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Kyoto University
LOGICAL FORMULAS FOR PETRI NET
\( \omega \)-LANGUAGES

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Abstract

In this paper, we study Petri net \( \omega \)-languages and logical formulas defining \( \omega \)-languages. We consider some accepting conditions for Petri nets, and characterize the classes of Petri net \( \omega \)-languages with these accepting conditions by logical formulas.

1 Preliminary

The set of integers \( \{0, 1, -1, 2, -2, \ldots \} \) is denoted by \( \mathbb{Z} \), and the set of nonnegative integers is denoted by \( \mathbb{N} \). For sets \( X \) and \( Y \), \( Y^X \) denotes the set \( \{f \mid f : X \to Y \} \) of all functions from \( X \) to \( Y \). For a finite set \( X = \{x_1, x_2, \ldots, x_n\} \), a function \( f \in \mathbb{Z}^X \) is identified with the \( n \)-dimensional vector \((f(x_1), f(x_2), \ldots, f(x_n))\). Then for functions \( f, g \in \mathbb{Z}^X \) and \( z \in \mathbb{Z} \), the addition \( f + g \), the scalar product \( zf \), and the partial ordering \( f \leq g \) are defined componentwise as usual.

Let \( \Sigma \) be an alphabet. We call a mapping \( \alpha \in \Sigma^\mathbb{N} \) an \( \omega \)-word over \( \Sigma \), and write \( \alpha = a_0a_1a_2\ldots \) where \( a_n = \alpha(n) \) for each \( n \). The set of all \( \omega \)-words over \( \Sigma \) is denoted by \( \Sigma^\omega \), and that of all finite words over \( \Sigma \) is denoted by \( \Sigma^* \) as usual.

If \( u = \alpha(0)\ldots\alpha(n) \) for some \( n \), then \( u \) is called a prefix of \( \alpha \) and we write \( u \prec \alpha \). For \( \alpha \in \Sigma^\omega \), we define \( \downarrow \alpha = \{v \in \Sigma^* \mid v \prec \alpha\} \); \( \alpha = \{a \mid a = \alpha(n) \text{ for some } n\} \), and \( \alpha = \{a \mid a = \alpha(n) \text{ for infinitely many } n\} \). For \( L \subseteq \Sigma^\omega \), we define \( \downarrow L = \bigcup_{\alpha \in L} \downarrow \alpha \).

For \( K \subseteq \Sigma^* \) and \( L \subseteq \Sigma^\omega \), we define \( KL = \{u\alpha \mid u \in L \text{ and } \alpha \in K\} \) and \( K^\omega = \{v_1v_2\ldots v_1, v_2, \ldots \in K - \{\epsilon\}\} \), where \( u\alpha \) is the \( \omega \)-word obtained by concatenating \( u \) before \( \alpha \), and \( v_1v_2\ldots \) is the \( \omega \)-word obtained by concatenating \( v_1, v_2, \ldots \) one after another.

We can consider \( \Sigma^\omega \) a metric space with the distance \( d \) defined by:

\[
d(\alpha, \beta) = \begin{cases} 
0, & \text{if } \alpha = \beta \\
2^{-k}, & \text{if } \alpha \neq \beta \text{ and } k = \text{Min}\{n \mid \alpha(n) \neq \beta(n)\}.
\end{cases}
\]

Then \( L \subseteq \Sigma^\omega \) is a closed set if and only if \( L = \{\alpha \mid \downarrow \alpha \subseteq \downarrow L\} \).

In this paper, when we mention a net or a Petri net \( N \), we mean a marked \( \lambda \)-free labelled Petri net \( N = (P, T, A, e, m_0, F) \), where \( P \) is a finite set of places, \( T \) a finite set of transitions, \( A : T \to \mathbb{N}^P \times \mathbb{N}^P, e \in \Sigma^T \) a \( \lambda \)-free labelling function, \( m_0 \in \mathbb{N}^P \) an initial marking, and \( F \subseteq \mathbb{N}^P \) a finite set of accepting markings.

