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Remarks on Fujiwara's Stationary Phase Method with a Phase Function Involving Electromagnetic Fields

金沢大学自然科学研究科　土田哲生　Tetsuo Tsuchida
Department of Mathematics, Faculty of Science
Kanazawa University, 920-11 Kanazawa, Japan

1. Introduction

We consider an oscillatory integral of the form

$$I(\{t_j\},S,a,\nu)(x_L,x_0)=\prod_{j=1}^{L}(\frac{\nu i}{2\pi t_j})^{d/2}\int_{\mathbf{R}^{d(L-1)}}e^{-i\nu S(x_L,\cdots,x_0)}a(x_L,\cdots,x_0)\prod_{j=1}^{L-1}dx_j. \tag{1.1}$$

Here each $x_j, j=0,1,\cdots,L$, runs in $\mathbf{R}^d$, $\nu>1$ is a constant and $t_j, j=1,\cdots,L$, are positive constants. Fujiwara [5] discussed this integral for L large and developed the stationary phase method with an estimate of the remainder term for the phase function $S(x_L,\cdots,x_0)$ coming from the action integral for a particle in electric fields. In this paper we extend his results to the case for the phase function involving both electric and magnetic fields.

We denote the l-th component of $x\in\mathbf{R}^d$ by $(x)_l$, and use the notations: $\partial_j^\alpha=\partial_{x_j}^\alpha=\partial_{(x_j)_1}^{\alpha_1}\cdots\partial_{(x_j)_d}^{\alpha_d}$ with a multi-index $\alpha=(\alpha_1,\cdots,\alpha_d)$, and $\partial_j f(x_j)=\partial_{x_j}f(x_j)$ as the gradient of $f(x_j)$.

Our assumption for the phase function $S(x_L,\cdots,x_0)$ is the following:

(H.1) $S(x_L, \cdots, x_0)$ is a real-valued function of the form

$$S(x_L, \cdots, x_0) = \sum_{j=1}^{L} S_j(t_j, x_j, x_{j-1}), \tag{1.2}$$

where

$$S_j(t_j, x_j, x_{j-1}) = \frac{|x_j - x_{j-1}|^2}{2t_j} + \omega_j(t_j, x_j, x_{j-1}), \quad j = 1, \cdots, L, \tag{1.3}$$

and $\omega_j(t_j, x_j, x_{j-1})$ satisfies the following conditions:

(i) For any $m \geq 2$ there exists a constant $\kappa_m > 0$ independent of j and t_j such that

$$\max_{2 \leq |\alpha+\beta| \leq m} \sup_{x,y \in \mathbf{R}^d} |\partial_x^\alpha \partial_y^\beta \omega_j(t_j, x, y)| \leq \kappa_m. \tag{1.4}$$

(ii) Let $(\bar{x}_L, \cdots, \bar{x}_0)$ be an arbitrary solution of the system of the equation

$$\partial_{x_j} S_{j+1}(t_{j+1}, \bar{x}_{j+1}, \bar{x}_j) + \partial_{x_j} S_j(t_j, \bar{x}_j, \bar{x}_{j-1}) = 0, \quad j = 1, \cdots, L-1. \tag{1.5}$$

For any $m \geq 1$, there exists a constant B_m independent of $(\bar{x}_L, \cdots, \bar{x}_0), L$ and $t_j, j = 1, \cdots, L$, but dependent on d such that

$$\sum_{j=1}^{L-1} \sum_{|\beta|=1, 1 \leq |\alpha| \leq m} |[(\partial_{x_{j-1}} + \partial_{x_j} + \partial_{x_{j+1}})^\alpha \partial_{x_j}^\beta (\omega_j + \omega_{j+1})](\bar{x}_{j-1}, \bar{x}_j, \bar{x}_{j+1})| \leq B_m, \tag{1.6}$$

where $(\partial_{x_{j-1}} + \partial_{x_j} + \partial_{x_{j+1}})^\alpha = \prod_{k=1}^d (\partial_{(x_{j-1})_k} + \partial_{(x_j)_k} + \partial_{(x_{j+1})_k})^{\alpha_k}$ for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$.

In the form of oscillatory integrals Yajima [10] constructed the propagator of Schrödinger evolution equation for a particle in a certain electromagnetic field. This case gives an example of the phase satisfying (H.1), in which $S_j(t_j, x_j, x_{j-1})$ is the action integral along the classical path of the particle.

