\Pi)^\infty = \{\alpha^\infty : \alpha \in Aut(\Pi)\}$ and $Aut(\Pi')^\infty = \{\alpha'^\infty : \alpha' \in Aut(\Pi')\}$. Clearly $Aut(\Gamma)$ is the subgroup of $Aut(\Pi) \times Aut(\Pi')$ consisting of the pairs (α,α') with $\alpha^\infty = \alpha'^\infty$. It is immediate to see that $Aut(\Gamma)$ is flag–transitive on Γ if and only if $Aut(\Gamma)$ acts transitively on the points of Π and Π' and moreover $Aut(\Pi)^\infty \cap Aut(\Pi')^\infty$ is transitive on I.

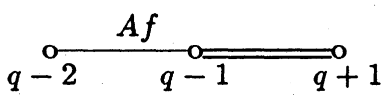
EXAMPLE 3.7 Let us consider the generalized quadrangle $T_2^*(\mathcal{O})$ of order $(q-1,q+1)$, obtained from a hyperoval $\mathcal{O}$ in $PG(3,q)$, $q = 2^h$ [29; 3]. In $T_2^*(\mathcal{O})$ two lines are called *parallel* if they meet in the same point of $\mathcal{O}$. Let f be any mapping from the set of parallelism classes of $T_2^*(\mathcal{O})$ to $I = \{0, 1, \cdots, q+1\}$. Given an affine plane A of order $q+1$ we consider a mapping f' from the set of parallelism classes of A to I. The (f, f') gluing of $T_2^*(\mathcal{O})$ and A has diagram



Since $T_2^*(\mathcal{O})$ is flag–transitive only for $q = 4$ and 16 (in this second case $\mathcal{O}$ is Lunelli–Sce's hyperoval), for $q \neq 4, 16$ this gluing cannot be flag–transitive. A straightforward computation proves that we have flag–transitivity only for $q = 4$, and then $Aut(\Gamma) = (5 \times 2^6).L_2(5)$.

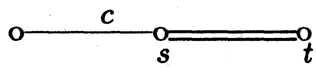
Also in the generalized quadrangle dual of $T_2^*(\mathcal{O})$ we can define a parallelism relation as follows. In $PG(3, q)$ let l be a line external to the hyperoval $\mathcal{O}$ on the plane containing $\mathcal{O}$. Two lines of the dual of $T_2^*(\mathcal{O})$ are called parallel if the two corresponding points in $PG(3, q)$ belong to the same plane containing l.

Given an affine plane A of order $q - 1$ we can glue A with the dual of $T_2^*(\mathcal{O})$ obtaining a geometry of diagram



Also in this case we have flag–transitivity only for $q = 4$ and moreover $Aut(\Gamma) = (2^4.S_5 \times 3^2).2^2$.

The previous are only some of the known examples. We point out that there are also many flag–transitive examples belonging to the diagram

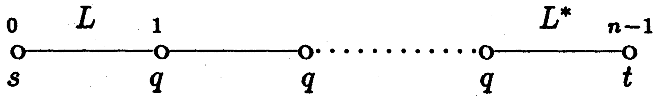


(see [23] [33]).

4. DIAGRAM $L.A_{n-2}.L^*$

In this section we illustrate the classification of the flag–transitive geometries belonging to the following diagram

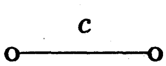
$(L.A_{n-2}.L^*)$



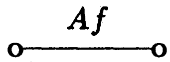
with rank $n \geq 4$.

It follows from a result of Delandtsheer [10] [11] that the stroke

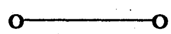
is either



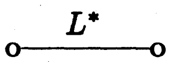
or



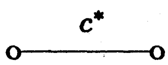
or



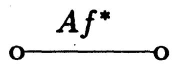
Analogously the stroke



is either



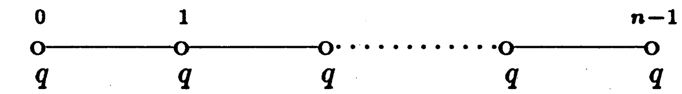
or



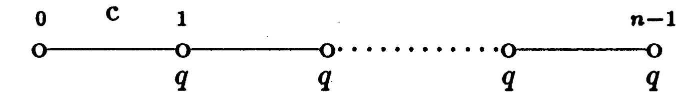
or

At first glance there would be nine different cases to study, but we can reduce them to six. In fact we can assume $s \leq t$ not to consider some cases twice. The six possibilities are

