Title: A group generated by the Levy Laplacian (White Noise Analysis and Quantum Probability)

Author(s): Saito, Kimiaki

Citation: 数理解析研究所講究録 (1994), 874: 192-201

Issue Date: 1994-06

URL: http://hdl.handle.net/2433/84105

Type: Departmental Bulletin Paper

Textversion: publisher
A group generated by the Lévy Laplacian

KIMIAKI SAITO

Department of Mathematics
Meijo University
Nagoya 468, Japan

1. INTRODUCTION

The Lévy Laplacian $\Delta_L$ is one of infinite dimensional Laplacians introduced by P. Lévy in his book [Lé 22]. In his book, he mentioned that $\Delta_L$ comes from the singular part $f''_s$ of the second derivative $f''$, i.e.,

$$\Delta_L f(x) = \int_0^1 f''_s(x; u) du.$$  

This Laplacian has been studied by many authors. In 1975, T. Hida introduced $\Delta_L$ into the theory of generalized white noise functionals in [Hi 75]. H.-H. Kuo [Ku 83, 89, 92a, 92b] defined the Fourier-Mehler transform on the space $(S)^*$ of generalized white noise functionals and gave a relation between its transform and $\Delta_L$. An interesting characterization of $\Delta_L$ in terms of rotation groups was obtained by N. Obata [Ob 90]. Recently, T. Hida [Hi 92b] applied $\Delta_L$ to S. Tomonaga's many time theory in quantum physics.

The purpose of this paper is to construct a group generated by $\Delta_L$.

In §2, we will explain a construction of the space of generalized white noise functionals and define the Lévy Laplacian $\Delta_L^T$ for a finite interval $T$ in $\mathbb{R}$ in that space. Moreover, we introduce an operator $\Delta$ and prove that $\Delta$ coincides with $2\Delta_L^T$ on a domain $D_L^T$ in $(S)^*$. In §3, we will construct a $(C_0)$-group $\{G_t\}_{t \in \mathbb{R}}$ generated by $\Delta_L^T$. In the last section, we will give a relation between the adjoint operator of Kuo's Fourier-Mehler transform and a group $\{G_{it}\}_{t \in \mathbb{R}}$.

2. THE LÉVY LAPLACIAN IN THE WHITE NOISE CALCULUS

In this section, we introduce a space of Hida distributions following [Hi 80], [KT 80-82] and [PS 91] (See also, [HKPS 93], [HOS 92] and [Ob 92]) and the Lévy Laplacian defined on a domain in this space.

1) Let $L^2(\mathbb{R})$ be the Hilbert space of real square-integrable functions on $\mathbb{R}$ with norm $| \cdot |_0$. Consider a Gel'fand triple

$$S = S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S^* = S^*(\mathbb{R}),$$
where $S(R)$ is the Schwartz space consisting of rapidly decreasing functions on $R$ and $S^*(R)$ is the dual space of $S(R)$.

Let $A$ be the following operator

$$A = -(d/dx)^2 + x^2 + 1.$$

For each $p \in \mathbb{Z}$, we define $|f|_p = |A^p f|_0$ and let $S_p$ be the completion of $S$ with respect to the norm $| \cdot |_p$. Then the dual space of $S'_p$ of $S_p$ is the same as $S_{-p}$.

2) Let $\mu$ be a probability measure on $S^*$ with the characteristic functional given by

$$C(\xi) \equiv \int_{S^*} \exp\{i <x, \xi>\} \, d\mu(x) = \exp\{-\frac{1}{2}||\xi||_0^2\}, \ \xi \in S.$$

Let $(L^2) = L^2(S^*, \mu)$ be the space of complex-valued square-integrable functionals defined on $S^*$ and define the $S$-transform by

$$S\varphi(\xi) = C(\xi) \int_{S^*} \exp\{<x, \xi>\} \varphi(x) \, d\mu(x), \ \varphi \in (L^2).$$

The Hilbert space admits the well-known Wiener-Itô decomposition:

$$(L^2) = \bigoplus_{n=0}^{\infty} H_n,$$

where $H_n$ is the space of multiple Wiener integrals of order $n \in \mathbb{N}$ and $H_0 = C$. From this decomposition theorem, each $\varphi \in (L^2)$ is uniquely represented as

$$\varphi = \sum_{n=0}^{\infty} I_n(f_n), \ f_n \in L^2_C(R)^{\otimes^n},$$

where $I_n \in H_n$ and $L^2_C(R)^{\otimes^n}$ denotes the $n$-th symmetric tensor product of the complexification of $L^2(R)$.