When $S(x_L, \cdots, x_0)$ satisfies (H.1), then if $T_L = t_1 + \cdots + t_L$ is small enough, for any $x_0, x_L \in \mathbf{R}^d$ there exists the unique critical point $(x_{L-1}^*, \cdots, x_1^*)$, i.e.

$$\partial_{x_j} S_{j+1}(t_{j+1}, x_{j+1}^*, x_j^*) + \partial_{x_j} S_j(t_j, x_j^*, x_{j-1}^*) = 0, \quad j = 1, \cdots, L-1, \tag{1.7}$$

where $x_L^* = x_L, x_0^* = x_0$ (The proof is in §3).

To state the assumption for the amplitude function, we use Fujiwara's notation:

$$a(\overline{x_L, x_0}) = a(x_L, x_{L-1}^*, \cdots, x_1^*, x_0).$$

Similarly, for any pair of integers k, m with $k+1 < m$ let $(x_{k+1}^*, \cdots x_{m-1}^*)$ be the partial critical point, i.e.

$$\partial_{x_j} S_{j+1}(t_{j+1}, x_{j+1}^*, x_j^*) + \partial_{x_j} S_j(t_j, x_j^*, x_{j-1}^*) = 0, \quad j = k+1, \cdots, m-1,$$

where $x_k^* = x_k, x_m^* = x_m$. Then we set

$$a(x_L, \cdots, \overline{x_m, x_k}, \cdots, x_0) = a(x_L, \cdots, x_m, x_{m-1}^*, \cdots, x_{k+1}^*, x_k, \cdots, x_0).$$

If $m = k+1$, we define

$$a(x_L, \cdots, \overline{x_{k+1}, x_k}, \cdots, x_0) = a(x_L, \cdots, x_{k+1}, x_k, \cdots x_0).$$

The assumption for the amplitude function is the following:

(H.2) $a(x_L, \cdots, x_0)$ is a real-valued function in $\mathcal{B}(\mathbf{R}^{d(L+1)})$. For any $K \geq 0$ there exist constants A_K and X_K with the following properties:

For any sequence of positive integers with $j_0 = 0 < j_1 - 1 < j_1 < j_2 - 1 < \cdots < j_s \leq L, s = 1, \ldots, L-1$,

$$|\partial_{x_0}^{\alpha_0} \partial_{x_L}^{\alpha_L} \prod_{u=1}^{s} \partial_{x_{j_u-1}}^{\alpha_{j_u-1}} \partial_{x_{j_u}}^{\alpha_{j_u}} a(\overline{x_L, x_{j_s}}, \overline{x_{j_s-1}, x_{j_{s-1}}}, \ldots, \overline{x_{j_1-1}, x_0})| \leq A_K X_K^s, \tag{1.8a}$$

if $|\alpha_j| \leq K$, $j = 0, j_1 - 1, j_1, \ldots, j_s - 1, j_s, L$. If $j_s = L$, then we read the above inequality as

$$|\partial_{x_0}^{\alpha_0} \prod_{u=1}^{s} \partial_{x_{j_u-1}}^{\alpha_{j_u-1}} \partial_{x_{j_u}}^{\alpha_{j_u}} a(x_L, \overline{x_{j_s-1}, x_{j_{s-1}}}, \ldots, \overline{x_{j_1-1}, x_0})| \leq A_K X_K^s. \tag{1.8b}$$

Let us state our main theorems. Let H be the $d(L-1)\times d(L-1)$ matrix

$$H=\begin{pmatrix} \frac{1}{t_1}+\frac{1}{t_2} & -\frac{1}{t_2} & 0 & 0 & \cdots & \\ -\frac{1}{t_2} & \frac{1}{t_2}+\frac{1}{t_3} & -\frac{1}{t_3} & 0 & \cdots & \\ 0 & -\frac{1}{t_3} & \frac{1}{t_3}+\frac{1}{t_4} & -\frac{1}{t_4} & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

and W the Hessian matrix of $\sum_{j=1}^{L}\omega_j(t_j,x_j,x_{j-1})$ at the critical point $(x_{L-1}^*,\dots,x_1^*)$.

