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ON REPRESENTATIONS OF SUPER-POINCARÉ ALGEBRA

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Abstract. A 4-component spinor covariant derivative representation of a new graded Lie algebra of infinite dimension in the general relativity is obtaind from a 4-component spinor partial derivative representation of a super-Poincaré algebra in the special relativity.

1. Poincaré Algebra. A Poincaré algebra $\mathcal{P}(P_\mu, M_{\mu\nu})$ is a Lie algebra with a set $(P_\mu, M_{\mu\nu})$ of ten generators P_μ, $M_{\mu\nu}$ ($M_{\mu\nu} = -M_{\nu\mu}$; $\mu = 0,1,2,3$) satisfying commutation relations

(1.1) $[P_\mu, P_\nu] = 0,$

(1.2) $[M_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu),$

(1.3) $[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho}),$

where we use $(\eta_{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)$. These relations satisfy the Jacobi identity, namely, the system of equations

(1.4) $\mathfrak{S}_{\mu\nu\rho}[P_\mu, [P_\nu, P_\rho]] = 0,$ (P,P,P),

(1.5) $[P_\mu, [P_\nu, M_{\rho\sigma}]] + [P_\nu, [M_{\rho\sigma}, P_\mu]] + [M_{\rho\sigma}, [P_\mu, P_\nu]] = 0,$ (P,P,M),

(1.6) $[P_\lambda, [M_{\mu\nu}, M_{\rho\sigma}]] + [M_{\mu\nu}, [M_{\rho\sigma}, P_\lambda]] + [M_{\rho\sigma}, [P_\lambda, M_{\mu\nu}]] = 0,$ (P,M,M),

(1.7) $\mathfrak{S}_{(\kappa\mu\rho)(\lambda\nu\sigma)}[M_{\kappa\lambda}, [M_{\mu\nu}, M_{\rho\sigma}]] = 0,$ (M,M,M),

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where $\mathfrak{S}_{\mu\nu\rho}$ denotes the cyclic summation with respect to μ,ν,ρ. The Dirac gamma-matrices γ^μ is connected with the metric $(\eta_{\mu\nu})$ in the anticommutation relations $\{\gamma_\mu,\gamma_\nu\} = 2\eta_{\mu\nu}$. Both spin angular momentum $s_{\mu\nu}$ and orbital angular momentum $\ell_{\mu\nu}$

(1.8) $$s_{\mu\nu}=\frac{1}{2}\sigma_{\mu\nu}, \qquad \ell_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$$

satisfy the equation (1.3), where $\sigma_{\mu\nu}=i[\gamma_\mu,\gamma_\nu]/2$ and (x^μ) denotes a coordinate system of the Minkowski space-time.

A matrix representation (1.9) of a Poincaré algebra $\mathcal{P}(P_\mu,M_{\mu\nu})$ can be given by the fundamental generators P_μ, $M_{\mu\nu}$ defined by

(1.9) $$P_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ 0 & 0 \end{pmatrix}, \qquad M_{\mu\nu} = s_{W\mu\nu}$$

in the full matrix algebra $M_4(\mathbb{C})$, where $s_{W\mu\nu}$ are determined by the Weyl representation of the γ-matrices

$$\gamma^\mu{}_W = \begin{pmatrix} 0 & \eta^{\mu\nu}\sigma_\nu \\ \sigma_\mu & 0 \end{pmatrix}$$

and σ_j (j=1,2,3) denote the Pauli matrices and $\sigma_0 = I_2$.

A partial derivative representation (1.10) of a Poincaré algebra $\mathcal{P}(P_\mu, M_{\mu\nu})$ can be given by the fundamental genetators P_μ, $M_{\mu\nu}$ defined by

(1.10) $$P_\mu = i\partial_\mu, \qquad M_{\mu\nu} = \ell_{\mu\nu}.$$

in a derivation algebra.

Consequently, we have an algebraic representation (1.9) and an analytic representation (1.10) of a Poincaré algebra.

