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A group generated by the Lévy Laplacian

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1. INTRODUCTION

The Lévy Laplacian $\Delta_L$ is one of infinite dimensional Laplacians introduced by P. Lévy in his book [Lé 22]. In his book, he mentioned that $\Delta_L$ comes from the singular part $f''_s$ of the second derivative $f''$, i.e.,

$$\Delta_L f(x) = \int_0^1 f''_s(x;u) du.$$

This Laplacian has been studied by many authors. In 1975, T.Hida introduced $\Delta_L$ into the theory of generalized white noise functionals in [Hi 75]. H.-H. Kuo [Ku 83,89,92a,92b] defined the Fourier-Mehler transform on the space $(S)^*$ of generalized white noise functionals and gave a relation between its transform and $\Delta_L$. An interesting characterization of $\Delta_L$ in terms of rotation groups was obtained by N. Obata [Ob 90]. Recently, T. Hida [Hi 92b] applied $\Delta_L$ to S. Tomonaga's many time theory in quantum physics.

The purpose of this paper is to construct a group generated by $\Delta_L$.

In §2, we will explain a construction of the space of generalized white noise functionals and define the Lévy Laplacian $\Delta_L^T$ for a finite interval $T$ in $\mathbb{R}$ in that space. Moreover, we introduce an operator $\Delta$ and prove that $\Delta$ coincides with $2\Delta_L^T$ on a domain $D_L^T$ in $(S)^*$. In §3, we will construct a $(C_0)$-group $\{G_t\}_{t \in \mathbb{R}}$ generated by $\Delta_L^T$. In the last section, we will give a relation between the adjoint operator of Kuo's Fourier-Mehler transform and a group $\{G_{it}\}_{t \in \mathbb{R}}$.

2. THE LÉVY LAPLACIAN IN THE WHITE NOISE CALCULUS

In this section, we introduce a space of Hida distributions following [Hi 80], [KT 80-82] and [PS 91] (See also, [HKPS 93], [HOS 92] and [Ob 92]) and the Lévy Laplacian defined on a domain in this space.

1) Let $L^2(\mathbb{R})$ be the Hilbert space of real square-integrable functions on $\mathbb{R}$ with norm $| \cdot |_0$. Consider a Gel'fand triple

$$S = S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S^* = S^*(\mathbb{R}),$$
where $S(\mathbb{R})$ is the Schwartz space consisting of rapidly decreasing functions on $\mathbb{R}$ and $S^*(\mathbb{R})$ is the dual space of $S(\mathbb{R})$.

Let $A$ be the following operator

$$A = -(d/dx)^2 + x^2 + 1.$$ 

For each $p \in \mathbb{Z}$, we define $|f|_p = |A^p f|_0$ and let $S_p$ be the completion of $S$ with respect to the norm $|\cdot|_p$. Then the dual space of $S_p'$ of $S_p$ is the same as $S_{-p}$.

2) Let $\mu$ be a probability measure on $S^*$ with the characteristic functional given by

$$C(\xi) \equiv \int_{S^*} \exp\{i < x, \xi >\} \, d\mu(x) = \exp\{-\frac{1}{2} |\xi|_0^2\}, \quad \xi \in S.$$

Let $(L^2) = L^2(S^*, \mu)$ be the space of complex-valued square-integrable functionals defined on $S^*$ and define the $S$-transform by

$$S\varphi(\xi) = C(\xi) \int_{S^*} \exp\{< x, \xi >\} \varphi(x) \, d\mu(x), \quad \varphi \in (L^2).$$

The Hilbert space admits the well-known Wiener-Itô decomposition:

$$(L^2) = \bigoplus_{n=0}^{\infty} H_n,$$

where $H_n$ is the space of multiple Wiener integrals of order $n \in \mathbb{N}$ and $H_0 = C$. From this decomposition theorem, each $\varphi \in (L^2)$ is uniquely represented as

$$\varphi = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in L^2_C(\mathbb{R}) \otimes^n,$$

where $I_n \in H_n$ and $L^2_C(\mathbb{R}) \otimes^n$ denotes the $n$-th symmetric tensor product of the complexification of $L^2(\mathbb{R})$.

