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TODA EQUATIONS AND HARMONIC MAPS

YOSHIHIRO OHNITA (大仁田義裕)

Department of Mathematics, Tokyo Metropolitan University,
Minami-Ohsawa 1-1, Hachioji, Tokyo 192-03, Japan

A harmonic map is defined as a critical point of energy functional for smooth maps between Riemannian manifolds. The excellent references for harmonic map theory are [EL1], [EL2]. Over the past few years Theory of Integarble Systems provides new approach and much progress in the theory of harmonic maps of Riemann surfaces into symmetric spaces. The purpose of this article is to give a survey on recent works due to [B-P], [BPW], [Mc1], [Mc2] and so on. Especially, we shall restrict our attention to the relationship between Toda field equations and harmonic maps.

This article consists of the following subjects :

(1) an elementary construction of a solution to (elliptic) Toda field equation of type A from a harmonic map of a Riemann surface into a complex projective space.

(2) to introduce the notion of primitive maps into k-symmetric spaces and primitive maps of finite type. Here the fundamental thorem is that a primitive map of a 2-torus satisfying certain semisimplicity condition is of finite type. By using this theorem it can be shown that harmonic 2-tori in some compact symmetric spaces are covered by primitive maps of finite type.

(3) a correspondence between solutions to an affine Toda field equation for a simple compact Lie group G and a certain class of harmonic maps into a symmetric space G/H, and its applications.

1. ELLIPTIC TODA EQUATION AND GAUSS BUNDLES

1.1. Elliptic Toda equation. The 2-dimensional Toda field equation (of type $\mathfrak{a}$) is a partial differential equation

$$2\frac{\partial^2}{\partial z\partial\bar{z}}\omega_p + e^{2(\omega_p-\omega_{p-1})} - e^{2(\omega_{p+1}-\omega_p)} = 0 \tag{1.1}$$

with the unknown functions $\{\omega_p \mid p \in \mathbf{Z}\}$. We shall restrict to the elliptic version, where it is assumed that each ω_p is a real-valued function defined on a domain of the Gauss plane $\mathbf{C}$. In this case, the left hand side of (1.1) becomes the Laplacian of ω_p.

In this section we shall indicate the relationship of the elliptic version of Toda equation with harmonic maps in a simple case. In order to do it, we consider harmonic maps into a complex projective space $\mathbf{C}P^n$.

1.2. Gauss bundles. One of the most fundamental method to make harmonic maps is to construct the Gauss bundles of a harmonic map. We begin with a brief explanation of the Gauss bundles ([BW],[EL2]). Let $\varphi : \Sigma \longrightarrow Gr(\mathbf{C}^N) = \coprod_{k=0}^N Gr_k(\mathbf{C}^N)$ be a smooth map of a Riemann surface Σ into the complex Grassmannian. The map φ can be identified with a subbundle $\underline{\varphi}$ of the trivial bundle $\underline{\mathbf{C}}^N = \Sigma \times \mathbf{C}^N$ in the natural way. Let $\{z\}$ be a local holomorphic coordinate of Σ.

The ∂'- and ∂''- second fundamental forms $A'_\varphi : \underline{\varphi} \longrightarrow \underline{\varphi}^\perp$ and $A''_\varphi : \underline{\varphi} \longrightarrow \underline{\varphi}^\perp$, of the subbundle $\underline{\varphi}$ are defined by

$$A'_\varphi(s) = \pi^\perp_\varphi(\frac{\partial s}{\partial z}) \text{ and } A''_\varphi(s) = \pi^\perp_\varphi(\frac{\partial s}{\partial \bar{z}}),$$

for each $s \in C^\infty(\underline{\varphi})$. The map φ is harmonic if and only if $A'_\varphi : \underline{\varphi} \longrightarrow \underline{\varphi}^\perp$ is holomorphic, i.e. $\nabla^{\varphi,\varphi^\perp}_{\frac{\partial}{\partial \bar{z}}} A'_\varphi \equiv 0$, or equivalently, $A''_\varphi : \underline{\varphi} \longrightarrow \underline{\varphi}^\perp$, is antiholomorphic. Set $G'(\varphi) = \underline{\mathrm{Im}A'_\varphi}$, the holomorphic subbundle of $\underline{\varphi}^\perp$ in $\underline{\mathbf{C}}^N$, which is called the *∂'-Gauss bundle* of φ and $G''(\varphi) = \underline{\mathrm{Im}A''_\varphi}$, the antiholomorphic subbundle of $\underline{\varphi}^\perp$ in $\underline{\mathbf{C}}^N$, which is called the *∂''-Gauss bundle* of φ. Then the subbundle $G'(\varphi)$ defines a harmonic map $\Sigma \longrightarrow Gr(\mathbf{C}^N)$. The sequence of harmonic maps