A marking \( m \) of a Petri net \( N \) is a function in \( \mathbb{N}^P \), i.e., an assignment of tokens to the places. We say that the place \( p \) has \( m(p) \) tokens at the marking \( m \). For each transition \( t \), \( A(t) = \langle \ast A(t), A(t)^\ast \rangle \) assigns a pair of functions \( \ast A(t) \) and \( A(t)^\ast \) called the input and output vector of \( t \), respectively.

Example 1

Let \( N = (\{p\}, \{s, t\}, A, e, \langle 0, \{0\}, \{\{2\}\} \rangle) \), where \( A(s) = \langle \{0\} \}, A(t) = \langle \{1\}, \{0\} \rangle, e(s) = a \)
and $e(t) = b$. Then the Petri net $N$ is illustrated as follows.

```
  a  b
   |   |
   p   t
```

A transition $t$ is fireable in a marking $m$ if $m \geq *A(t)$, and if so, $t$ may be fired at $m$ resulting in the marking

$$m' = m - *A(t) + A(t)^*.$$ 

In this case, we write $m[t]$ or $m[t]m'$. Intuitively, $t$ removes $*A(t)(p)$ tokens from the place $p$, and distributes $A(t)^*(p)$ tokens to $p$, when $t$ fires.

The definitions and notations are extended to finite or infinite sequences of transitions. That is, $m[t_1t_2\ldots t_n]$ or $m[t_1t_2\ldots t_n]m'$ if $m[t_1]m_2\ldots m_{n-1} [t_n]m'$, and $m(\alpha)$ if $m(\alpha(0))m_1(\alpha(1))m_2\ldots$.

We define infinite behaviour of a Petri net $N$ as the homomorphic image of infinite firing sequences by the $\lambda$-free labelling function $e$. For a Petri net $N = (P,T,A,e,m_0,F)$ and $\alpha \in T^\omega$, we define $N(\alpha) = m_0m_1m_2\ldots$ if $m(\alpha(0))m_1(\alpha(1))m_2\ldots$. Let

$$\uparrow F = \{m' | m' \geq m \text{ for some } m \in F\}.$$ 

Then we consider the following five types of $\omega$-languages accepted by $N$:

$$L_0(N) = \{e(\alpha) | m_0(\alpha)\},$$

$$L_1(N) = \{e(\alpha) | N(\alpha) \cap \uparrow F \neq \phi\},$$

$$L_2(N) = \{e(\alpha) | \overline{N(\alpha)} \subseteq \uparrow F\},$$

$$L_3(N) = \{e(\alpha) | N(\alpha) \cap \uparrow F \neq \phi\},$$

$$L_4(N) = \{e(\alpha) | \overline{N(\alpha)} \subseteq \uparrow F\}.$$ 

We define $P_i = \{L_i(N) | N \text{ is a Petri net over } \Sigma\}$ ($i = 0, \ldots, 4$). The accepting conditions considered in [2, 3] are defined by $F$ instead of $\uparrow F$.

**Example 2** For the Petri net $N$ in the previous example, $L_0(N) = \{\alpha | \#_a(u) \geq \#_b(u) \text{ for any } u < \alpha\}$, $L_1(N) = L_0(N) - (ab)^\omega$, $L_2(N) = \phi$, $L_3(N) = L_0(N) - D(ab)^\omega$, and $L_4(N) = \{u | \#_a(u) = \#_b(u) + 2\}L_0(N) \cap L_0(N)$, where $\#_a(u)$ is the number of occurrence of the letter $a$ in the string $u$.

Let $M = (Q, \Sigma, \delta, s, F)$ be a nondeterministic finite automaton with the finite set $Q$ of states, the input alphabet $\Sigma$, the transition relation $\delta \subseteq Q \times \Sigma \times Q$, the initial state $s$, and the set $F$ of accepting states. Any $\alpha = \langle q_0, a_0, p_0 \rangle \langle q_1, a_1, p_1 \rangle \langle q_2, a_2, p_2 \rangle \ldots \in \delta^\omega$ is called a run of $M$, if $q_0 = s$ and $q_i = q_{i+1}$ for any $i$. For a run $\alpha$ of $M$, we define $M(\alpha) = q_0q_1q_2q_3\ldots$ and $\Sigma(\alpha) = a_0a_1a_2\ldots$.

