Extremal Problems and Ramsey Properties of Ball, Box or Orthant containing many points in $\mathbb{R}^d$ — And Combinatorics of Permutations

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1 Ball and Box

For any points $x = (x_1, \cdots, x_d), y = (y_1, \cdots, y_d) \in \mathbb{R}^d$, let $Box_d(x, y)$ be the smallest $d$-dimensional standard box in $\mathbb{R}^d$ which contains the two point $x, y$, i.e.

$$Box_d(x, y) := \{z = (z_i) \in \mathbb{R}^d | x_i \leq z_i \leq y_i \text{ or } x_i \geq z_i \geq y_i \text{ for any } 1 \leq i \leq d\} - \{x, y\}.$$ 

And let $Ball_d(x, y)$ be the smallest $d$-dimensional ball in $\mathbb{R}^d$ which contains the two points $x, y \in \mathbb{R}^d$, i.e.

$$Ball_d(x, y) := \left\{\frac{1}{2}(x + y) + r \mid \|r\| \leq \frac{1}{2}\|x - y\|\right\} - \{x, y\},$$

where $\|\cdot\|$ means the euclidean norm.

For any positive integers $d, n$, if $F = Box$ or $Ball$, then we define $\Pi^F(n, d)$ the largest number which satisfies the condition (*) "For any set $P$ of $n$ points in $\mathbb{R}^d$, there exist two points $x, y \in P$ such that $F_d(x, y)$ contains $\Pi^F(n, d)$ points of $P."$

When "For any set $P$ " is replaced by "For any convex set $P$" in (*), we denote $\Pi^F(n, d)$ by $\overline{\Pi}^F(n, d)$.

Clearly, $\Pi^{Box}(n, 1) = \Pi^{Ball}(n, 1) = n$.

**Proposition 1** $\Pi^{Box}(n, 2) = \left\lceil \frac{n - 4}{5} \right\rceil$, $\Pi^{Box}(n, 2) = \left\lceil \frac{n}{2} \right\rceil - 1.$

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Theorem 2 \( \Pi^{Ball}(n, 2) = \overline{\Pi}^{Ball}(n, 2) = \left\lceil \frac{n}{3} \right \rceil - 1 \).

J. Urrutia conjectured \( \overline{\Pi}^{Ball}(n) \geq n/2 \). Theorem 2 disprove it.

Theorem 3 For any integer \( n, d( \geq 1) \),
\[
\left( \frac{2}{8^{d-1}} \right) n \leq \Pi^{Box}(n, d) \leq \left( \frac{2}{1.15^{d}} \right) n.
\]

Theorem 4 For any integer \( n, d( \geq 1) \),
\[
\left( \frac{2}{8^{d}} \right) n - 2 \leq \Pi^{Ball}(n, d) \leq \left( \frac{2}{1.15^{d}} \right) n.
\]

It is interesting to compare Theorem 3 with Erdős-Szekeres Theorem (\( R^{d+1} \)-version).

2 Orphant and Permutaion

N.G. de Bruijn extended the Erdős-Szekeres Theorem “Any sequence of integers of length \( n \) contains a monotone subsequence of length \( \left\lceil \sqrt{n} \right \rceil \) (best possible)” to a result about sequences of \( d \)-dimensional vectors, which includes the following proposition:

Let \( r(d) \) be the largest number such that there is a set \( P \) of \( r(d) \) points of \( R^{d} \) whose boxes are empty, i.e. \( Box_{d}(x, y) \cap P = \emptyset \) for any \( x, y \in P \). Then \( r(d) = 2^{2^{d-1}} \).

N. Alon, Z. Füredi and M. Katchalski studied a set of \( n \) points of \( R^{d} \) having many empty boxes.

When \( P \) is a finite set of points of \( R^{d} \), for \( x = (x_{i})_{i} \in P \) and for \( \varepsilon \in \{-1, 1\}^{d} \), consider the \( \varepsilon \)-th orthant having \( x \) as the origin,

\[
Orth_{d}(x, \varepsilon) := \{ z \in R^{d} \mid \forall i, \text{ if } \varepsilon = 1, z_{i} \geq x_{i}, \text{ and if } \varepsilon = -1, z_{i} \leq x_{i} \} - \{ x \}.
\]

Theorem 5 Let \( l(d) \) be the largest number such that there is a set \( P \) of \( l(d) \) points of \( R^{d} \) whose orthants contains at most one point, i.e. \( |Orth_{d}(x, \varepsilon) \cap P| \leq 1 \) for \( \forall x \in P \) and \( \forall \varepsilon \in \{-1, 1\}^{d} \). Then
\[
1.47^{d} \leq l(d) \leq c \left( d \right)^{d/4} < 1.76^{d}
\]
for an absolute constant \( c \) and any sufficiently large \( d \). (The lower bound can be shown constructively.)
Let $t, n(t \leq n)$ be positive integers and $A$ a set of $n$ elements. A finite sequence $\sigma = \sigma(1)\sigma(2)\cdots\sigma(t)$ is a $t$-permutation of $A$ if and only if $\sigma(i) \in A$ for any $1 \leq i \leq t$ and $\sigma(i) \neq \sigma(j)$ for $1 \leq i < \forall j \leq t$. The inverse of $\sigma$ is the sequence $\sigma^{-1} = \sigma(t)\sigma(t-1)\cdots\sigma(1)$. Note that the inverse of a $t$-permutation is a $t$-permutation. A $n$-permutation $\sigma$ of $A$ contains a $t$-parmutation of $A$ if $\tau$ is a subsequence of $\sigma$. Let $n_t(d) \; [n^*_t(d)]$ be the largest number $n$ having $d$ $n$-permutations $\{\sigma_1, \sigma_2, \ldots, \sigma_d\}$ of $A$ such that for any $t$-permutation $\tau$ of $A$, there exists $\sigma_i (1 \leq \exists i \leq d)$ containing $\tau$ [$\tau$ or $\tau^{-1}$]. A simple argument show that

$$l(d) = n^*_3(d).$$

For example, the five orders 1643275, 2654371, 3615472, 4621573, 5632174 of \{1, 2, \cdots, 7\} yields $n^*_3(5) \geq 7$. We will obtain bounds of $n_t(d)$ and $n^*_t(d)$.

**Theorem 6** (i) For $t \geq 4$ and $d \geq t!$,

$$\left(1 - \frac{1}{t}\right) \left(\frac{1}{t}\right)^{\frac{1}{t-1}} \left(\frac{t!}{t! - 1}\right)^{\frac{d}{t-1}} \leq n_t(d) \leq t - 3 + \left(\frac{d}{\left\lfloor \frac{d}{(t-2)!}\right\rfloor}\right)^{\frac{1}{t-2}}.$$

(ii) For $t \geq 6$ and $d \geq t!$,

$$\left(1 - \frac{1}{t}\right) \left(\frac{2}{t}\right)^{\frac{1}{t-1}} \left(\frac{t!}{t! - 2}\right)^{\frac{d}{t-1}} \leq n^*_t(d) \leq t - 4 + 2 \left(\frac{d}{\left\lceil \frac{d}{(t-3)!}\right\rceil}\right)^{\frac{1}{t-3}}.$$

**References**


[5] Ishigami, Y., An extremal problem of orthants containing at most one point besides the origin. Discrete Mathematics (to appear)