For each $p \in \mathbb{Z}, p \geq 0$, we define the norm $||\varphi||_p$ of $\varphi = \sum_{n=0}^{\infty} I_n(f_n)$, by

$$||\varphi||_p = \left( \sum_{n=0}^{\infty} n! |f_n|_{p,n} \right)^{1/2},$$

where $| \cdot |_{p,n}$ is the norm of $S^*_{C(p)}$ (the $n$-th symmetric tensor product of the complexification of $S_p$). The norm $|| \cdot ||_0$ is nothing but the $(L^2)$-norm. We put

$$(S)_p = \{ \varphi \in (L^2); ||\varphi||_p < \infty \}.$$
for \( p \in \mathbb{Z}, p \geq 0 \). Let \((S)^*_{p}\) be the dual space of \((S)_{p}\). Then \((S)_{p}\) and \((S)^*_{p}\) are Hilbert spaces with the norm \( \| \cdot \|_{p} \) and the dual norm of \( \| \cdot \|_{p} \), respectively.

Denote the projective limit space of the \((S)_{p}, p \in \mathbb{Z}, p \geq 0\), and the inductive limit space of the \((S)^*_{p}, p \in \mathbb{Z}, p \geq 0\), by \((S)\) and \((S)^*\), respectively. Then \((S)\) is a nuclear space and \((S)^*\) is nothing but the dual space of \((S)\). The space \((S)^*\) is called the space of \textit{Hida distributions} (or \textit{generalized white noise functionals}).

Since \( \exp<\cdot, \xi> \in (S) \), the \( S \)-transform is extended to an operator \( U \) defined on \((S)^*\):

\[
U\Phi(\xi) = C(\xi) \ll \Phi, \exp<\cdot, \xi>, \xi \in S,
\]

where \( \ll \cdot, \cdot \gg \) is the canonical pairing of \((S)\) and \((S)^*\). We call \( U\Phi \) the \textit{U-functional} of \( \Phi \).

3) We next introduce the definition of the Lévy Laplacian following Kuo [Ku 92] (see also [HKPS 93]). Let \( U \) be a \textit{Fréchet differentiable} function defined on \( S \), i.e. we assume that there exists a map \( U' \) from \( S \) to \( S^* \) such that

\[
U(\xi + \eta) = U(\xi) + U'(\xi)(\eta) + o(\eta), \eta \in S,
\]

where \( o(\eta) \) means that there exists \( p \in \mathbb{Z}, p \geq 0 \), depending on \( \xi \) such that \( o(\eta)/|\eta|_{p} \to 0 \) as \( |\eta|_{p} \to 0 \). Then the first variation

\[
\delta U(\xi; \eta) = dU(\xi + \lambda \eta)/d\lambda|_{\lambda=0}
\]

is expressed in the form

\[
\delta U(\xi; \eta) = \int_{\mathbb{R}} U'(\xi; u)\eta(u)du
\]

for every \( \eta \in S \) by using the generalized function \( U'(\xi; \cdot) \). We define the \textit{Hida derivative} \( \partial_{t}\Phi \) of \( \Phi \) to be the generalized white noise functional whose \( U \)-functional is given by \( U'(\xi; t) \).

Definition. (I) A Hida distribution \( \Phi \) is called an \textit{L-functional} if for each \( \xi \in S \), there exist \((U\Phi)'(\xi; \cdot) \in L_{loc}^{1}(\mathbb{R}), (U\Phi)''_{s}(\xi; \cdot) \in L_{loc}^{1}(\mathbb{R})\) and \((U\Phi)''_{r}(\xi; \cdot, \cdot) \in L_{loc}^{1}(\mathbb{R}^{2})\) such that the first and second variations are uniquely expressed in the forms:

\[
(U\Phi)'(\xi)(\eta) = \int_{\mathbb{R}}(U\Phi)'(\xi; u)\eta(u)du,
\]

and

\[
(U\Phi)''(\xi)(\eta, \zeta) = \int_{\mathbb{R}}(U\Phi)''_{s}(\xi; u)\eta(u)\zeta(u)du + \int_{\mathbb{R}^{2}}(U\Phi)''_{r}(\xi; u, v)\eta(u)\zeta(v)dudv,
\]

for each \( \eta, \zeta \in S \), respectively and for any finite interval \( T \), \( \int_{T}(U\Phi)''_{s}(\cdot; u)du \) is a \( U \)-functional.
Let $D_L$ denote the set of all $L$-functionals. For $\Phi \in D_L$ and any finite interval $T$ in $\mathbb{R}$, the Lévy Laplacian $\Delta_L^T$ is defined by

$$\Delta_L^T \Phi = U^{-1} \left[ \frac{1}{|T|} \int_T (U\Phi)''(\cdot; u) \, du \right].$$

Remark. Explicit conditions for the uniqueness of the above decomposition (2.1) is given in [HKPS 93, chapter 6].