Then we can also define the following five types of $\omega$-languages accepted by $M$:

$$L_0(M) = \{\Sigma(\alpha) | \alpha \text{ is a run of } M\},$$

$$L_1(M) = \{\Sigma(\alpha) | M(\alpha) \cap F \neq \phi\},$$

$$L_2(M) = \{\Sigma(\alpha) | \overline{M(\alpha)} \subseteq F\},$$

$$L_3(M) = \{\Sigma(\alpha) | M(\alpha) \cap F \neq \phi\},$$

$$L_4(M) = \{\Sigma(\alpha) | \overline{M(\alpha)} \subseteq F\}.$$ 

We define $E_i = \{L_i(M) | M \text{ is a nondeterministic finite automaton over } \Sigma\}$ ($i = 0, \ldots, 4$).

## 2 Inclusion relations

In the case of $\omega$-languages accepted by nondeterministic finite automata, it is known that

$$E_0 = E_2 \subset E_1 = E_4 \subset E_3 \subset \{4, 5, 7\}.$$ 

We show the similar results for the classes $P_i$ of Petri net $\omega$-languages.

As a tool of the proofs in this section, we define a new accepting condition for a Petri net, which is described by a language over
transitions. Let \( N = (P, T, A, e, m_0, \phi) \) and \( R \subseteq T^\omega \). We define

\[
L(N, R) = \{e(\alpha) \mid m_0[\alpha] \text{ and } \alpha \in R\}.
\]

In the proof of the following theorems, we use the following notations to simplify the description. For \( f \in \mathbb{Z}^X \) and \( g \in \mathbb{Z}^Y \), \( f \oplus g \) denotes the function in \( \mathbb{Z}^{X \cup Y} \), defined by

\[
f \oplus g(z) = \begin{cases} 
  f(z) + g(z), & \text{if } z \in X \cap Y \\
  f(z), & \text{if } z \in X \\
  g(z), & \text{if } z \in Y.
\end{cases}
\]

For \( n \in \mathbb{N} \) and a set \( X \), \( n^X \) denote the constant function in \( \mathbb{N}^X \) such that \( n^X(z) = n \) for any \( z \in X \). If \( X \) is a singleton \( \{z\} \), then we write \( n^z \) instead of \( n^{\{z\}} \). Thus, for example, for \( p_0 \in P \),

\[
0^p \oplus 1^p(p) = \begin{cases} 
  1, & \text{if } p = p_0 \\
  0, & \text{if } p \neq p_0.
\end{cases}
\]

**Theorem 1** For any \( i = 0, \ldots, 4 \), \( P_i = \{L(N, R) \mid N \) is a Petri net and \( R \in E_i\} \).

**Proof.** Let \( N = (P, T, A, e, m_0, \phi) \) and \( M = (Q, \Sigma, \delta, s, F) \) be a finite automaton such that \( L_i(M) = R \). We define the Petri net \( N' = (P \cup Q, \delta, A', e', m_0 \oplus 1^S \oplus 0^Q, \{0^p \oplus 1^q \oplus 0^Q \mid q \in E \}) \), where \( A'(\langle q, t, q' \rangle) = \langle A(t) \oplus 1^q \oplus 0^Q, A(t)^* \oplus 1^q \oplus 0^Q \rangle \), and \( e'(\langle q, t, q' \rangle) = e(t) \) for any \( \langle q, t, q' \rangle \). Intuitively, \( N' \) is a product of \( N \) and \( M \), and simulates \( N \) and \( M \), simultaneously. Thus it is clear that \( L(N, R) = L(N, L_i(M)) = L_i(N') \).

Let \( N = (P, T, A, e, m_0, F) \) and \( L = L_i(N) \). For each \( t \in T \) and \( m \in F \), we add new transition \( t_m \) to \( N \), such that \( m_1[t_m]m_2 \) if and only if \( m_1 \geq m \) and \( m_1[t_m]m_2 \). Since \( m_1 \geq m \in F \) means \( m_1 \in \uparrow F \), \( t_m \) works same as \( t \), and can check whether the current marking is in \( \uparrow F \) or not.