Theorem 1. *Assume (H.1) and (H.2). There exists a positive constant δ such that if $T_L=t_1+\cdots+t_L<\delta$ then*

$$I(\{t_j\},S,a,\nu)(x_L,x_0)=\left(\frac{\nu i}{2\pi T_L}\right)^{d/2}\exp\{-i\nu \overline{S}(x_L,x_0)\}\det(I+H^{-1}W)^{-1/2} \times(\overline{a}(x_L,x_0)+r(x_L,x_0)), \tag{1.9}$$

and for any $K\geq 0$ there exist positive constants C_K and $M(K)$ such that if $|\alpha_0|$, $|\alpha_L|\leq K$,

$$|\partial_{x_L}^{\alpha_L}\partial_{x_0}^{\alpha_0}r(x_L,x_0)|\leq A_{M(K)}\left(\prod_{j=1}^{L}(1+C_K X_{M(K)}\nu^{-1}t_j)-1\right). \tag{1.10}$$

Constants δ and C_K are independent of a, $L,\{t_j\},x_L,x_0$ and ν but depend on the dimension d of space $\mathbf{R}^d$ and $\{\kappa_m\}$ and $\{B_m\}$. $M(K)$ depends only on K and d.

Theorem 2. *Assume that $a\equiv 1$ and (H.1) and let δ be the constant as in Theorem 1. Then for any $K\geq 0$ there exists a constant C_K such that if $|\alpha_0|$, $|\alpha_L|\leq K$,*

$$|\partial_{x_L}^{\alpha_L}\partial_{x_0}^{\alpha_0}r(x_L,x_0)|\leq \prod_{j=1}^{L}(1+C_K\nu^{-1}t_jT_L)-1. \tag{1.11}$$

Fujiwara [5] treated the case that the phase function is of the form

$$S(x_L,\cdots,x_0)=\sum_{j=1}^{L}S_j(t_j,x_j,x_{j-1}),$$

with

$$S_j(t_j,x_j,x_{j-1})=\frac{|x_j-x_{j-1}|^2}{2t_j}+t_j\omega_j(t_j,x_j,x_{j-1}),\quad j=1,\cdots,L,$$

where $\omega_j(t_j, x_j, x_{j-1})$ satisfies (1.4). The estimate of $r(x_L, x_0)$ in Theorem 1 is the same as his, but that in Theorem 2 differs from his in the power of T_L: our power is 1 while his power is 2.

In §2 we see that the phase function coming from the action integral for a particle in electromagnetic fields satisfies (H.1). In §3 the existence of the critical point of the phase function is proved. We refer the other properties of the critical point and the proofs of Theorems 1 and 2 to [9].

2. Piecewise classical path in electromagnetic fields

We give an example of $S(x_L, \cdots, x_0)$ which satisfies the assumption (H.1). We consider a particle in electromagnetic fields in $\mathbf{R}^d$. In this section we denote the l-th component of $x \in \mathbf{R}^d$ by x_l. We make the following assumption for the vector and scalar potentials $A(t,x)$ and $V(x)$:

Assumption (A). For $k = 1, \ldots, d$, $A_k(t,x)$ is a real-valued function of $(t,x) \in \mathbf{R} \times \mathbf{R}^d$, and for any α, $\partial_x^\alpha A_k(t,x)$ is C^1 in $(t,x) \in \mathbf{R} \times \mathbf{R}^d$. There exists $\varepsilon > 0$ such that

$$|\partial_x^\alpha A_k(t,x)| + |\partial_x^\alpha \partial_t A_k(t,x)| \le C_\alpha, \quad |\alpha| \ge 1, \quad (t,x) \in \mathbf{R} \times \mathbf{R}^d, \tag{2.1}$$

$$|\partial_x^\alpha B(t,x)| \le C_\alpha (1+|x|)^{-1-\varepsilon}, \quad |\alpha| \ge 1, \tag{2.2}$$

where $B(t,x)$ is the skew symmetric matrix with (k,l)-component $B_{kl}(t,x) = (\partial A_l/\partial x_k - \partial A_k/\partial x_l)(t,x)$ and $|B|$ denotes the norm of matrix B regarded as an operator on $\mathbf{R}^d$. $V(x)$ is a real-valued C^∞ function which satisfies

$$|\partial_x^\alpha V(x)| \le C_\alpha, \quad |\alpha| \ge 2. \tag{2.3}$$

This assumption for vector potentials was used by Yajima [10]. In particular, constant magnetic fields satisfy this assumption.