$$\ldots, G^{(-2)}(\varphi), G''(\varphi), \varphi, G'(\varphi), G^{(2)}(\varphi), \ldots$$

is said to be a *harmonic sequence* of φ. Here $G^{(k+1)}(\varphi) = G'(G^{(k)}(\varphi))$ and $G^{(-(k+1))}(\varphi) = G''(G^{(-k)}(\varphi))$ for each nonnegative integer k. The harmonic map φ is called *strongly isotropic* if $\underline{\varphi} \perp G^{(\ell)}(\varphi)$ for each positive integer ℓ. In the case of a map into a complex projective space $\mathbf{C}P^n$, we say it simply *isotropic* The *isotropy order* of φ is the maximal positive integer k such that $\underline{\varphi} \perp G^{(\ell)}(\varphi)$ for each $1 \leq \ell \leq k$. It is known ([BW, Lemma 3.1]) that if $\underline{\varphi} \perp G^{(\ell)}(\varphi)$ for each $1 \leq \ell \leq k$, then $G^{(i)}(\varphi) \perp G^{(j)}(\varphi)$ for each $1 \leq |i-j| \leq k$.

1.3. Construction of a solution to Toda equation from a harmonic map. Let $\varphi : \Sigma \longrightarrow \mathbf{C}P^n$ be a harmonic map into a complex projective space. We consider its harmonic sequence

$$\ldots, G^{(-2)}(\varphi), G''(\varphi), \varphi, G'(\varphi), G^{(2)}(\varphi), \ldots.$$

Choose a local nonzero holomorphic section f_p of $G^{(p)}(\varphi)$ for each $p \in \mathbf{Z}$, i.e.

$$\nabla^{G^{(p)}(\varphi)}_{\frac{\partial}{\partial \bar{z}}} f_p = 0$$

such that

$$f_{p+1} = A'_{G^{(p)}(\varphi)}(f_p)$$

for each $p \in \mathbf{Z}$. We define a local real-valued function ω_p as $|f_p| = e^{\omega_p}$ for each $p \in \mathbf{Z}$. By a simple computation we see that $\{f_p \mid p \in \mathbf{Z}\}$ satisfy

$$\frac{\partial f_p}{\partial z} = (2\frac{\partial}{\partial z}\omega_p)f_p + f_{p+1},$$
$$\frac{\partial f_p}{\partial \bar{z}} = -e^{2(\omega_p - \omega_{p-1})}f_{p-1}$$

for each $p \in \mathbf{Z}$. The complete integrability condition for the above linear partial differential equation becomes

$$2\frac{\partial^2}{\partial z \partial \bar{z}}\omega_p + e^{2(\omega_p - \omega_{p-1})} - e^{2(\omega_{p+1}-\omega_p)} = 0$$

for each $p \in \mathbf{Z}$. Thus the functions $\{\omega_p\}$ gives a solution to the elliptic Toda field equation (1.1) of type $\mathfrak{a}$.

If φ is isotropic, then we obtain the finite lattice $\{\omega_p\}$ with $\omega_p = 0$ for each $p < -\ell$ and each $k < p$. If φ has orthogonally periodic harmonic sequence, i.e. $\underline{\varphi} \perp G^{(p)}(\varphi)$ for $0 \leq p \leq n$ and $G^{(p+n+1)}(\varphi) = G^{(p)}(\varphi)$ for each $p \in \mathbf{Z}$, then we obtain the periodic lattice $\{\omega_p\}$ with $\omega_{p+n+1} = \omega_p$ for each $p \in \mathbf{Z}$. In the case of periodic lattice, such a harmonic map is called *superconformal* (see 3.2).

2. Theory of harmonic tori

In this section we shall provide briefly a review on the theory of harmonic tori.

2.1 Primitive maps. Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Let τ be an automorphism of G of order k and set $K = \{a \in G \mid \tau(a) = a\}$. Denote also by τ the automorphism of the Lie algebra $\mathfrak{g}$ induced by τ. Set $\omega = e^{2\pi\sqrt{-1}/k}$. We have a decomposition of $\mathfrak{g}^{\mathbf{C}}$ into eigenspaces of τ :

$$\mathfrak{g}^{\mathbf{C}} = \bigoplus_{i \in \mathbf{Z}_k} \mathfrak{g}_i,$$

where $\mathfrak{g}_i$ denotes the ω^i-eigenspace of τ. Note that $\mathfrak{g}_0 = \mathfrak{k}^{\mathbf{C}}$. Then the homogeneous space $N = G/K$ is called a *k-symmetric space.* In the case of $k = 2$, it is nothing but a symmetric space.