We construct \( N' = (P, T \cup T_F, A', e', m_0, \phi) \), where \( T_F = \{t_m \mid t \in T \) and \( m \in F \} \), \( A'(t) = A(t) \) and \( e'(t) = e'(t_m) = e(t) \), for each \( t \in T \) and \( m \in F \). Moreover, \( A'(t_m) = \langle A'(t_m), A'(t_m)^* \rangle \) with

\[
A(t_m)(p) = \text{Max}(A(t)(p), m(p)),
\]

\[
A(t_m)^*(p) = A(t_m)(p) + A(t)^*(p) - A(t)(p),
\]

for any \( p \in P \).

Then it is clear that \( L_0(N) = L(N', T^\omega) \), \( L_1(N) = L(N', T^*T_F T^\omega) \), \( L_2(N) = L(N', T^F) \), \( L_3(N) = L(N', T^*T_F T^\omega) \), \( L_4(N) = L(N', T^*T^*_F \omega) \). \( \square \)

**Corollary 2** \( P_0 \subseteq P_1 \subseteq P_4 \subseteq P_3 \).

**Proof.** It is clear from the Theorem 1 and the results for \( E_i \)'s. \( \square \)

In the sequel, we only consider the case \( i = 0, 1, 3 \). To prove the strict inclusions between these classes, we prove the following topological properties of the classes \( P_0 \) and \( P_1 \).

**Lemma 3** For any Petri net \( N \), \( L_0(N) \) is a closed set, and \( L_1(N) \) is a denumerable union of closed sets.

**Proof.** Let \( N = (P, T, A, e, m_0, F) \), and \( \downarrow \alpha \subseteq \downarrow L_0(N) \). We will show that \( \alpha \in L_0(N) \). Consider the set \( C = \{w \mid e(w) < \alpha, \) and \( m_0[w]\} \) of all the fireable finite sequences generating the prefixes of \( \alpha \). Then \( C \) is infinite. By König's Lemma, there exists \( \beta \in T^\omega \) such that \( \downarrow \beta \subseteq C \). It means that \( m_0[\beta] \) and \( e(\beta) = \alpha \). Hence \( \alpha \in L \).

Let \( N_m = (P, T, A, e, m, F) \) for \( m \in \mathbb{N} \).

Then, \( L_1(N) = \bigcup \{e(w)L_0(N_m) \mid m_0[w]m \in \uparrow F\} \), which is a denumerable union of closed sets. \( \square \)

Then the next theorem follows from the topological characterizations of \( \omega \)-regular languages [4, 5].

**Theorem 4** \( P_0 = P_2 \subseteq P_1 = P_4 \subseteq P_3 \).
Theorem 5 The classes $P_i$ ($i = 0, 1, 3$) of Petri net $\omega$-languages are closed under union, intersection, and projection.

Proof. Let $N_j = (P_j,T_j,A_j,e_j,m_j,\phi)$ for $j = 1, 2$. We define a Petri net $N$ which can simulate $N_1$ and $N_2$ simultaneously, as follows. $N = (P_1 \cup P_2, T, A, e, m_1 \oplus m_2, \phi)$, where $T = \{ (t_1, t_2) \in T_1 \times T_2 \mid e_1(t_1) = e_2(t_2) \}$, $A'((t_1, t_2)) = \langle *A_1(t_1) \oplus *A_2(t_2), A_1(t_1)^* \oplus A_2(t_2)^* \rangle$, $e((t_1, t_2)) = e_1(t_1)$, for any $(t_1, t_2) \in T$. For any $R_j \subseteq T_j^\omega$ ($j = 1, 2$), let $R_0 = \{ \alpha \mid \alpha \in R_1 \text{ or } \alpha \in R_2 \}$, $R_n = \{ \alpha \mid \alpha \in T^\omega \text{ and } \alpha_i \in R_1 \text{ or } \alpha_i \in R_2 \}$, where $\alpha_i$ is the $\omega$-word over $T_j$ obtained by concatenating $j$-th elements of $\alpha(i)$ for $i = 0, 1, \ldots$. Then it is clear that $L(N_1, R_1) \cup L(N_2, R_2) = L(N, R_0)$, $L(N_1, R_1) \cap L(N_2, R_2) = L(N, R_n)$.

The closure under projection is clear from the definition. □

3 Normal form of Petri nets

We define a normal form of Petri nets and show that any Petri net can be transformed into a normal form Petri net.