For each $p \in \mathbb{Z}, p \geq 0$, we define the norm $||\varphi||_p$ of $\varphi = \sum_{n=0}^{\infty} I_n(f_n)$, by

$$||\varphi||_p = \left( \sum_{n=0}^{\infty} n! |f_n|_{p,n} \right)^{1/2},$$

where $|\cdot|_{p,n}$ is the norm of $S^*_{p,n}$ (the $n$-th symmetric tensor product of the complexification of $S_p$). The norm $||\cdot||_0$ is nothing but the $(L^2)$-norm. We put

$$(S)_p = \{ \varphi \in (L^2); ||\varphi||_p < \infty \}$$
for $p \in \mathbb{Z}, p \geq 0$. Let $(S)_{p}^{*}$ be the dual space of $(S)_{p}$. Then $(S)_{p}$ and $(S)_{p}^{*}$ are Hilbert spaces with the norm $\| \cdot \|_{p}$ and the dual norm of $\| \cdot \|_{p}$, respectively.

Denote the projective limit space of the $(S)_{p}, p \in \mathbb{Z}, p \geq 0$, and the inductive limit space of the $(S)_{p}^{*}, p \in \mathbb{Z}, p \geq 0$, by $(S)$ and $(S)^{*}$, respectively. Then $(S)$ is a nuclear space and $(S)^{*}$ is nothing but the dual space of $(S)$. The space $(S)^{*}$ is called the space of Hida distributions (or generalized white noise functionals).

Since $\exp < \xi > \in (S)$, the $S$-transform is extended to an operator $U$ defined on $(S)^{*}$:

$$U \Phi(\xi) = C(\xi) \ll \Phi, \exp < \cdot, \xi > \gg, \xi \in S,$$

where $\ll \cdot, \cdot \gg$ is the canonical pairing of $(S)$ and $(S)^{*}$. We call $U \Phi$ the $U$-functional of $\Phi$.

3) We next introduce the definition of the Lévy Laplacian following Kuo [Ku 92] (see also [HKPS 93]). Let $U$ be a Fréchet differentiable function defined on $S$, i.e. we assume that there exists a map $U'$ from $S$ to $S^{*}$ such that

$$U(\xi + \eta) = U(\xi) + U'(\xi)(\eta) + o(\eta), \eta \in S,$$

where $o(\eta)$ means that there exists $p \in \mathbb{Z}, p \geq 0$, depending on $\xi$ such that $o(\eta)/|\eta|_{p} \to 0$ as $|\eta|_{p} \to 0$. Then the first variation

$$\delta U(\xi; \eta) = dU(\xi + \lambda \eta)/d\lambda|_{\lambda=0}$$

is expressed in the form

$$\delta U(\xi; \eta) = \int_{\mathbb{R}} U'(\xi; u) \eta(u) du$$

for every $\eta \in S$ by using the generalized function $U' (\xi; \cdot)$. We define the Hida derivative $\partial_{t} \Phi$ of $\Phi$ to be the generalized white noise functional whose $U$-functional is given by $U'(\xi; t)$.

Definition. (I) A Hida distribution $\Phi$ is called an $L$-functional if for each $\xi \in S$, there exist $(U \Phi)'(\xi; \cdot) \in L_{loc}^{1}(\mathbb{R}), (U \Phi)'_{s}(\xi; \cdot) \in L_{loc}^{1}(\mathbb{R})$ and $(U \Phi)'_{r}(\xi; \cdot, \cdot) \in L_{loc}^{1}(\mathbb{R}^{2})$ such that the first and second variations are uniquely expressed in the forms:

$$(U \Phi)'(\xi)(\eta) = \int_{\mathbb{R}} (U \Phi)'(\xi; u) \eta(u) du,$$

and

$$(U \Phi)'_{s}(\xi)(\eta, \zeta) = \int_{\mathbb{R}} (U \Phi)'_{s}(\xi; u) \eta(u) \zeta(u) du$$