Let $T$ be a finite interval in $\mathbb{R}$. Take a smooth function $e$ defined on $\mathbb{R}$ satisfying $0 \leq e(u) \leq 1$ for all $u \in \mathbb{R}$, $e(u) = 1$ for $|u| \leq 1/2$ and $e(u) = 0$ for $|u| \geq 1$. Let $\rho_n^*$ be the Friedrichs mollifier. Put $e_n(u) = e(u/n)$ and $\theta_n^T = \sqrt{2}\rho_n|_{0}^{-1}|T|^{-1/2}$, $n = 1, 2, \ldots$

We define an operator $\Delta$ for a Hida distribution $\Phi$ by

$$U[\Delta \Phi](\xi) = \lim_{n \to \infty} \int_{S^s} U\Phi''(\xi)(\theta_n^T e_n(\rho_n^* x), \theta_n^T e_n(\rho_n^* x)) \, d\mu(x),$$

if the limit exists in $U[(S)^*]$. From now on, we denote $e_n(\rho_n^* x)$ by $j_n(x)$. Let $D_L^T$ denote the set of all $L$-functionals $\Phi$ satisfying $U\Phi(\eta) = 0$ for $\eta$ with $\text{supp}(\eta) \subset T^c$. In [Sa 94], we obtained the following result. (For the proof, see [Sa 94].)

**Theorem 1.** Let $T$ be a finite interval in $\mathbb{R}$ and $\Phi$ an $L$-functional in $D_L^T$. Then, we have $\Delta \Phi = 2\Delta_L^T \Phi$.

3. **The Lévy Laplacian as the infinitesimal generator**

A generalized functional $\Phi$ is called a normal functional if its $U$-functional $U\Phi$ is given by a finite linear combination of

$$\int_{A^k} f(u_1, \ldots, u_k)\xi(u_1)^{p_1} \cdots \xi(u_k)^{p_k} \, du_1 \cdots du_k,$$

where $f \in L^1(A^k), p_1, \ldots, p_k \in \mathbb{N} \cup \{0\}, k \in \mathbb{N}$, and $A$ : a finite interval in $\mathbb{R}$. This functional $\Phi$ is in $D_L$. Let $\mathcal{N}_T$ denote the set of all normal functionals in $D_L^T$. For $p > 1$ and $\Phi \in D_L^T$, we define a $-p$-norm $\| \cdot \|_{-p}$ by

$$\| \Phi \|_{-p}^2 = \sum_{k=0}^{\infty} \| (\Delta_L^T)^k \Phi \|_{-p}^2 (\in [0, \infty])$$

and denote the completion of $\mathcal{N}_T$ with respect to the norm $\| \cdot \|_{-p}$ by $D_L^{(-p)}$. Then $D_L^{(-p)}$ is the Hilbert space with the norm $\| \cdot \|_{-p}$ and $\Delta_L^T$ is a bounded linear operator.
on $D_{L}^{(-p)}$. Hence a $(C_{0})$-group $\{G_{t}, t \in \mathbb{R}\}$ is given by

$$G_{t} = \lim_{n \rightarrow \infty} \sum_{k=0}^{n} \frac{t^{k}}{k!} (\Delta_{L}^{T})^{k},$$

in the sense of the operator norm. It is easily checked that $\|G_{t}\| \leq e^{\|t\|}$ for any $t \in \mathbb{R}$.

Define an operator $g_{t}$ on $\mathcal{N}_{T}$ for $t \geq 0$ by

$$U[g_{t} \Phi](\xi) = \lim_{n \rightarrow \infty} \int_{S^{n}} U\Phi(\xi + \sqrt{t} \theta_{n}^{T} j_{n}(x)) d\mu(x), \quad \Phi \in \mathcal{N}_{T}.$$ 

For a normal functional $\Phi$ which $U\Phi$ is given as in (3.1) with the domain $A^{k} \subset T^{k}$, it is easily checked that

$$U[g_{t} \Phi](\xi) = \sum_{\nu_{1}=0}^{[p_{1}/2]} \cdots \sum_{\nu_{k}=0}^{[p_{k}/2]} \frac{p_{1}! \cdots p_{k}!}{(2\nu_{1})!!(p_{1}-2\nu_{1})! \cdots (2\nu_{k})!!(p_{k}-2\nu_{k})!} \left(\frac{2t}{|T|}\right)^{\nu_{1}+\cdots+\nu_{k}} \int_{A^{k}} f(u_{1}, \ldots, u_{k}) \xi(u_{1})^{p_{1}-2\nu_{1}} \cdots \xi(u_{k})^{p_{k}-2\nu_{k}} du_{1} \cdots du_{k}.$$ 