Let $H(t,x,\xi)$ be the Hamiltonian

$$H(t,x,\xi) = 2^{-1}(\xi - A(t,x))^2 + V(x).$$

Then Hamilton's equation is

$$\dot{x} = \partial_\xi H(t,x,\xi), \quad \dot{\xi} = -\partial_x H(t,x,\xi)$$

with $\dot{x} = dx/dt$ and $\dot{\xi} = d\xi/dt$. When we introduce the position-velocity variables by $(q(t), v(t)) = (x(t), \xi(t) - A(t, x(t)))$, the equation is Lagrange equation:

$$\dot{q}(t) = v(t), \quad \dot{v}(t) = B(t, q(t))v(t) + F(t, q(t)), \tag{2.4}$$

where $F(t,x) = -(\partial_t A)(t,x) - (\partial_x V)(x)$. The next lemma is a result of Yajima [10].

Lemma 2.1. *Let* $|t - s| \leq 1$.

(i) For any α *with* $|\alpha| \geq 1$*, there exists a constant* C'_α *such that for any solution* $(q(\tau), v(\tau))$, $s \leq \tau \leq t$, *of (2.4),*

$$\int_s^t |(\partial_x^\alpha B)(\tau, q(\tau))||v(\tau)|d\tau \leq C'_\alpha.$$

(ii) There exists a constant $T > 0$ *such that if* $0 < |t - s| < T$*, then for any* $x, y \in \mathbf{R}^d$ *there exists a unique solution* $(q(\tau), v(\tau))$, $s \leq \tau \leq t$, *of (2.4) with* $q(s) = y$ *and* $q(t) = x$.

Proof. We refer the proof to Yajima [10, Lemma 2.1 and Proposition 2.6]. ∎

Let $T > 0$ be as in Lemma 2.1(ii) and $|t - s| \leq T$. We write the unique solution $q(\tau)$ of (2.4) with $q(s) = y$ and $q(t) = x$ as

$$q(\tau) = q^0(\tau) + q^1(\tau)$$

where $q^0(\tau) = \dfrac{\tau - s}{t - s}(x - y) + y$. Then we have

$$\ddot{q}^1(\tau) = B(\tau, q(\tau))v(\tau) + F(\tau, q(\tau)), \tag{2.5}$$

and

$$q^1(s) = q^1(t) = 0.$$

Lemma 2.2. *There exists a constant* $0 < T^0 < \min(T, 1)$ *such that if* $|t - s| \leq T^0$ *then for any* α, β *with* $|\alpha + \beta| \geq 1$,

$$\|\partial_x^\alpha \partial_y^\beta q^1\|_{L^\infty} \leq \|\partial_x^\alpha \partial_y^\beta \dot{q}^1\|_{L^1} \leq C_{\alpha\beta}|t - s|. \tag{2.6}$$

The proof is omitted.

Let $S(t, s, x, y)$ be the action of the classical path $(q(\tau), v(\tau))$ joining (s, y) to (t, x):

$$S(t, s, x, y) = \int_s^t L(\tau, q(\tau), v(\tau))d\tau, \tag{2.7}$$

where $L(\tau, q, v)$ is the Lagrangian corresponding to $H(\tau, x, \xi)$:

$$L(\tau, q, v) = v\xi - H(\tau, x, \xi) = \frac{v^2}{2} + A(\tau, q)v - V(q).$$

For any sequence $0 = T_0 < T_1 < \cdots < T_L < T^0$ and any points $x^j \in \mathbf{R}^d$, $j = 0, \ldots, L$, we put

$$S_j(t_j, x^j, x^{j-1}) = S(T_j, T_{j-1}, x^j, x^{j-1}), \quad j = 1, \ldots, L,$$

where $t_j = T_j - T_{j-1}$. We denote by $q_\Delta = q_\Delta^0 + q_\Delta^1$ the piecewise classical path joining (T_j, x^j), $j = 0, \ldots, L$, i.e. q_Δ^0 is

$$q_\Delta^0(\tau) = \frac{\tau - T_{j-1}}{t_j}(x^j - x^{j-1}) + x^{j-1}, \quad T_{j-1} \leq \tau \leq T_j, \quad j = 1, \ldots, L,$$