$$+ \int_{\mathbb{R}^{2}} (U \Phi)'_{r}(\xi; u, v) \eta(u) \zeta(v) dudv,$$

(2.1)

for each $\eta, \zeta \in S$, respectively and for any finite interval $T$, $\int_{T} (U \Phi)'_{s}(\cdot; u) du$ is a $U$-functional.
(II) Let $D_{L}$ denote the set of all $L$-functionals. For $\Phi \in D_{L}$ and any finite interval $T$ in $\mathbb{R}$, the Lévy Laplacian $\Delta_{L}^{T}$ is defined by

$$\Delta_{L}^{T}\Phi = U^{-1}\left[\frac{1}{|T|} \int_{T} (U\Phi)^{''}_{s}(\cdot;u) \, du\right].$$

Remark. Explicit conditions for the uniqueness of the above decomposition (2.1) is given in [HKPS 93, chapter 6].

Let $T$ be a finite interval in $\mathbb{R}$. Take a smooth function $e$ defined on $\mathbb{R}$ satisfying $0 \leq e(u) \leq 1$ for all $u \in \mathbb{R}$, $e(u) = 1$ for $|u| \leq 1/2$ and $e(u) = 0$ for $|u| \geq 1$. Let $\rho_{n}^{*}$ be the Friedrichs mollifier. Put $e_{n}(u) = e(u/n)$ and $\theta_{n}^{T} = \sqrt{2}|\rho_{n}|_{0}^{-1}|T|^{-1/2}$, $n = 1,2,\ldots$.

We define an operator $\Delta$ for a Hida distribution $\Phi$ by

$$U[\Delta\Phi](\xi) = \lim_{n \to \infty} \int_{S} U\Phi''(\xi)(\theta_{n}^{T}e_{n}(\rho_{n}^{*}x), \theta_{n}^{T}e_{n}(\rho_{n}^{*}x)) \, d\mu(x),$$

if the limit exists in $U[(S)^{*}]$. From now on, we denote $e_{n}(\rho_{n}^{*}x)$ by $j_{n}(x)$. Let $D_{L}^{T}$ denote the set of all $L$-functionals $\Phi$ satisfying $U\Phi(\eta) = 0$ for $\eta$ with $\text{supp}(\eta) \subset T^{c}$. In [Sa 94], we obtained the following result. (For the proof, see [Sa 94].)

**Theorem 1.** Let $T$ be a finite interval in $\mathbb{R}$ and $\Phi$ an $L$-functional in $D_{L}^{T}$. Then, we have $\Delta\Phi = 2\Delta_{L}^{T}\Phi$.

### 3. The Lévy Laplacian as the Infinitesimal Generator

A generalized functional $\Phi$ is called a normal functional if its $U$-functional $U\Phi$ is given by a finite linear combination of

$$\int_{A^{k}} f(u_{1}, \ldots, u_{k})\xi(u_{1})^{p_{1}} \cdots \xi(u_{k})^{p_{k}} \, du_{1} \cdots du_{k},$$

where $f \in L^{1}(A^{k}), p_{1}, \ldots, p_{k} \in \mathbb{N} \cup \{0\}, k \in \mathbb{N},$ and $A$ : a finite interval in $\mathbb{R}$. This functional $\Phi$ is in $D_{L}$. Let $\mathcal{N}_{T}$ denote the set of all normal functionals in $D_{L}^{T}$. For $p > 1$ and $\Phi \in D_{L}^{T}$, we define a $-p$-norm $\cdot \cdot$ by

$$\|\Phi\|_{-p}^{2} = \sum_{k=0}^{\infty} \|(\Delta_{L}^{T})^{k}\Phi\|_{-p}^{2}(\in [0, \infty))$$

and denote the completion of $\mathcal{N}_{T}$ with respect to the norm $\cdot \cdot_{-p}$ by $D_{L}^{(-p)}$. Then $D_{L}^{(-p)}$ is the Hilbert space with the norm $\cdot \cdot_{-p}$ and $\Delta_{L}^{T}$ is a bounded linear operator.
on $D^{(-p)}_L$. Hence a $(C_0)$-group $\{G_t, t \in \mathbb{R}\}$ is given by

$$G_t = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{t^k}{k!} (\Delta^T_L)^k,$$  \hspace{1cm} (3.2)$$

in the sense of the operator norm. It is easily checked that $\|G_t\| \leq e^{|t|}$, for any $t \in \mathbb{R}$.

