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A solution for a polygon containment problem using sorting $X + Y$.

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Abstract

Here an algorithm is presented which shows that the problem of computing the path that a rectilinear non-decreasing polygonal takes as it slides along the edges of another one can be reduced to the problem of sorting $X + Y$. The existence of such an algorithm allows us to say that the polygon containment problem under translation in the case of rectilinearly convex polygons can be also reduced to sort $X + Y$.

1 Introduction

The problem of deciding if a given polygon $P$, which is allowed to have some kind of motions, can be placed into another fixed $Q$, has been studied extensively in the last ten years. This problem, related to those referred to in the Computational Geometry literature as the polygon placement problem and the motion planning problem, has been attacked using several assumptions about the kind of polygons and motions allowed. Here we briefly mention some of these results.

Suppose $P$ and $Q$ are $p$-gon and $q$-gon respectively; $P$ can move while $Q$ remains fixed. In 1983 Chazelle ([CH]) proposed an $O(pq^2)$ algorithm for the case where $Q$ is convex and it is possible for $P$ to translate and rotate, and gave an $O(p + q)$ procedure valid if $P$ is restricted to translate. Also assuming just translations appeared in [BFM] an algorithm that runs in $O(pq \log p)$ when both $P$ and $Q$ are rectilinearly convex polygons and, another restricted problem that considered $P$ as convex was solved in [F] using $O(pq \log pq)$. For both $P$ and $Q$ being non convex and even not connected, Avnaim and Boissonnat proposed in [AB] an $O(p^2q^2 \log pq)$ algorithm in the case of only translations, and showed that the method can be generalized to an $O(p^2q^3 \log pq)$ procedure when rotations are also possible. Other versions of the polygon containment problem (PCP) have been studied. See for example [CK] and [AIIT].

If the size of the output in each specific case is taken into account, it might be said that most of the above-mentioned results are nearly optimal. For example, when both $P$ and $Q$ are rectilinearly convex polygons, it was proved in [BFM] that the boundary of the set of feasible placements of $P$ inside $Q$ is a rectilinearly convex polygonal region
that could reach $\Omega(pg)$ edges, while the algorithm used there to obtain that region is $O(pq \log p)$. Another close to optimal result can be found in [AB] as the solution may consist of $\Omega(p^3q^3)$ faces when translations as well as rotations are allowed. Except for the case in which Q is convex and P can only translate, we are not aware of either the existence of an optimal in the worst case algorithm for any of the above-stated problems or a proof of the non existence of such optimal algorithms.

Here a procedure is presented which shows that the problem of computing the path that a rectilinear non decreasing polygonal takes as it slides along the edges of another one can be reduced to the problem of sorting sets of numbers of the form $X + Y$. That result enables us to say that the PCP under translation in the case of rectilinearly convex polygons can be also reduced to sort $X + Y$. For this problem of sorting, the question of how much computation time is really needed is still open.

2 Preliminaries

In what follows, we denote the $x$-coordinate and the $y$-coordinate of a point $p$ in the plane as $p.x$ and $p.y$ respectively.

Let $A$ and $B$ be rectilinear polygonals, i.e. with all their edges being horizontal or vertical and described through the list of their vertices given in order (every vertex and its successor are the extremes of an edge). Suppose that $A$ ($B$) refers to $O_A$ ($O_B$), origin of the coordinate system. We will consider that $B$ and its coordinate system are fixed and $A$ with its coordinate system can translate.

Let us think of $A$ and $B$ as non decreasing polygonals, that is, if both $v$ and $u$ are vertices in one of them and $v$ precedes $u$ in the list of vertices then $v.x \leq u.x$ and $v.y < u.y$, or $v.x < u.x$ and $v.y \leq u.y$. We suppose that the bottommost edge of $B$ (the topmost edge of $A$) is horizontal and its left (right) extreme is in some point in the infinite. On the other hand, the rightmost edge of $B$ (the leftmost edge of $A$) is vertical and its upper (lower) extreme is also in some point in the infinite.

Let's denote the $i^{th}$ vertex in one of the polygonals as $v_i$. We say that $v_i$ is convex if the clockwise angle between segments $\overline{v_{i-1}v_i}$ and $\overline{v_iv_{i+1}}$ is $90^\circ$, and concave if that angle is $270^\circ$.

It is clear that $B$ would be completely described through the list of its concave vertices as well as $A$ through its convex vertices. Therefore, from now we assume that $B$ is represented by the concave vertices $b_1, b_2, \ldots, b_n$ and $A$ by the convex vertices $a_1, a_2, \ldots, a_m$.

We wish to obtain the vertices of the rectilinear polygonal that $O_A$ would describe in the fixed coordinate system as $A$ slides along the edges of $B$. In other words, we want a description of all the positions that $O_A$ would reach if, beginning in some point in the
infinite with the topmost edge of $A$ "touching" the bottommost edge of $B$, $A$ translates to the right until going upward without intersecting $B$ is possible, then $A$ translates in that direction until the distance between some pair of horizontal segments is zero, etc. Notice that in essence, what we want is to solve a polygon containment problem in which both regions are not bounded.

For a certain position $O$ of $O_A$ referred to $O_B$, the coordinates of the vertex $a_j$ ($1 \leq j \leq m$) in the fixed coordinate system is $(O.x+a_j.x, O.y+a_j.y)$ that we will denote as $a_j^o$.

3 The algorithm

Initially suppose $O_A$ is placed in a position $O$ so that there is contact between $b_1$ and $a_m$, that is, $a_m^o = b_1$.