The above decomposition of $\mathfrak{g}^{\mathbf{C}}$ induces the decomposition of the complexified tangent bundle

$$TN^{\mathbf{C}} = \bigoplus_{i \in \mathbf{Z}_k \setminus \{0\}} [\mathfrak{g}_i].$$

Definition. A smooth map $\psi : \Sigma \longrightarrow N = G/K$ is called *primitive* if the differential $d\psi$ of ψ satisfies $d\psi(T\Sigma^{1,0}) \subset [\mathfrak{g}_1]$.

We shall mention the harmonicity of primitive maps.

Proposition 2.1 [Bl]. *Any primitive map is harmonic with respect to any G-invariant Riemannian metric on N whose corresponding $Ad(K)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ satisfies*

$$(*) \qquad \langle \mathfrak{g}_i, \mathfrak{g}_j \rangle = \langle \mathfrak{g}_i, \bar{\mathfrak{g}}_{-j} \rangle = 0$$

for each $i, j \in \mathbf{Z}_k \setminus \{0\}$ with $i + j \not\equiv 0$ (mod k).

Remark. A map $\psi : \Sigma \longrightarrow G/K$ is called *equiharmonic* if ψ is harmonic with respect to any G-invariant Riemannian metric on G/K. If one of the following conditions is assumed

(1) τ is an inner automorphism,
(2) for each $i, \ell \in \mathbf{Z}_k \setminus \{0\}$ with $i \not\equiv \ell$ (mod k), as K-modules, $\mathfrak{g}_i$ contains no irreducible component isomorphic to one of $\mathfrak{g}_\ell$,

then any G-invariant Riemannian metric on G/K satisfies the condition (*). In these cases, any primitive map into G/K is equiharmonic.

Let H be a closed subgroup with $K \subset H$. We define a homogeneous projection $p : G/K \longrightarrow G/H$.

Proposition 2.2 [Bl]. *If $\psi : \Sigma \longrightarrow G/K$ is equiharmonic, then $\varphi = p \circ \psi : \Sigma \longrightarrow G/H$ is equiharmonic.*

2.2 Primitive maps of finite type. We define the twisted loop algebra

$$\Lambda \mathfrak{g}_\tau = \{\xi : S^1 \longrightarrow \mathfrak{g} \mid \tau(\xi(\lambda)) = \xi(\omega\lambda)\}.$$

If we express $\xi \in \Lambda\mathfrak{g}_\tau$ as $\xi = \sum \lambda^n \xi_n$, then we have $\xi_n \in \mathfrak{g}_n$ for each n. Let $d \equiv 1(\mathrm{mod}\ k)$. Define a finite dimensional vector subspace $\Lambda_d = \{\xi \in \Lambda\mathfrak{g}_\tau \mid \xi_n = 0\ (|n| > d)\}$ of $\Lambda\mathfrak{g}_\tau$. We consider

$$\frac{\partial \xi}{\partial z} = [\xi, \lambda\xi_d + r(\xi_{d-1})], \tag{2.1}$$

where $r(\cdot)$ denotes some component of $(\cdot)$ (see [BP]). [BP, Bu] proved that (2.1) is completely integrable, and for each $\xi_0 \in \Lambda_d$, there exists a unique solution $\xi : \mathbf{R}^2 \longrightarrow \Lambda_d$ to (2.1) satisfying the initial condition $\xi(0) = \xi_0$.

Define a 1-form α with values in $\mathfrak{g}$ by

$$\alpha = (\xi_d + r(\xi_{d-1}))dz + \overline{(\xi_d + r(\xi_{d-1}))}d\bar{z}.$$

Moreover, [BP, Bu] proved that the form α satisfies the Maurer-Cartan equation

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.$$

Hence there exists a smooth map $F : \mathbf{R}^2 \longrightarrow G$ satisfying $F^{-1}dF = \alpha$. It is possible to show that the map F projects to a primitive map $\psi : \mathbf{R}^2 \longrightarrow G/K$. The primitive map so obtained is said to be *of finite type.*

2.3 Harmonic tori. The following is the fundamental result on characterization of harmonic tori. It was proved by differential geometric method.