and q_Δ^1 satisfies

$$\ddot{q}_\Delta^1(\tau) = B(\tau, q_\Delta(\tau))\dot{q}_\Delta(\tau) + F(\tau, q_\Delta(\tau)), \quad T_{j-1} \leq \tau \leq T_j,$$

and $q^1_\Delta(T_j) = 0$, $j = 0, \dots, L$. The action along the piecewise classical path can be written as

$$S(q_\Delta) = S(x^L, \cdots, x^0) = \sum_{j=1}^{L} S_j(t_j, x^j, x^{j-1}).$$

Theorem 2.3. *Let $T_L < T^0$. Then $S(x^L, \cdots, x^0) = \sum_{j=1}^{L} S_j(t_j, x^j, x^{j-1})$ satisfies Assumption (H.1).*

Proof. First we verify (H.1). Let $q(\tau) = q^0(\tau) + q^1(\tau)$ be the classical path joining (s, y) to (t, x). We have

$$\begin{aligned} S(t,s,x,y) &= \int_s^t \Big(\frac{|\dot q^0(\tau) + \dot q^1(\tau)|^2}{2} + A(\tau, q(\tau))\dot q(\tau) - V(q(\tau))\Big)\, d\tau \\ &= \frac{|x-y|^2}{2(t-s)} + \int_s^t \Big(\frac{|\dot q^1(\tau)|^2}{2} + A(\tau, q(\tau))\dot q(\tau) - V(q(\tau))\Big)\, d\tau \\ &= \frac{|x-y|^2}{2(t-s)} + \omega(t,s,x,y) \end{aligned}$$

where

$$\omega(t,s,x,y) = \int_s^t \Big(\frac{|\dot q^1(\tau)|^2}{2} + A(\tau, q(\tau))\dot q(\tau) - V(q(\tau))\Big)\, d\tau. \tag{2.8}$$

Since q satisfies (2.5), it follows that

$$(\partial_{y_k}\omega)(t,s,x,y) = \int_s^t \partial_{y_k} q^0 (B(\tau, q(\tau))\dot q(\tau) + F(\tau, q(\tau))\, d\tau - A_k(s,y).$$

Noting $\partial_{y_k} q^0_m = (t-\tau)(t-s)^{-1}\delta_{km}$, we obtain

$$\begin{aligned} (\partial_{y_l}\partial_{y_k}\omega)(t,s,x,y) = \int_s^t \frac{t-\tau}{t-s}\Big(&\sum_{m=1}^{d} B_{km}\partial_{y_l}\dot q_m + \sum_{n,m=1}^{d} \partial_{y_l} q_n \partial_{x_n} B_{km}\dot q_m \\ &+ \sum_{m=1}^{d} \partial_{y_l} q_m \cdot \partial_{x_m} F_k\Big)\, d\tau - (\partial_{y_l} A_k)(s,y). \end{aligned}$$

So from Assumption (A), Lemma 2.1(i) and Lemma 2.2, we have

$$|\partial_{y_l}\partial_{y_k}\omega| \le C_1(1 + C|t-s|) + C_1'(1 + C|t-s|) + C_1|t-s|(1 + C|t-s|) + C_1 \le \kappa_2,$$

where κ_2 is independent of x, y and $t-s$. For the other higher derivatives of ω, similar arguments hold. So we have proved (H.1)(i).