Define an operator $g_t$ on $\mathcal{N}_T$ for $t \geq 0$ by

$$U[g_t \Phi](\xi) = \lim_{n \to \infty} \int_{\mathcal{S}} U\Phi(\xi + \sqrt{t} \theta_n^T j_n(x)) \, d\mu(x), \Phi \in \mathcal{N}_T.$$  

For a normal functional $\Phi$ which $U\Phi$ is given as in (3.1) with the domain $A^k \subset T^k$, it is easily checked that

$$U[g_t \Phi](\xi) = \sum_{\nu_1=0}^{[p_1/2]} \cdots \sum_{\nu_k=0}^{[p_k/2]} \frac{p_1! \cdots p_k!}{(2\nu_1)!!(p_1-2\nu_1)! \cdots (2\nu_k)!!(p_k-2\nu_k)!} \left( \frac{2t}{|T|} \right)^{\nu_1+\cdots+\nu_k} \int_{A^k} f(u_1, \ldots, u_k) \xi(u_1)^{p_1-2\nu_1} \cdots \xi(u_k)^{p_k-2\nu_k} \, du_1 \cdots du_k.$$  

Therefore, $g_t$ is a linear operator from $\mathcal{N}_T$ to itself. By Theorem 1, it can be checked that $G_t = g_t$ on $\mathcal{N}_T$. Since $\mathcal{N}_T$ is dense in $D^{(-p)}_L$, we have the following

**Theorem 2.** For any $t \geq 0$, $g_t$ is extended to the operator $G_t$.

### 4. The Fourier-Mehler Transform and the Lévy Laplacian

An characterization of Hida distributions was obtained by J. Potthoff and L. Streit [PS 91]. From [PS 91], we see that for any $U$-functional $F$, and $\xi, \eta$ in $\mathcal{S}$, the function $F(\lambda \xi + \eta)$, $\lambda \in \mathbb{R}$, extends to an entire function $F(z \xi + \eta)$, $z \in \mathbb{C}$. Then we can define an operator $g_{it}$, $t \in \mathbb{R}$, by

$$U[g_{it} \Phi](\xi) = \lim_{n \to \infty} \int_{\mathcal{S}} U\Phi(\xi + \sqrt{it} \theta_n^T j_n(x)) \, d\mu(x),$$  

if the limit exists. Since $\mu$ is symmetric, the integral is defined independent of choices of the branch of $\sqrt{it}$. As in (3.2), we can naturally define $G_{it}$, $t \in \mathbb{R}$, by

$$G_{it} = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{(it)^k}{k!} (\Delta^T_L)^k,$$  

on $D^{(-p)}_L$. 

An infinite dimensional Fourier-Mehler transform $F_\theta$, $\theta \in \mathbb{R}$, on $(S)^*$ was defined by H.-H. Kuo [Ku 91] as follows. The transform $F_\theta \Phi$, $\theta \in \mathbb{R}$ of $\Phi \in (S)^*$ is defined by the unique generalized white noise functional with the $U$-functional

$$U[F_\theta \Phi](\xi) = U\Phi(e^{i\theta} \xi) \exp \left[ \frac{i}{2} e^{i\theta} \sin \theta |\xi|_0^2 \right], \quad \xi \in S.$$ 