Theorem 2.3 [BFPP, BP, Bu]. *Let $\psi : T^2 \longrightarrow G/K$ be a primitive map of a 2-torus into a k-symmetric space. If $d\psi(\frac{\partial}{\partial z}) \subset [\mathfrak{g}_1]$ is contained in an $Ad(K^{\mathbf{C}})$-orbit of a semisimple element, then ψ is of finite type.*

Problem. Let $\varphi : T^2 \longrightarrow G/H$ be a harmonic 2-torus in a symmtric space G/H. Does there exist a primitive map $\psi : T^2 \longrightarrow G/K$ into a k-symmetric space G/K and a homogeneous fibration $\pi : G/K \longrightarrow G/H$ such that $\varphi = \pi \circ \phi$?

In the case $G/H = S^n$ and $G/H = \mathbf{C}P^n$, it was proved affirmatively by ([Bu]). Very recently the case of $G/H = Gr_2(\mathbf{C}^4)$ is studied by Udagawa [Ud].

3. Affine Toda field equations and harmonic maps

3.1. A special class of primitive maps of finite type are related with solutions to Toda equations. Bolton, Pedit and Woodward ([BPW]) clarified the relationship between affine Toda field equations for general compact simple Lie groups and special class of primitive maps.

Let T be a maximal torus of G with Lie algebra $\mathfrak{t}$. Let

$$\mathfrak{g}^{\mathbf{C}} = \mathfrak{t}^{\mathbf{C}} + \sum_{\alpha \in \Delta} \mathfrak{g}^{\alpha}$$

be the root decomposition of $\mathfrak{g}^{\mathbf{C}}$ and $\{\xi_\alpha \in \mathfrak{g}^\alpha \mid \alpha \in \Delta\}$ be the Cartan-Weyl basis satisfying

$$\begin{cases} \bar{\xi}_\alpha = -\xi_{-\alpha}, \\ [\xi_\alpha, \xi_{-\alpha}] = \alpha^\sharp, \\ (\xi_\alpha, \xi_\beta) = \delta_{\alpha,-\beta}. \end{cases} \tag{3.1}$$

Let $\{\alpha_1, \dots, \alpha_\ell\}$ be the fundamental root system and $\theta = \sum_{p=1}^{\ell} m_p \alpha_p$ be the highest root where $\ell = \operatorname{rank} G$. Define $m_0 = 1$. We denote by $(\ ,\ )$ the Killing-Cartan form of $\mathfrak{g}^{\mathbf{C}}$ and an element $\alpha^\sharp \in \sqrt{-1}\mathfrak{t}$ is defined by $\alpha(X) = (\alpha^\sharp, X)$ for each $X \in \sqrt{-1}\mathfrak{t}$.

The flag manifold $N = G/T$ has an m-symmetric space structure with the automorphism τ of G of order m, where $m = \sum_{p=0}^{\ell} m_p$, and the automorphism τ is given by $\tau = Ad(exp(2\pi\sqrt{-1}Z))$, where $Z = \frac{1}{m}\sum_{k=1}^{\ell} \eta_k$ and $\eta_k \in \sqrt{-1}\mathfrak{t}$, $\alpha_j(\eta_k) = \delta_{j,k}$. The eigenspace decomposition of $\mathfrak{g}^{\mathbf{C}}$ with respect to τ becomes

$$\mathfrak{g}^{\mathbf{C}} = \mathfrak{t}^{\mathbf{C}} + \sum_{i \in \mathbf{Z}_m \setminus \{0\}} \mathfrak{g}_i.$$

Then we have $\mathfrak{g}_1 = \sum_{p=0}^{\ell} \mathfrak{g}^{\alpha_p}$. We call $\xi \in \mathfrak{g}_1$ *cyclic* if $\xi = \sum_{p=0}^{\ell} a_p \xi_{\alpha_p}$ with $a_p \neq 0$.

The affine Toda field equation for $\mathfrak{g}$ is

$$2\frac{\partial^2 \Omega}{\partial z \partial \bar{z}} + \sum_{p=0}^{\ell} m_p e^{2\alpha_p(\Omega)} \alpha_p^\sharp = 0, \tag{3.2}$$

where $\Omega : U \longrightarrow \sqrt{-1}\mathfrak{t}$ is a unknown function and U is a simply connected domain in $\mathbf{C}$.

The following is fundamental in the treatment of Toda equation.