Next we show (H.1)(ii). We put

$$\omega_j(x^j, x^{j-1}) = \omega(T_j, T_{j-1}, x^j, x^{j-1}). \tag{2.9}$$

We have similarly to the above

$$\begin{aligned}
&(\partial_{x_l^{j-1}} + \partial_{x_l^j} + \partial_{x_l^{j+1}})\partial_{x_k^j}(\omega_j + \omega_{j+1})(x^{j-1}, x^j, x^{j+1}) \\
&= \int_{T_j}^{T_{j+1}} \frac{T_{j+1}-\tau}{t_{j+1}} \Big(\sum_{m=1}^d B_{km}(\partial_{x_l^{j+1}} + \partial_{x_l^j})\dot{q}_{\Delta m}^1 \\
&\quad + \sum_{n,m=1}^d (\partial_{x_l^{j+1}} + \partial_{x_l^j})(q_\Delta)_n \partial_{x_n} B_{km}(\dot{q}_\Delta)_m + \sum_{m=1}^d (\partial_{x_l^{j+1}} + \partial_{x_l^j})(q_\Delta)_m \cdot \partial_{x_m} F_k\Big) d\tau \\
&\quad + \int_{T_{j-1}}^{T_j} \frac{\tau - T_{j-1}}{t_j} \Big(\sum_{m=1}^d B_{km}(\partial_{x_l^j} + \partial_{x_l^{j-1}})\dot{q}_{\Delta m}^1 \\
&\quad + \sum_{n,m=1}^d (\partial_{x_l^j} + \partial_{x_l^{j-1}})(q_\Delta)_n \partial_{x_n} B_{km}(\dot{q}_\Delta)_m + \sum_{m=1}^d (\partial_{x_l^j} + \partial_{x_l^{j-1}})(q_\Delta)_m \cdot \partial_{x_m} F_k\Big) d\tau.
\end{aligned}$$

When $(\bar{x}^L, \cdots, \bar{x}^0)$ is a critical point of $S(q_\Delta)$, the piecewise classical path $q_\Delta(\tau)$ coincides with the classical path $q(\tau)$ joining $(0, \bar{x}^0)$ and $(T_L, \bar{x}^L)$. So we have from Lemma 2.2

$$\begin{aligned}
&|(\partial_{x_l^{j-1}} + \partial_{x_l^j} + \partial_{x_l^{j+1}})\partial_{x_k^j}(\omega_j + \omega_{j+1})(\bar{x}^{j-1}, \bar{x}^j, \bar{x}^{j+1})| \\
&\le C(t_{j+1} + t_j) + C\int_{T_{j-1}}^{T_{j+1}} |(\partial B)(\tau, q(\tau))||v(\tau)| d\tau.
\end{aligned}$$

Therefore, we have by Lemma 2.1(i)

$$\begin{aligned}
&\sum_{j=1}^{L-1} |(\partial_{x_l^{j-1}} + \partial_{x_l^j} + \partial_{x_l^{j+1}})\partial_{x_k^j}(\omega_j + \omega_{j+1})(\bar{x}^{j-1}, \bar{x}^j, \bar{x}^{j+1})| \\
&\le CT_L + C\int_0^{T_L} |(\partial B)(\tau, q(\tau))||v(\tau)| d\tau \\
&\le B_1,
\end{aligned}$$

where B_1 is independent of $(\bar{x}^L, \ldots, \bar{x}^0)$, L and T_L if $T_L < T^0$. Similar discussions hold for other differentiation $(\partial_{x^{j-1}} + \partial_{x^j} + \partial_{x^{j+1}})^\alpha$. Thus we have proved (H.1)(ii). ∎

3. Phase functions

In this section we discuss the unique existence of the critical point of S (Lemma 3.5). The method is similar to Yajima [10]. In what follows, we assume (H.1) and abbreviate $S_j(t_j, x_j, x_{j-1})$ as $S_j(x_j, x_{j-1})$ and $\omega_j(t_j, x_j, x_{j-1})$ as $\omega_j(x_j, x_{j-1})$. To avoid additional complexity we put $d = 1$. **Lemma 3.1.** *Let $2t_j\kappa_2 \leq 1$, $j = 1, \cdots, L$. Then for any y and $k \in \mathbf{R}$, there exists a unique $(x_0^\sharp, \cdots, x_L^\sharp) = (x_0^\sharp(y,k), \cdots, x_L^\sharp(y,k))$ which satisfies $x_0^\sharp = y$, $\dfrac{x_1^\sharp - y}{t_1} = k$ and*

$$\frac{x_{j+1}^\sharp - x_j^\sharp}{t_{j+1}} - \frac{x_j^\sharp - x_{j-1}^\sharp}{t_j} = \partial_j\omega_j(x_j^\sharp, x_{j-1}^\sharp) + \partial_j\omega_{j+1}(x_{j+1}^\sharp, x_j^\sharp), \quad j = 1, \cdots, L-1. \tag{3.1}$$

Proof. We have $x_1^\sharp = x_1^\sharp(y,k) = t_1 k + y$. Put

$$k_j^\sharp = \frac{x_j^\sharp - x_{j-1}^\sharp}{t_j}, \quad j = 1, \cdots, L. \tag{3.2}$$