Moreover, the adjoint operator $F_\theta^*$ of $F_\theta$ is given by

$$F_\theta^* \Phi = \sum_{n=0}^{\infty} I_n(h_n(\Phi; \theta)) \text{ for } \Phi = \sum_{n=0}^{\infty} I_n(f_n) \in (S),$$

where

$$h_n(\Phi; \theta) = \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!} \frac{i}{2} \sin \theta)^m \exp \left[ \frac{i}{2} (m+n) \theta \right] \tau^\otimes m * f_{n+2m};$$

$$\tau^\otimes m = \int_{\mathbb{R}^m} \delta_{t_1} \otimes \delta_{t_1} \otimes \cdots \otimes \delta_{t_m} \otimes \delta_{t_m} dt_1 \cdots dt_m.$$ 

This operator $F_\theta^*$ is a continuous linear operator on $(S)$. (For details, see [Ku 91] and also [HKO 90]) On $(S)$, the Gross Laplacian $\Delta_G$ (See [Gr 65, 67]) and the number operator $N$ is given by

$$\Delta_G \Phi = \int_{\mathbb{R}} \partial_t^2 \Phi dt$$

and

$$N \Phi = \int_{\mathbb{R}} \partial_t^* \partial_t \Phi dt,$$

respectively (see [Ku 86]). The operator $e^{i\theta N}$ is called the Fourier-Wiener transform (see [HKO 90]). Now, we introduce an operator $e^{\frac{i}{2} \theta \Delta_G}$ from $(S)$ into itself given by

$$e^{\frac{i}{2} \vartheta \Delta_G} \Phi = \sum_{n=0}^{\infty} I_n(\ell_n(\Phi; \theta));$$

(4.1)

$$\ell_n(\Phi; \theta) = \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!} \left( \frac{i}{2} \theta \right)^m \tau^\otimes m * f_{n+2m},$$

for $\Phi = \sum_{n=0}^{\infty} I_n(f_n) \in (S)$. Then we have the followings.

**Lemma 1.**

$$F_\theta^* = e^{i\theta N} \circ e^{\frac{i}{2} (e^{i\theta} \sin \theta) \Delta_G}.$$ 

**Proof:** Take $\Phi = \sum_{n=0}^{\infty} I_n(f_n) \in (S)$. From (4.1), we see that

$$e^{\frac{i}{2} (e^{i\theta} \sin \theta) \Delta_G} \Phi = \sum_{n=0}^{\infty} I_n(\ell_n(\Phi; e^{i\theta} \sin \theta)).$$

Hence,

$$e^{i\theta N} \left( e^{\frac{i}{2} (e^{i\theta} \sin \theta) \Delta_G} \Phi \right) = \sum_{n=0}^{\infty} I_n(e^{i\theta} \ell_n(\Phi; e^{i\theta} \sin \theta)).$$
Since $e^{in\theta}l_n(\Phi; e^{i\theta} \sin \theta) = h_n(\Phi; \theta)$, we obtain (4.2).}

**Lemma 2.** For any $\Phi \in (S)$, we have

$$U[e^{\frac{i}{2} \theta \Delta_G \Phi}](\xi) = \int_{S^*} U\Phi(\xi + \sqrt{i\theta}y) \, d\mu(y). \tag{4.3}$$

**Remark.** For any $\Phi \in (S), \xi \in S$ and $z_1, z_2 \in \mathbb{C}$, the functional $U\Phi(z_1 \xi + z_2 \eta), \eta \in S$, can be extended to a functional $U\Phi(z_1 \xi + z_2 y)$, same symbol $U\Phi(z_1 \xi + z_2 y)$.

**Proof:** For $\Phi = \sum_{n=0}^\infty I_n(f_n) \in (S)$, the right-hand side of (4.3) has the following expansion:

$$\sum_{n=0}^\infty \int_{\mathbb{R}^n} f_n(u) \int_{S^*} \{\xi(u_1) + \sqrt{i\theta}x(u_1)\} \cdots \{\xi(u_n) + \sqrt{i\theta}x(u_n)\} d\mu(x) du$$

$$= \sum_{n=0}^\infty \sum_{\nu=0}^{\lfloor n/2 \rfloor} \frac{n!}{(2\nu)!!(n-2\nu)!} (i\theta)^\nu <\xi^{\otimes(n-2\nu)}, \tau^\nu * f_n> = \sum_{m=0}^\infty <\xi^{\otimes m}, l_m(\Phi; \theta)>.$$  