We say that a Petri net $N = (P, T, A, e, m_0, F)$ is in normal form if
1) there exists a place $p_0 \in P$ such that $m_0 = 1^{p_0} \oplus 0^P$,
2) there exists a place $p_f$ such that $F = \{ 1^{p_f} \oplus 0^P \}$,
3) for any transition $t$ fireable at markings in $\uparrow F$, $*A(t)(p_f) = 1$,
4) for any $p \in P$, and $t \in T$, $*A(t)(p) \leq 1$ and $A(t)^*(p) \leq 1$, that is, each place $p$ gets or lose at most one token at once.

Theorem 6 For any Petri net $N$, we can construct a Petri net $N'$ in normal form such that $L_i(N) = L_i(N')$ for any $i = 0, 1, 3$.

Proof. First we show that any Petri net $N = (P, T, A, e, m_0, F)$ can be transformed into a Petri net $N' = (P', T', A', m'_0, e', F')$ which satisfies the conditions 1), 2) and 3). Let $P' = P \cup \{ p_0, p_e, p_f \}$ and $T' = T \cup \{ t' \mid m_0(t) \} \cup \{ t'' \mid t \in T \} \cup \{ t_m \mid t \in T$ and $m \in N \}$. We define

$A'(t) = \langle *A(t) \oplus 0^{p_0} \oplus 1^{p_c} \oplus 0^{p_f}, A'(t) \rangle$

$A'(t') = \langle 0^P \oplus 1^{p_0} \oplus 0^{p_c} \oplus 0^{p_f} \rangle$

$(m_0 - \cdot A(t) + A(t)^* \oplus 0^{p_0} \oplus 1^{p_c} \oplus 0^{p_f})$

$A'(t'') = \langle *A(t) \oplus 0^P \oplus 1^{p_c} \oplus 1^{p_f}, A(t)^* \oplus 0^P \oplus 0^{p_c} \oplus 0^{p_f} \rangle$

$A'(t_m) = \langle *A(t_m) \oplus 0^P \oplus 1^{p_c} \oplus 1^{p_f}, A(t_m)^* \oplus 0^P \oplus 0^{p_c} \oplus 1^{p_f} \rangle$

$e'(t) = e'(t) = e'(t') = e'(t_m) = e(t)$

$m'_0 = 1^{p_0} \oplus 0^P$, and $F = \{ 1^{p_f} \oplus 0^P \}$.

Then the Petri net $N'$ satisfies 1), 2) and 3), and it is clear from the construction that $L_i(N) = L_i(N')$ for $i = 0, 1, 3$.

Next we show that we can decrease the number of places $q \in P'$ such that $Max \{ *A'(t)(q), A'(t)^*(q) \mid t \in T \} = n > 1$. Repeating the process, we can transform $N'$ into a Petri net in normal form.

To construct $N'' = (P'', T'', A'', e'', m''_0, F'')$, we replace $q$ by $n$ new places $q_1, q_2, \ldots, q_n$. For each transition $t$, let $D_i (1 \leq i \leq k_t)$ and $E_j (1 \leq j \leq l_t)$ be the enumerations of the subsets of $\{ q_1, q_2, \ldots, q_n \}$ with $*A'(t)(q)$ and $A'(t)^*(q)$ elements, respectively. Then we also replace the transition $t$ by $n_t \times m_t$ transitions $t_{i,j}$ ($1 \leq i \leq k_t$, $1 \leq j \leq l_t$) such that

$A''(t_{i,j})(p) = \begin{cases} *A'(t)(p), & \text{if } p \neq q \\ 1, & \text{if } p \in D_i \\ 0, & \text{if } p \not\in D_i \end{cases}$

$A''(t_{i,j})^*(p) = \begin{cases} A'(t)^*(p), & \text{if } p \neq q \\ 1, & \text{if } p \in E_j \\ 0, & \text{if } p \not\in E_j \end{cases}$

and $e''(t_{i,j}) = e(t)$. 
Note that on the Petri net $N''$, the tokens in $q$ on $N'$ are distributed to the places $q_1, q_2, \cdots, q_n$, and the arcs from or to $q$ in $N'$ are also distributed to these places.