Let's consider the set $D_V = \{(b_i, a_j^o)_V/1 \leq i \leq n, 1 \leq j \leq m\}$ where the pair $(b_i, a_j^o)_V$ denotes the difference $b_i.y - a_j^o.y$ and the set $D_H = \{(b_k, a_l^o)_H/1 \leq k \leq n, 1 \leq l \leq m\}$, where $(b_k, a_l^o)_H$ denotes $b_k.x - a_l^o.x$. For the sake of simplicity we assume that no two pairs in $D_V$ are the same, i.e. there is not $i, j, r, s$, $1 \leq i, r \leq n$, $1 \leq j, s \leq m$, such that $(b_i, a_j^o)_V = (b_r, a_s^o)_V$. We assume a similar condition for $D_H$. These limitations result in no loss of generality.

Loosely speaking, our algorithm computes all contacts (in the order they take place) between vertices in $B$ and vertices in $A$ as the latter translates. Note that, if for a certain location of $O_A$, $p$, $b_i$ contacts $a_j$, and we know that the next contact as $A$ slides first upward and after to the right is $b_r$ and $a_s$, then the next two positions of $O_A$ are $(p.x, p.y + (b_r.y - a_s^o.y))$ and $(p.x + (b_r.x - a_s^o.x), p.y + (b_r.y - a_s^o.y))$, where the last one is the position for $O_A$ in which $b_r$ contacts $a_s$. This observation tell us how to obtain all the positions of $O_A$.

So, the problem is to obtain $b_r$ and $a_s$. Let $D_V^r$ and $D_H^r$ be the sorted sets $D_V$ and $D_H$ respectively.

Lemma

If $b_i$ touches $a_j$ ($1 \leq i \leq n, 1 \leq j \leq m$) for a certain location $p$ of $O_A$ then, the next contact as $A$ slides upward and to the right is between $b_r$ and $a_s$ if, $(b_r, a_s^o)_V$ is the next pair in $D_V^r$ following $(b_i, a_j^o)_V$ on the condition that its corresponding pair $(b_r, a_s^o)_H$ is located in $D_H^r$ following $(b_i, a_j^o)_H$.

Proof

It is clear that the pair we are looking for must succeed $(b_i, a_j^o)_V$ in $D_V$. Let $(b_k, a_l^o)_H$ be a pair preceding $(b_i, a_j^o)_H$ in $D_H^r$. Then it is easy to realize that $b_k.x < a_l^o.x$ so it is impossible for $a_l$ to make contact with $b_k$. □
The previous lemma enables us to design the following algorithm to compute the set of positions which \( O_A \) can be at. In the algorithm it is considered that there is a label associated to each element in \( D_H^* \) which can hold the values unvisited (there could be a contact) or visited (definitely no contact in the future). The contacts will be stored in \( L \).

**Procedure** Compute-Contacts

begin
  Compute \( D_V \) and \( D_H \);
  Compute \( D_V^* \) and \( D_H^* \);
  Mark every pair in \( D_H^* \) as unvisited;
  Move \( A \) until \( a_m \) contact \( b_1 \) and mark as visited \( (b_1, a_m) \) in \( D_H^* \);
  \( L \leftarrow (b_1, a_m) \);
  For every pair \( (b_i, a_j^o) \) in \( D_V^* \) do
    if \( (b_i, a_j^o) \) in \( D_H^* \) is visited then go to analysis next pair in \( D_V^* \)
    else
      \( L \leftarrow (b_i, a_j) \);
      Mark \( (b_i, a_j^o) \) as visited as well as every pair in \( D_H^* \) that precedes it down to the last marked pair
  end
end

We should notice that the running time of the previous procedure is dominated by sorting. If we use a standard sorting algorithm an \( O(nm \log_2 m) \) upper bound would be obtained, which equals the one showed in [BFM]. Could \( D_V \) and \( D_H \) be sorted in less time, taking into consideration the particular structure of these sets? The problem of sorting \( X + Y \), where \( X \) and \( Y \) are sets of real numbers, has been studied before ([HPSS], [FR] and [LA]). It was proved in [LA] that, for two given sequences of numbers \( (x_i)_{1 \leq i \leq N} \) and \( (y_j)_{1 \leq j \leq N} \), there exists an algorithm to compute the \( N^2 \) sums \( (x_i + y_j)_{1 \leq i,j \leq N} \) in \( O(N^2) \) comparisons. In fact, such an algorithm was presented there, but unfortunately, its performance was analyzed just in terms of comparisons and the existence of an algorithm with a similar bound in the case of a more general study remains still unknown.

The above procedure shows that the problem of sliding a rectilinear non decreasing polygonal of \( m \) edges along the \( n \) edges of another one can be reduced to sort \( X + Y \), \( X \) and \( Y \) with \( n \) and \( m \) elements respectively; if there is an \( O(nm) \) algorithm for the last problem then the solution for the former could be also obtained in \( O(nm) \). On the other hand, the polygon containment problem under translation in the case of rectilinear-early convex polygons can be reduced to compute such a sliding-path ([BFM]) hence...
our algorithm proves the reducibility of that specific PCP to sort $X+Y$. More precisely, suppose $Q$, $P$ are rectilinearly convex polygons with $n$, $m$ vertices respectively. Determining all the positions at which $P$ is contained by $Q$ can be done within the same bound as sorting $X+Y$.

4 Conclusions

We have shown that the problem of sliding a non decreasing rectilinear polygonal along the edges of another one can be reduced to solve the problem of sorting $X+Y$, which in fact proves that the PCP under translation is also reduced to sort $X+Y$ when both polygons are rectilinearly convex. As determining lower bounds for both problems has yet to be solved, it will be interesting to see if the contrary relation holds true in further research.

5 References