From (4.1), we see that the last series is equal to $U[e^{\frac{i}{2} \theta \Delta_G \Phi}](\xi)$.}

Define an operator $J_n$ by

$$U[J_n \Phi](\xi) = U\Phi \circ j_n(\xi), \quad \Phi \in D_L^{(-p)}, \quad \xi \in S.$$  

For all $n \in \mathbb{N}$ and $\Phi \in D_L^{(-p)}$, we can easily check $J_n \Phi \in (S)$. Then we have the following.

**Theorem 3.** Let $\Phi \in D_L^{(-p)}$ be a generalized white noise functional with the $U$-functional given by $\psi(F_1, \ldots, F_n)$, where $\psi$ is an entire function on $\mathbb{C}$ and $F_1, \ldots, F_n \in U[\mathcal{N}_T]$. We assume the condition

$$\sum_{k_1, \ldots, k_n=0}^\infty \frac{1}{k_1! \cdots k_n!} |\partial_{u_1}^{k_1} \cdots \partial_{u_1}^{k_n} \psi(0, \ldots, 0)|.$$  

$$\sup_N \int_{S^*} \left| (F_1 \circ j_N)^{k_1} \cdots (F_n \circ j_N)^{k_n} (ie^{i\alpha_N(t)} \xi + \sqrt{ie^{i\alpha_N(t)} \sin \alpha_N(t)} x) \right| \, d\mu(x) < \infty$$  

holds for all $t > 0$ and $\xi \in S$, where $\alpha_N(t) = t(\theta_N^2)^{2}$. Then

$$\lim_{N \to \infty} U[F_{\alpha_N(t)} J_N \Phi](\xi) = U[G_{it} \Phi](\xi), \quad \xi \in S. \tag{4.4}$$

**Proof:** From Lemma 2, we have

$$U[e^{\frac{i}{2} i\alpha_N(t) \sin \alpha_N(t) \Delta_G J_N \Phi}](\xi) = \int_{S^*} U[J_N \Phi](\xi + \sqrt{ie^{i\alpha_N(t)} \sin \alpha_N(t)} y) \, d\mu(y).$$
This functional is expressed in the form given by
\[ \sum_{\ell=0}^{\infty} \langle \xi^{\otimes \ell}, f_{N,\ell} \rangle, \]
where \( f_{N,\ell} \in S_{C}^{\otimes \ell} \). Hence, from Lemma 1, we get
\[ U[F_{N}(t)](\xi) = \sum_{\ell=0}^{\infty} e^{i\alpha N(t)\ell} \langle \xi^{\otimes \ell}, f_{N,\ell} \rangle. \]

From the condition of this theorem and the Lebesgue convergence theorem, we can calculate as follows:

\[ \lim_{N \to \infty} U[F_{N}(t)](\xi) = \lim_{N \to \infty} U[e^{i\alpha N(t)} \Delta g J_{N} \Phi](e^{i\alpha N(t)} \xi) \]
\[ = \sum_{k_{1}, \ldots, k_{n}=0}^{\infty} \frac{1}{k_{1}! \cdots k_{n}!} \partial_{\xi_{1}}^{k_{1}} \cdots \partial_{\xi_{n}}^{k_{n}} \psi(0, \ldots, 0). \]

By the direct calculations, it is easily checked that
\[ \lim_{N \to \infty} \int_{S^{*}} ((F_{1} \circ j_{N})^{k_{1}} \cdots (F_{n} \circ j_{N})^{k_{n}})(i e^{i\alpha N(t)} \xi + \sqrt{i e^{i\alpha N(t)} \sin \alpha N(t)} x) d\mu(x) \]
\[ = U[g_{it} U^{-1} F_{1}](\xi)^{k_{1}} \cdots U[g_{it} U^{-1} F_{n}](\xi)^{k_{n}}. \]

Consequently, we obtain (4.4).

\textbf{REFERENCES}