It is easy to see that the $L_i(N') = L_i(N'')$ for $i = 0, 1, 3$. □

4 Characterizations by logical formulas

We define the monadic second-order theory $K$ over an alphabet $\Sigma$ for natural numbers, which is introduced by Parigot and Pez [2, 3]. $K$ has two sorts of variables, number variables $x, y, \ldots$ ranging over $\mathbb{N}$, and set variables $X, Y, \ldots$ ranging over the power set of $\mathbb{N}$. $K$ also has set constants $P_a$ for each $a \in \Sigma$.

The terms of $K$ are expressions of form $n$ or $x + n$, where $x$ is a number variable and $n$ is a constant in $\mathbb{N}$. The atomic formulas of $K$ are expressions of form $u \leq t$, $t \in W$ or $V \leq W$, where $u, t$ are terms and $V, W$ are set variables or $P_a$ for some $a \in \Sigma$. Here, $\leq$ and $\in$ are usual 'less than or equal to' and 'belong to' relations, and $V \leq W$ is true if and only if there exists a one to one function $f: W \to V$ such that $f(x) \leq x$ for any $x \in W$.

The formulas of $K$, called $K$-formulas, are defined as usual. That is, $\varphi \land \psi, \varphi \lor \psi, \lnot \varphi, \forall x \varphi, \exists x \varphi, \forall X \varphi, \exists X \varphi$ are formulas for any formula or atomic formula $\varphi, \psi$, number variable $x$ and set variable $X$. We use bold-face quantifier symbols $\forall$ and $\exists$ for set variables to distinguish from those for number variables.

Note that the $K$-formulas not containing the symbol $\leq$ is the S1S-formulas considered in Büchi [1].

We say that an $\omega$-word $\alpha \in \Sigma^\omega$ satisfies $K$-sentence (i.e., formulas without free variables) $\psi$, if $\psi$ is true under the interpretation $P_a = \{n | \alpha(n) = a\}$. Then, $K$-sentence $\psi$ define the set $L(\psi)$ of all $\omega$-words satisfying $\psi$. For a set of $K$-formulas $\Delta$, we define that $L(\Delta) = \{L(\psi) | \psi \in \Delta\}$, the class of $\omega$-languages defined by the sentences in $\Delta$.

For a language $R$ over quantifier symbols $\{\forall, \exists, \forall, \exists\}$, $[R]$ denotes the set of S1S-formulas of the prenex normal form

$\Xi_1 \xi_1 \Xi_2 \xi_2 \cdots \Xi_i \xi_i$, $\psi(\xi_1, \xi_2, \cdots, \xi_i)$,

where $\Xi_1 \Xi_2 \cdots \Xi_i$ is a string in $R$, and $\psi$ is a quantifier-free formula.

On the relation between S1S-formulas and $\omega$-regular languages, we have shown the following theorem [6].

Theorem 7 $E_0 = L(\exists^{*} \forall)$,

$E_1 = L(\exists^{*} \exists^{*} \forall^{*} \exists)$,

$E_3 = L(\exists^{*} \forall^{*} \exists^{*} \forall)$.

For any $\alpha \in (\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n)^\omega$, $\alpha_i$ is defined to be the $\omega$-words obtained by concatenating the $i$-th elements of $\alpha(j)$ for $j = 0, 1, 2, \ldots$. We say that $\alpha \in (\{0, 1\}^{n+k} \times \Sigma)^\omega$ satisfies the formula $\psi(X_1, \ldots, X_n, x_1, \ldots, x_k)$, if $\alpha_{n+i} \in \Sigma^\omega$ satisfies $\psi(C_1, \ldots, C_n, d_1, \ldots, d_k)$, where $C_i = \{j | \alpha_i(j) = 1\}$ for $i = 1, \ldots, n$ and $\alpha_{n+i+1} = 1$ if and only if $j = d_i$ for $i = 1, \ldots, k$. We write $L(\alpha) = \{\alpha | \alpha$ satisfies $\psi\}$.

Now, we show the main theorem. Let $\Delta$ be a set of formulas and $\overline{\Delta}$ be the smallest set of formulas constructed from the atomic formulas $V \leq W$ and formulas in $\Delta$ using $\land, \lor, \forall, \exists$.