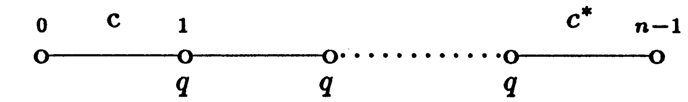
(A_n)



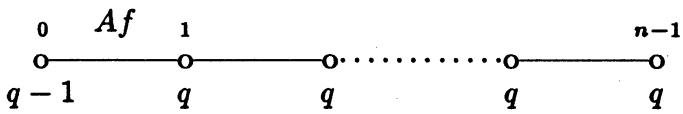
$(c.A_{n-1})$



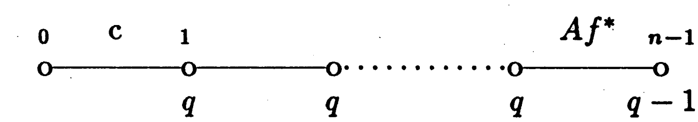
$(c.A_{n-2}.c^*)$



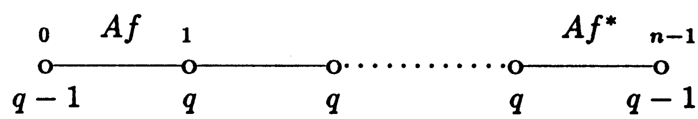
$(Af.A_{n-1})$



$(c.A_{n-2}.Af^*)$



$(Af.A_{n-2}.Af^*)$



The classification of the flag–transitive geometries belonging to the previous diagrams is given in the following subsections. We warn that flag–transitivity is not necessary in cases 4.1, 4.2, 4.4.

4.1 *Case* A_n

As we said in section 2, only projective spaces of dimension n and order q exist for this diagram [30].

4.2 *Case* $c.A_{n-1}$

By a result of Hughes [18], the only possible cases are thin projective spaces ($q = 1$) or affine spaces of order $q = 2$. Note that $c = A_2$ if $q = 1$ and $c = Af$ if $q = 2$.

4.3 *Case* $c.A_{n-2}.Af^*$

Pasini and Yoshiara proved in [28] that in this case there are only bi–affine geometries of order 2 (see 4.6 for the definition), or a sporadic geometry with diagram

for the group HS.

4.4 *Case* $Af.A_{n-1}$

As we said in section 2, only affine spaces of dimension n and order q exist for this diagram.

4.5 *Case* $c.A_{n-2}.Af^*$

We can assume $q > 2$. In fact if $q = 1$ or 2 turn back to 4.3. By a result of Hughes [18] we have $q = 4$ and $n = 4$, namely

Del Fra and Pasini proved in [15] that no flag–transitive such geometry exists. No non flag–transitive examples are known for the time being.

4.6 *Case* $Af.A_{n-2}.Af^*$

Also in this case we can assume $q > 2$. Del Fra and Pasini proved in [15] that the only possible cases are quotients of bi–affine geometries of order q. Let us describe them.

A *bi–affine geometry* Γ *of order* q *and rank* n is obtained by removing from $PG(n, q)$ the residues of a hyperplane π and a point P. We obtain two non isomorphic geometries called of *flag type* or *anti–flag type* according to whether $P \in \pi$ or $P \neq \pi$, respectively.

In the case of flag type $Aut(\Gamma) = (Z.E).(Z_{q-1} \times GL(n-1, q)).F$, with Z elementary abelian of order q, E elementary abelian of order $q^{2(n-1)}$ and $F \cong Aut(GF(q))$. However $Z.E.(Z_d \times GL(n-1, q))$ with $d = (n+1, q-1)$ also acts flag–transitively on Γ.

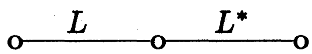
In the case of anti–flag type $Aut(\Gamma) = \Gamma L(n, q)$. However $SL(n, q)$ also acts flag–transitively on Γ.

In both of these cases we denote by $G_0 < Aut(\Gamma)$ the group of homologies with center P and axis π. Given any subgroup $H \neq 1$ of G_0 the geometry Γ/H still belongs to the diagram $Af.A_{n-2}.Af^*$, but not in all cases is flag–transitive. In fact if Γ is of flag type, then Γ/H is flag–transitive if and only if $H = G_0$ and in this case the geometry is flat. If Γ is of anti–flag type, Γ/H is flag–transitive for any $H \leq G_0$.