Proposition 3.1. *The complete integrability condition of the linear partial differential equation*

$$\begin{cases} F^{-1}\frac{\partial F}{\partial z} &= \frac{\partial \Omega}{\partial z} + (Adexp(\Omega))(B), \\ F^{-1}\frac{\partial F}{\partial \bar{z}} &= -\frac{\partial \Omega}{\partial \bar{z}} + (Adexp(-\Omega))(\bar{B}), \end{cases}$$

where $B = \sum_{p=0}^{\ell} \sqrt{m_p}\xi_{\alpha_p} \in \mathfrak{g}_1$ and $\Omega : U \longrightarrow \sqrt{-1}\mathfrak{t}$, is that Ω satisfies the Toda equation (3.2).

Indeed, using (3.1) we compute

$$\frac{\partial}{\partial z}(-\frac{\partial\Omega}{\partial\bar{z}} - \sum_{p=0}^{\ell}\sqrt{m_p}e^{\alpha_p(\Omega)}\xi_{-\alpha_p}) - \frac{\partial}{\partial\bar{z}}(\frac{\partial\Omega}{\partial z} + \sum_{p=0}^{\ell}\sqrt{m_p}e^{\alpha_p(\Omega)}\xi_{\alpha_p})$$
$$+[\frac{\partial\Omega}{\partial z} + \sum_{p=0}^{\ell}\sqrt{m_p}e^{\alpha_p(\Omega)}\xi_{\alpha_p}, -\frac{\partial\Omega}{\partial\bar{z}} - \sum_{p=0}^{\ell}\sqrt{m_p}e^{\alpha_p(\Omega)}\xi_{-\alpha_p}]$$
$$= -2\frac{\partial^2\Omega}{\partial z\partial\bar{z}} - \sum_{p=0}^{\ell} m_p e^{2\alpha_p(\Omega)}\alpha_p^\sharp = 0.$$

Definition. A framing $F : U \longrightarrow G$ is called a *Toda frame* ([BPW]) if F satisfies

$$F^{-1}\frac{\partial F}{\partial z} = \frac{\partial\Omega}{\partial z} + (\mathrm{Ad}exp(\Omega))(B) \in \mathfrak{t}^{\mathbf{C}} \oplus \mathfrak{g}_1, \tag{3.3}$$

for some $\Omega : U \longrightarrow \sqrt{-1}\mathfrak{t}$.

The relation between a Toda frame and a primitive map is described as follows. From (3.3) we see immediately

Proposition 3.2. *If F is a Toda frame, then $\psi = F \cdot T : U \longrightarrow G/T$ is a primitive map such that $d\psi(\frac{\partial}{\partial z}) \in [\mathfrak{g}_1]$ is cyclic.*

[BPW] proved the following by using the argument of [FPPS].

Proposition 3.3. *If $\psi : U \longrightarrow G/T$ is a primitive map from a simply connected domain U such that $d\psi(\frac{\partial}{\partial z}) \in [\mathfrak{g}_1]$ is cyclic, then there exists a Toda frame F such that $\pi \circ F = \psi$.*

The following result was proved first by [BPW] as extension of results of [FPPS]. Theorem 2.1 can be considered as its generalization.

Theorem 3.4 [BPW]. *Let $\psi : T^2 \longrightarrow G/T$ be a primitive map and $d\psi(\frac{\partial}{\partial z})$ is cyclic. Then ψ is of finite type.*

This result implies that any double periodic solution to (T) can be obtained from finite dimensional Hamiltonian ODE system (2.1) for G/T.

3.2 Differential geometric characterization. We suppose that G/K is a symmetric space with $T \subset K$ and the projection $\pi : G/T \longrightarrow G/K$. By a result of [Bl], a primitive map ψ into G/T with cyclic $d\psi(\frac{\partial}{\partial z})$ projects a harmonic map $\varphi = \pi \circ \psi$ into G/K. It is a very interesting question how can harmonic maps obtained so from solutions of affine Toda equation for each $\mathfrak{g}$ be characterized in the sense of differential geometry.

In [BPW], in the case when $\mathfrak{g}$ is of type $\mathfrak{a}_n, \mathfrak{b}_n, \mathfrak{d}_n$ or $\mathfrak{g}_2$, they gave differential geometric characterization of harmonic maps so obtained, which were called *superconformal* harmonic maps.