Theorem 8 If $L(\Delta) = E_i$, then $L(\overline{\Delta}) = P_i$ for $i = 0, 1, 3$.

Proof. $E_i \subseteq P_i$ from Theorem 1. Moreover the $\omega$-language $L(X_1 \leq X_2) \subseteq (\{0, 1\}^2)^\omega$ is accepted by the following Petri net without the accepting condition.

\[
\begin{array}{cccc}
\langle 0, 0 \rangle & \langle 1, 1 \rangle & \langle 1, 0 \rangle & \langle 0, 1 \rangle \\
\end{array}
\]
Since each class of Petri net ω-languages is closed under union, intersection and projection, we have shown that the half part of the theorem.

Let \( N = (P, T, A, e, 0^P \oplus 1^P, \{0^P \oplus 1^P\}) \) be a Petri net in normal form. Note that in normal form Petri net, each place \( p \) can get or lose at most one token at once. To describe the infinite behaviour of the net \( N \), we use the following set variables,
- \( X_t \) to represent the time \( t \) fires,
- \( E^p \) to represent the time \( p \) gets a new token,
- \( S^p \) to represent the time \( p \) loses one token,
for each \( t \in T \) and \( p \in P \). Let

\[
\psi_1(z) = \bigvee_{t \in T} ((z \in X_t) \land (z \in P_{d(t)}))
\]
\[
\quad \land \left( \bigwedge_{t' \neq t} \neg(z \in X_{t'}) \right)
\]

which means that there exists a unique transition \( t \) that fires at time \( z \).

\[
\psi_2(z) = \bigwedge_{p \in P} ((z \in S_p) \land (z \in X_{t}))
\]
\[
\quad \leftrightarrow \bigvee_{t \in T} \ast A(t)(p) = 1
\]

which means that a place \( p \) loses a token at time \( z \) if and only if a transition \( t \) with \( \ast A(t)(p) = 1 \) fires at the same time \( z \).

\[
\psi_3(z) = \bigwedge_{p \in P} ((z + 1 \in E_p) \land (z \in X_{t}))
\]
\[
\quad \leftrightarrow \bigvee_{t \in T} \ast A(t)(p) = 1
\]

which means that a place \( p \) gets a token at time \( z + 1 \) if and only if a transition \( t \) with \( \ast A(t)(p) = 1 \) fires at time \( z \).

\[
\psi_4 = (0 \in E_{p_0}) \land (\bigwedge_{p \neq p_0} \neg(0 \in E_p))
\]

which represents the condition for the initial marking.

\[
\varphi_0 = true,
\]

\[
\varphi_1(y) = (y \in E_{p_f}),
\]

\[
\varphi_3(z, y) = (x \leq y) \land (y \in E_{p_f}),
\]

here \( \varphi_i \) is a formula to represent the accepting condition of type \( i, (i = 0, 1, 3) \). Finally,

\[
\psi_5 = \bigwedge_{p \in P} (E_p \preceq S_p)
\]

which means that each place \( p \) can lose only tokens which got previously.

Then, \( L_0(N) \) is defined by

\[
\exists_t \in T X_t \exists_p \in P E_p \exists_p \in P S_p (\forall x)\psi_5
\]
\[
(\psi_1(x) \land \psi_2(x) \land \psi_3(x) \land \psi_4 \land \varphi_0 \land \psi_5).
\]

\( L_1(N) \) is defined by

\[
\exists_t \in T X_t \exists_p \in P E_p \exists_p \in P S_p (\forall x \exists y)\psi_5
\]
\[
(\psi_1(x) \land \psi_2(x) \land \psi_3(x) \land \psi_4 \land \varphi_1(y) \land \psi_5).
\]

\( L_3(N) \) is defined by

\[
\exists_t \in T X_t \exists_p \in P E_p \exists_p \in P S_p (\forall y \exists x)\psi_5
\]
\[
(\psi_1(x) \land \psi_2(x) \land \psi_3(x) \land \psi_4 \land \varphi_3(y) \land \psi_5).
\]

Note that all \( \psi_1, \ldots, \psi_4 \) and \( \varphi_i \) (\( i = 0, 1, 3 \)) are S1S-formulas with no quantifiers. From Theorem 7, this completes the proof. □

References