In both of these cases, the property LL is not satisfied (being $H \neq 1$) (see section 2).

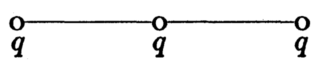
5. DIAGRAM $L.L^*$

The previous classification leaves out the case of rank 3, that is the diagram

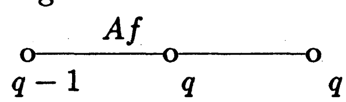


There are many examples. Here are some of them:

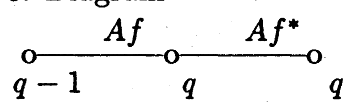
1) Projective 3–spaces. Diagram



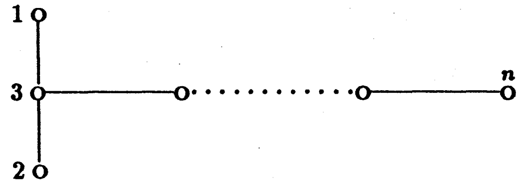
2) Affine 3–spaces. They have diagram



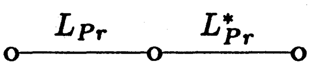
3) Bi–affine geometries of rank 3. Diagram



4) The geometries obtained by truncating all types greater than 3 in the building D_n:

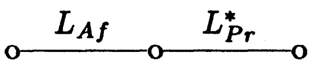


They have diagram



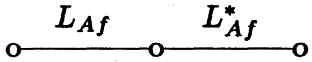
where L_{Pr} denotes the linear space of points and lines of a projective space.

If we delete in the D_n building a geometric hyperplane with respect to the types 1 and 3 and then we truncate as above, we obtain a geometry with diagram



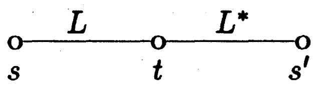
where L_{Af} denotes the linear geometry of points and lines of an affine space.

If we also delete a geometric hyperplane with respect to the types 2 and 3, by truncating we obtain a geometry with diagram

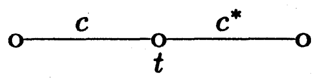


We now give some "less regular" examples.

5) Gluings of two linear spaces Π and Π' of order (s,t), (s',t) supplied with parallelism (see section 3). Their diagram is



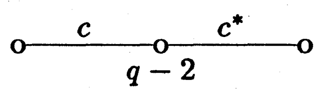
In particular if Π and Π' are circular geometries ($s = s' = 1$) we obtain



Flag–transitive examples of this type exist. For istance if we consider a flag–transitive linear space with parallelism and choose $\Pi' = \Pi$ (with $f' = f$: see section 3), then we obtain a flag transitive gluing.

6) In PG(3,q), q odd, let us consider a point p and a conic $\mathcal{C}$ on a plane π not passing through p. Define the following geometry Γ:
points are those of the cone projecting $\mathcal{C}$ from p;
lines are the pairs of points not on the same line through p;
planes are those meeting π in a line tangent to $\mathcal{C}$.

Γ has diagram



properties IP and FT hold. The group $Z_{q-1}.PGL(2,q)$ acts flag–transitively on Γ.

7) In $PG(3,q)$, $q = 2^h$, let us consider a point p and a hyperoval $\mathcal{O}$ on a plane π not passing through p. Let Γ be the geometry in which points and lines are defined in the same way as in the previous case, planes are those meeting π in a line external to $\mathcal{O}$. Γ belongs to the diagram

$$\circ\overset{c}{\text{———}}\underset{q}{\circ}\overset{L^*}{\text{———}}\underset{\frac{q}{2}-1}{\circ}$$

In particular if $q = 4$ we have a geometry belonging to the diagram

$$\circ\overset{c}{\text{———}}\underset{4}{\circ}\overset{c^*}{\text{———}}\circ$$

and the group $3.S_6$ acts flag–transitively on it.