The case $\mathfrak{a}_n$: A harmonic map $\varphi : \Sigma \longrightarrow \mathbf{C}P^n$ is called *superconformal* if φ has isotroy order n. This condition is equivalent to that φ has orthogonally periodic harmonic sequence, that is, $G^{(i+n+1)}(\varphi) = G^{(i)}(\varphi)$ for each $i \in \mathbf{Z}$. Any harmonic map $\varphi : \Sigma \longrightarrow \mathbf{C}P^1$ is holomorphic, anti-holomorphic or superconformal. Any weakly conformal, harmonic map (branched minimal immersion) $\varphi : \Sigma \longrightarrow \mathbf{C}P^2$ is isotropic or superconformal. The solutions to affine Toda field equations of type $\mathfrak{a}_n$ correspond to superconformal harmonic maps into $\mathbf{C}P^n$.

The case $\mathfrak{b}_n$ and $\mathfrak{d}_n$: A full harmonic map $\varphi : \Sigma \longrightarrow S^n$ is called *superconformal* if φ has isotropy order $2m-1$ in the case of $n = 2m$ and φ has isotropy order $2m+1$ in the case $n = 2m+1$. Any weakly conformal harmonic map $\varphi : \Sigma \longrightarrow S^3$ or S^4 is isotropic or superconformal.

In the case of $n = 2m+1$, φ is superconformal if and only if φ has periodic harmonic sequence, that is,

$$\underline{\varphi} \oplus G'(\varphi) \oplus \cdots \oplus G^{(2m+1)}(\varphi) = \underline{\mathbf{C}}^{2m+2}$$

with $\underline{\varphi} \perp G^{(p)}$ for each $1 \leq p \leq$ and $G^{(p)}(\varphi) = G^{(2m+2+p)}(\varphi)$ for each $p \in \mathbf{Z}$. Note that we have $G^{(i)}(\varphi) = \overline{G^{(-i)}(\varphi)}$ for each $p \in \mathbf{Z}$. The branched minimal surface in S^{2m+1} defined by $G^{(m+1)}(\varphi) = G^{(-(m+1))}(\varphi)$ is called a *polar surface* of φ. In the case of $n = 2m$, we should remark that a superconformal harmonic map φ does not always have periodic harmonic sequence. It was shown that the solutions to affine Toda field equations of type $\mathfrak{b}_n$ or $\mathfrak{d}_n$ correspond to superconformal harmonic maps into S^{2n}.

The case $\mathfrak{g}_2$: It is well-known that the 6-dimensional sphere S^6 has the standard nearly Kähler manifold structure. Any almost complex curve S^6 is isotropic or superconformal (see [BPW]). It is shown in [BPW] that the solutions to the affine Toda equation of type $\mathfrak{g}_2$ correspond to superconformal, almost complex curves in S^6, and any non-isotropic almost complex 2-tori in S^6 is of finite type.

The case $(\mathfrak{bc})_1$: More generally, the affine Toda field equation can be defined for each root system, particularly also for nonreduced root systems $(\mathfrak{bc})_\ell$. It is interesting to examine what kind of class of harmonic maps corresponds to solutions of affine Toda equation for a nonreduced root system in the sense of differential geometry. The solutions to the affine Toda field equation of type $(\mathfrak{bc})_1$

$$2\frac{\partial^2}{\partial z \partial \bar{z}}\omega + e^{2\omega} - e^{-4\omega} = 0$$

correspond to non-isotropic totally real minimal surfaces in $\mathbf{C}P^2$. This is studied by J. Inoguchi, who is a graduate student of Tokyo Metropolitan University.

Problem. Classify totally real minimal tori in $\mathbf{C}P^2$.

Some constructions of totally real minimal tori in $\mathbf{C}P^2$ are already known.

Problem. Characterize harmonic maps corresponding to the solutions to affine Toda field equation for other root systems in the sense of differential geoemtry.

Problem. It is known that there is a bijective correspondence between simple root systems and quaternionic Kähler symmetric spaces. Is there a good relationship between a certain class of harmonic maps into a quaternionic Kähler symmetric space and solutions to affine Toda field equation for the correspondsing simple root systems ?

3.3 Soliton theory for elliptic Toda field equations. Theory of solutions to Toda field equation were already established as integrable systems. For applications to harmonic maps, we need to develop theory of solutions to ELLIPTIC Toda field equation. When $\mathfrak{g}$ is of type $\mathfrak{a}_n$, I.McIntosh [Mc1],[Mc2] has discussed soliton theory for elliptic Toda field equations. As the application, he gave a description of solutions to elliptic Toda field equation in terms of θ-functions and a correspondence between superconformal harmonic 2-tori in $\mathbf{C}P^n$ and pairs of spectral curves and certain rational functions (X, π).

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