2. Super-Poincaré Algebra. A super-Poincaré algebra $\mathfrak{P}_s(P_\mu, M_{\mu\nu}, Q_\alpha)$ is a graded Lie algebra with a set $(P_\mu, M_{\mu\nu}, Q_\alpha)$ of ten bosonic (even) generators P_μ, $M_{\mu\nu}$ and four fermionic (odd) generators Q_α (α=1,2,3,4) satisfying the relations (1.1)∿(1.3) and

(2.1) $\quad [P_\mu, Q_\alpha] = 0,$

(2.2) $\quad [M_{\mu\nu}, Q_\alpha] = -\frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta ,$

(2.3) $\quad \{Q_\alpha, Q_\beta\} = \frac{1}{2}(\gamma^\mu C)_{\alpha\beta} P_\mu .$

These relations satisfy the super-Jacobi identity, namely, the system of equations (1.4)∿(1.7) and

(2.4) $\quad [P_\mu,[P_\nu,Q_\alpha]]+[P_\nu,[Q_\alpha,P_\mu]]+[Q_\alpha,[P_\mu,P_\nu]] = 0,$ $\quad$ (P,P,Q),

(2.5) $\quad [P_\rho,[Q_\alpha,M_{\mu\nu}]]+[Q_\alpha,[M_{\mu\nu},P_\rho]]+[M_{\mu\nu},[P_\rho,Q_\alpha]] = 0,$ $\quad$ (P,M,Q),

(2.6) $\quad [Q_\alpha,[M_{\mu\nu},M_{\rho\sigma}]]+[M_{\mu\nu},[M_{\rho\sigma},Q_\alpha]]+[M_{\rho\sigma},[Q_\alpha,M_{\mu\nu}]] = 0,$ $\quad$ (M,M,Q),

(2.7) $\quad [P_\mu,\{Q_\alpha,Q_\beta\}]+\{Q_\alpha,[Q_\beta,P_\mu]\}-\{Q_\beta,[P_\mu,Q_\alpha]\} = 0,$ $\quad$ (P,Q,Q),

(2.8) $\quad [M_{\mu\nu},\{Q_\alpha,Q_\beta\}]+\{Q_\alpha,[Q_\beta,M_{\mu\nu}]\}-\{Q_\beta,[M_{\mu\nu},Q_\alpha]\} = 0,$ $\quad$ (M,Q,Q),

(2.9) $\quad \mathfrak{S}_{\alpha\beta\gamma}[Q_\alpha,\{Q_\beta,Q_\gamma\}] = 0,$ $\quad$ (Q,Q,Q).

A matrix representation (2.10) of a super-Poincaré algebra $\mathfrak{P}_s(P_\mu, M_{\mu\nu}, Q_\alpha)$ can be given by P_μ, $M_{\mu\nu}$, Q_α defined by

$$P_\mu = \begin{pmatrix} 0 & \sigma_\mu & 0 \\ & & 0 \\ 0 & 0 & 0 \\ & & 0 \\ 0\;0\;0\;0 & & 0 \end{pmatrix}, \qquad M_{\mu\nu} = \begin{pmatrix} & & 0 \\ & {}^sW_{\mu\nu} & 0 \\ & & 0 \\ & & 0 \\ 0\;0\;0\;0 & & 0 \end{pmatrix},$$

(2.10)
$$Q_1 = \begin{pmatrix} & & 0 \\ & & -1 \\ & 0 & 0 \\ & & 0 \\ 0\;0\;0\;0 & & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} & & 1 \\ & & 0 \\ & 0 & 0 \\ & & 0 \\ 0\;0\;0\;0 & & 0 \end{pmatrix},$$

$$Q_3 = \begin{bmatrix} & & & & 0 \\ & & 0 & & 0 \\ & & & & 0 \\ & & & & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \qquad Q_4 = \begin{bmatrix} & & & & 0 \\ & & 0 & & 0 \\ & & & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

in the full matrix algebra $M_5(\mathbb{C})$. Here we use the charge conjugation C of the form $C = i\gamma_W^0 \gamma_W^2$. We note that only units ± 1, $\pm i$ of the Gauss' complex integer ring $\mathbb{Z}[\mathbb{C}]$ and their halves have chosen as non-zero components of these matrices (cf.1)∿5)).

A 4-component spinor partial derivative representation (2.11) of a super-Poincaré algebra $\mathfrak{P}_s(P_\mu, M_{\mu\nu}, Q_\alpha)$ can be given by P_μ, $M_{\mu\nu}$, Q_α defined by

(2.11)
$$P_\mu = i\partial_\mu, \qquad Q_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{i}{4}(\gamma^\mu C)_{\alpha\beta}\theta^\beta \partial_\mu .$$
$$M_{\mu\nu} = \ell_{\mu\nu} + \underset{\sim}{s}_{\mu\nu}, \qquad \text{where } \underset{\sim}{s}_{\mu\nu} = (s_{\mu\nu})_\alpha{}^\beta \theta^\alpha \frac{\partial}{\partial\theta^\beta}$$

with Grassmann variables θ^α in a derivation-antiderivation algebra of ∂_μ and $\partial/\partial\theta^\alpha$.