8) A biplane $\mathcal{D}$ (namely a $2-(v,k,2)$ symmetric design) defines a geometry Γ of rank 3 in which *points* are the points of $\mathcal{D}$, *lines* are the pairs of points of $\mathcal{D}$ and *planes* are the blocks of $\mathcal{D}$. Such a geometry belongs to the diagram

$$\circ\overset{c}{\text{———}}\underset{k-2}{\circ}\overset{c^*}{\text{———}}\circ$$

and satisfies IP. In particular if $k = 5$ (in this case $\mathcal{D}$ is also a Hadamard design) the group $L_2(11)$ acts flag–transitively on it [4; (26)].

9) In general geometries belonging to the diagram

$$\circ\overset{c}{\text{———}}\underset{t}{\circ}\overset{c^*}{\text{———}}\circ$$

satisfying IP are called *semi–biplanes*. An example of semi–biplane with $t = 10$, which is not a biplane is described in [4; (32)]. The group M_{12} acts flag–transitively on it.

10) Let us consider the geometry of rank 3 obtained from the Steiner system $S(3,6,22)$ in a way similar to that of example 8. It belongs to the diagram

$$\circ\overset{c}{\text{———}}\underset{4}{\circ}\text{———}\underset{4}{\circ}$$

The group M_{22} acts flag–transitively on it. If we delete a point and its residue we obtain a geometry with diagram

$$\circ\overset{c}{\text{———}}\underset{4}{\circ}\overset{Af^*}{\text{———}}\underset{3}{\circ}$$

and the group $L_3(4).2$ acts flag–transitively on it.

11) We consider the graph Λ of 100 vertices on which the group J_2 acts transitively. There is a geometry of rank 3 defined by starting from Λ belonging to the diagram

$$\circ\overset{c}{\text{———}}\underset{8}{\circ}\overset{L^*}{\text{———}}\underset{3}{\circ}$$

where the point–residues are dual hermitian arcs in $PG(2,9)$. The group J_2 acts flag–transitively on this geometry [4; (104)].

The broad variety of examples suggests that a classification of flag–transitive geometries with diagram $L.L^*$ is very hard, maybe hopeless. It is more realistic to look for a classification in some particular cases. To illustrate one of them we need some definitions.

Given a geometry Γ belonging to a string diagram we say that two points of Γ are *collinear* if there is a line incident with both of them. We denote the collinearity relation by $\perp$. The *collinearity graph* is the graph defined by $\perp$ on the set of points of Γ. The diameter of this graph is called *collinearity diameter* of Γ.

Given two distinct points a, b, $\nu_{a,b}$ denotes the number of lines incident with both a and b. If $\nu_{a,b} = \nu$ for any pair of collinear points a, b we call ν the *collinearity index* of Γ and we say that the collinearity relation is *uniform*.

Given two different lines l, l' we set $\mu_{l,l'} = |\sigma_0(l) \cap \sigma_0(l')|$. If $\mu_{l,l'} > 1$ we say that l and l' are *multi–secant*. If $\mu_{l,l'}$ always takes the same value μ for any two multi–secant lines l and l', then we say that Γ has *uniform line–intersection of index* μ.

Del Fra and Pasini recently studied the diagram $Af.Af^*$ and obtained the following partial results [16].

THEOREM 5.1 *Let Γ be a geometry belonging to the diagram*

$$(Af.Af^*) \qquad \begin{array}{ccccc} & Af & & Af^* & \\ \circ & \text{———————} & \circ & \text{———————} & \circ \\ q-1 & & q & & q-1 \end{array}$$

Then

i) Γ has uniform line–intersection and uniform collinearity of the same index ν with $1 \leq \nu \leq q$ and ν dividing q or $q-1$;

ii) the collinearity diameter of Γ is ≤ 2;

iii) Γ is flat if and only if $\nu = q$ or $q-1$.

Moreover if Γ is flag–transitive, then

j) Γ has diameter 1 if and only if $\nu = q$ or $q-1$: in the first case Γ is a gluing of two affine planes of order q, in the second case Γ is the minimal quotient of the bi–affine geometry of anti–flag type of rank 3 and order q;

jj) if ν divides $q-1$ and $q \geq \sqrt{q}/2 + 1$, then Γ is a quotient of the bi–affine geometry of anti–flag type of rank 3 and order q;

jjj) if ν divides q and $q \geq \sqrt{q}/2 + 1$, then Γ admits as quotient a flat geometry (namely a gluing of two affine planes of order q).

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