Then the system of the equation (3.1) is equivalent to

$$\begin{aligned} k_{j+1}^\sharp - k_j^\sharp &= \partial_j\omega_j(x_{j-1}^\sharp + t_j k_j^\sharp, x_{j-1}^\sharp) \\ &+ \partial_j\omega_{j+1}(x_{j-1}^\sharp + t_j k_j^\sharp + t_{j+1}k_{j+1}^\sharp, x_{j-1}^\sharp + t_j k_j^\sharp), \quad j = 1, \cdots, L-1. \end{aligned} \tag{3.3}$$

If $2t_2\kappa_2 \leq 1$, for any $y, k \in \mathbf{R}$, the map Φ_1 :

$$k_2 \mapsto \Phi_1(k_2) = k + (\partial_1\omega_1)(y + t_1 k, y) + (\partial_1\omega_2)(y + t_1 k + t_2 k_2, y + t_1 k)$$

is a contraction. So there exists a unique $k_2^\sharp = k_2^\sharp(y,k)$ which satisfies (3.3) for $j = 1$. Hence we have $x_2^\sharp(y,k) = x_1^\sharp(y,k) + t_2 k_2^\sharp(y,k)$. Similarly we have the unique existence of $k_3^\sharp, \cdots, k_L^\sharp$ and $x_3^\sharp, \cdots, x_L^\sharp$, successively. ∎

As in the proof of Lemma 3.1, we set $k_j^\sharp(y,k) = \dfrac{x_j^\sharp(y,k) - x_{j-1}^\sharp(y,k)}{t_j}$, $j = 1, \cdots, L$, where $k_1^\sharp = k$ and $x_0^\sharp = y$. Let $T_j = t_1 + \cdots + t_j$.

Lemma 3.2. *If $2t_j\kappa_2 \le 1$, $j = 1, \cdots, L$, then for $|\alpha+\beta| \ge 1$,*

$$|\partial_y^\alpha \partial_k^\beta (x_j^\sharp(y,k) - y - T_j k)| \le C_{\alpha\beta} T_j^{|\beta|+1}, \tag{3.4}$$

$$|\partial_y^\alpha \partial_k^\beta (k_j^\sharp(y,k) - k)| \le C_{\alpha\beta} T_j^{|\beta|}. \tag{3.5}$$

Proof. We can prove this by induction on $l = |\alpha+\beta|$. Here we show (3.4,5) for the case $l = 1$ only. We denote $x_j^\sharp(y,k)$ by x_j, $k_j^\sharp(y,k)$ by k_j, $\partial_y^\alpha\partial_k^\beta x_j^\sharp$ by $x_j^{\alpha\beta}$ and $\partial_y^\alpha\partial_k^\beta k_j^\sharp$ by $k_j^{\alpha\beta}$.

Let $l = 1$. Then we have from (3.2,3),

$$\begin{aligned} &x_j^{\alpha\beta} - x_{j-1}^{\alpha\beta} = t_j k_j^{\alpha\beta}, \quad j = 1, \cdots, L, \\ &k_{j+1}^{\alpha\beta} - k_j^{\alpha\beta} = (\partial_{j-1} + \partial_j + \partial_{j+1})\partial_j(\omega_j + \omega_{j+1}) x_{j-1}^{\alpha\beta} \\ &\quad + (\partial_j^2(\omega_j + \omega_{j+1}) + \partial_{j+1}\partial_j\omega_{j+1}) t_j k_j^{\alpha\beta} + \partial_{j+1}\partial_j\omega_{j+1} t_{j+1} k_{j+1}^{\alpha\beta}, \quad j = 1, \ldots, L-1. \end{aligned} \tag{3.6}$$

So we obtain with $\phi_j^1 = (\partial_{j-1} + \partial_j + \partial_{j+1})\partial_j(\omega_j + \omega_{j+1})(x_{j-1}, x_j, x_{j+1})$

$$(1 - \kappa_2 t_{j+1})|k_{j+1}^{\alpha\beta}| + |x_j^{\alpha\beta}| \le (1 + (3\kappa_2 + 1)t_j)|k_j^{\alpha\beta}| + (1 + |\phi_j^1|)|x_{j-1}^{\alpha\beta}|.$$