3. Covariant Derivation Algebra $\mathfrak{P}_{gs}$ on a Curved Space-time.

We solve a problem: Can one find a covariant derivative representation, $(P_\mu(\equiv i\nabla_\mu)$, M's, Q's), of a graded Lie algebra on a curved space-time from $(P_\mu(\equiv i\partial_\mu), M_{\mu\nu}, Q_\alpha)$ of (2.11) on the flat space-time ? For this purpose we assume that $\gamma^\mu(x)$ satisfies

(3.1)
$$\gamma_\mu(x)\gamma_\nu(x) + \gamma_\nu(x)\gamma_\mu(x) = 2\, g_{\mu\nu}(x) ,$$

and a covariant derivation ∇_μ satisfies

(3.2)
$$\nabla_\mu \gamma_\nu(x) = 0,$$

on a curved space-time with a metric $g_{\mu\nu}(x)$.[6)]

By a straightfoward calculation we see that the following generators are solutions of the super-Jacobi identity, the system of equations (1.4)∿(1.7) and (2.4)∿(2.9):

$$P_\mu = i\nabla_\mu , \qquad Q_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{i}{4}(\gamma^\mu(x)C)_{\alpha\beta}\theta^\beta\nabla_\mu ,$$

(3.3)

$$M(\xi)_{\mu\nu} = \ell(\xi)_{\mu\nu} + \underline{s}_{\mu\nu}(x), \quad \text{where} \quad \ell(\xi)_{\mu\nu} \equiv i(\xi_\mu\nabla_\nu - \xi_\nu\nabla_\mu)$$

for an arbitrary vector field ξ^μ. The commutation and anticommutation relations of these generators are written as

(3.4) $[P_\mu , P_\nu] = -[\nabla_\mu , \nabla_\nu]$,

(3.5) $[M(\xi)_{\mu\nu} , P_\rho] = \mathcal{A}_{\mu\nu}\{(P_\rho\xi_\nu)P_\mu + \xi_\nu[P_\rho , P_\mu]\}$,

(3.6) $[M(\xi)_{\mu\nu} , M(\xi)_{\rho\sigma}] = [s_{\mu\nu} , s_{\rho\sigma}] + \mathcal{A}_{\mu\nu}\mathcal{A}_{\rho\sigma}\{\mathcal{A}_{(\mu\rho)(\nu\sigma)}(P_\mu\xi_\sigma)\xi_\nu P_\rho + \xi_\mu\xi_\rho[P_\nu , P_\sigma]\}$,

(3.7) $[P_\mu , Q_\alpha] = \frac{1}{4}(\gamma^\rho C)_{\alpha\beta}\theta^\beta[P_\mu , P_\rho]$,

(3.8) $[M(\xi)_{\mu\nu} , Q_\alpha] = -(s_{\mu\nu})_\alpha{}^\beta\frac{\partial}{\partial\theta^\beta} + \frac{1}{4}(s_{\mu\nu}\gamma^\lambda C)_{\beta\alpha}\theta^\beta P_\lambda + \frac{1}{4}(\gamma^\lambda C)_{\alpha\beta}\theta^\beta[M(\xi)_{\mu\nu} , P_\lambda]$,

(3.9) $\{Q_\alpha , Q_\beta\} = \frac{1}{2}(\gamma^\mu C)_{\alpha\beta}P_\mu + \frac{1}{16}(\gamma^\mu C)_{\alpha\gamma}(\gamma^\nu C)_{\beta\delta}\theta^\gamma\theta^\delta[P_\mu , P_\nu]$,

where $\mathcal{A}_{\mu\nu}$ denotes the alternating summation with respect to μ and ν, and

$$[s_{\mu\nu} , s_{\rho\sigma}] \equiv ([s_{\mu\nu} , s_{\rho\sigma}])_\alpha{}^\beta\theta^\alpha\frac{\partial}{\partial\theta^\beta} .$$

Consequently, we have obtained a 4-component spinor covariant derivative representation (3.3) of a graded Lie algebra. We note that this covariant derivation algebra, say quasi-super-Poincaré algebra $\mathfrak{P}_{qs}$ ($P_\mu (\equiv i\nabla_\mu)$, $M(\xi)_{\mu\nu}$, Q_α), on the curved space-time is of infinite dimension because $[P_\mu , P_\nu] \neq 0$, and that the cyclic sum $\mathfrak{S}_{\mu\nu\rho}[\nabla_\mu , [\nabla_\nu , \nabla_\rho]] = 0$ is nothing but the Bianchi identity of the connection ∇_μ .

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