Hence if $1 - \kappa_2 t_{j+1} \ge \frac{1}{2}$, then

$$|k_{j+1}^{\alpha\beta}| + |x_j^{\alpha\beta}| \le (1 + 2|\phi_j^1| + 2(3\kappa_2 + 1)t_j + 2\kappa_2 t_{j+1})(|k_j^{\alpha\beta}| + |x_{j-1}^{\alpha\beta}|).$$

Here we have used $(1+b)(1-a)^{-1} \le 1 + 2(a+b)$ for $0 \le 2a \le 1$. Since $k_1^{\alpha\beta}, x_0^{\alpha\beta} = 0$ or 1, it follows from Assumption (H.1)(ii) that $|k_{j+1}^{\alpha\beta}| + |x_j^{\alpha\beta}| \le C$. So we have

$$|\partial_y(k_j - k)| \le C \text{ and } |\partial_y(x_j - y - T_j k)| = |\sum_{l=1}^{j} t_l \partial_y k_l| \le CT_j.$$

Moreover since we have

$$|\partial_k x_j| = |\partial_k (x_j - y)| = |\sum_{l=1}^{j} t_l \partial_k k_l| \leq CT_j,$$

we obtain by summing (3.6) for j

$$|\partial_k (k_j - k)| \leq CT_j \text{ and } |\partial_k (x_j - y - T_j k)| = |\sum_{l=1}^{j} t_l \partial_k (k_l - k)| \leq CT_j^2.$$

∎

We need the inverse of the map $(y,k) \mapsto (y, x_L^\sharp(y,k))$. To this end we introduce the new variables:

$$\tilde{x}_j(y,k) = x_j^\sharp(y, k/T_j) \text{ and } \tilde{k}_j(y,k) = T_j k_j^\sharp(y, k/T_j), \quad j = 1, \cdots, L. \tag{3.7}$$

Lemma 3.3. *For any α and β, there exists $C_{\alpha\beta}$ such that*

$$\begin{aligned} &|\partial_y^\alpha \partial_k^\beta (\partial_y \tilde{x}_j - 1)| + |\partial_y^\alpha \partial_k^\beta (\partial_k \tilde{x}_j - 1)| \\ &+ |\partial_y^\alpha \partial_k^\beta (\partial_y \tilde{k}_j)| + |\partial_y^\alpha \partial_k^\beta (\partial_k \tilde{k}_j - 1)| \leq C_{\alpha\beta} T_j. \end{aligned}$$

Proof. This follows from Lemma 3.2. ∎

Lemma 3.4. *There exists a constant $T > 0$ such that if $T_L < T$, then the map $(y,k) \mapsto (y,x) = (y, \tilde{x}_L(y,k))$ is a global diffeomorphism on* $\mathbf{R} \times \mathbf{R}$.

Proof. Let T satify $2C_{00}T \leq 1$ with the constant C_{00} in Lemma 3.3 and $2\kappa_2 T \leq 1$. Then by Lemma 3.3 the map $k \mapsto U(k) = x + k - \tilde{x}_L(y,k)$ is a contraction. So Lemma 3.4 is proved. ∎

Let $(y, \tilde{k}(y,x))$ be the inverse of the map $(y,k) \mapsto (y,x) = (y, \tilde{x}_L(y,k))$ in Lemma 3.4 and set $k(y,x) = \tilde{k}(y,x)/T_L$. Put

$$\begin{aligned} x_j^*(y,x) &= x_j^\sharp(y, k(y,x)), \quad j = 1, \cdots, L-1, \\ k_j^*(y,x) &= \frac{x_j^*(y,x) - x_{j-1}^*(y,x)}{t_j}, \quad j = 1, \cdots, L, \end{aligned} \tag{3.8}$$

where $x_0^* = y$ and $x_L^* = x$.

Lemma 3.5. *If $T_L < T$, then $x_j^*(y,x), j = 1, \cdots, L-1$ is the unique critical point of S with $x_0^* = y$ and $x_L^* = x$, i.e. it satisfies (1.7).*

Proof. Let y, $x \in \mathbf{R}$. Then by Lemma 3.1, for y, $k = k(y,x)$ there exists a unique $(x_0^\sharp(y,k), \cdots, x_L^\sharp(y,k))$ which satisfies (3.1). And we have $x_L^\sharp(y, k(y,x)) = x$ by Lemma 3.4. These $x_j^\sharp(y,k(y,x))$ are nothing but the desired $x_j^*(y,x)$. ∎

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