Therefore, $g_{t}$ is a linear operator from $\mathcal{N}_{T}$ to itself. By Theorem 1, it can be checked that $G_{t} = g_{t}$ on $\mathcal{N}_{T}$. Since $\mathcal{N}_{T}$ is dense in $D_{L}^{(-p)}$, we have the following

**THEOREM 2.** For any $t \geq 0$, $g_{t}$ is extended to the operator $G_{t}$.

### 4. The Fourier-Mehler Transform and the Lévy Laplacian

An characterization of Hida distributions was obtained by J. Potthoff and L. Streit [PS 91]. From [PS 91], we see that for any $U$-functional $F$, and $\xi, \eta$ in $\mathcal{S}$, the function $F(\lambda \xi + \eta)$, $\lambda \in \mathbb{R}$, extends to an entire function $F(z\xi + \eta)$, $z \in \mathbb{C}$. Then we can define an operator $g_{it}$, $t \in \mathbb{R}$, by

$$U[g_{it} \Phi](\xi) = \lim_{n \rightarrow \infty} \int_{S^{n}} U\Phi(\xi + \sqrt{it} \theta_{n}^{T} j_{n}(x)) d\mu(x),$$

if the limit exists. Since $\mu$ is symmetric, the integral is defined independent of choices of the branch of $\sqrt{it}$. As in (3.2), we can naturally define $G_{it}$, $t \in \mathbb{R}$, by

$$G_{it} = \lim_{n \rightarrow \infty} \sum_{k=0}^{n} \frac{(it)^{k}}{k!} (\Delta_{L}^{T})^{k},$$

on $D_{L}^{(-p)}$. 
An infinite dimensional Fourier-Mehler transform \( F_{\theta}, \theta \in \mathbb{R} \), on \((\mathcal{S})^*\) was defined by H.-H. Kuo [Ku 91] as follows. The transform \( F_{\theta} \Phi, \theta \in \mathbb{R} \) of \( \Phi \in (\mathcal{S})^* \) is defined by the unique generalized white noise functional with the \( U \)-functional

\[
U[F_{\theta}\Phi](\xi) = U\Phi(e^{i\theta}\xi) \exp \left[ \frac{i}{2} e^{i\theta} \sin \theta |\xi|_0^2 \right], \quad \xi \in \mathcal{S}.
\]

Moreover, the adjoint operator \( F_{\theta}^* \) of \( F_{\theta} \) is given by

\[
F_{\theta}^* \Phi = \sum_{n=0}^{\infty} I_n(h_n(\Phi; \theta)) \quad \text{for} \quad \Phi = \sum_{n=0}^{\infty} I_n(f_n) \in (\mathcal{S}),
\]

where

\[
h_n(\Phi; \theta) = \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!} \left( \frac{i}{2} \sin \theta \right)^m e^{i(m+n)\theta} \tau^\otimes m * f_{n+2m};
\]

\[
\tau^\otimes m = \int_{\mathbb{R}^m} \delta_{t_1} \otimes \delta_{t_1} \otimes \cdots \otimes \delta_{t_m} \otimes \delta_{t_m} dt_1 \cdots dt_m.
\]

This operator \( F_{\theta}^* \) is a continuous linear operator on \((\mathcal{S})\). (For details, see [Ku 91] and also [HKO 90]) On \((\mathcal{S})\), the Gross Laplacian \( \Delta_G \) (See [Gr 65, 67]) and the number operator \( N \) is given by

\[
\Delta_G \Phi = \int_{\mathbb{R}} \partial_t^2 \Phi dt
\]

and

\[
N \Phi = \int_{\mathbb{R}} \partial_t^* \partial_t \Phi dt,
\]

respectively (see [Ku 86]). The operator \( e^{i\theta N} \) is called the Fourier- Wiener transform (see [HKO 90]). Now, we introduce an operator \( e^{i\theta \Delta_G} \) from \((\mathcal{S})\) into itself given by

\[
e^{i\theta \Delta_G} \Phi = \sum_{n=0}^{\infty} I_n(\ell_n(\Phi; \theta)); \quad (4.1)
\]

\[
\ell_n(\Phi; \theta) = \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!} \left( \frac{i}{2} \theta \right)^m \tau^\otimes m * f_{n+2m},
\]

for \( \Phi = \sum_{n=0}^{\infty} I_n(f_n) \in (\mathcal{S}) \). Then we have the followings.

**Lemma 1.**

\[
F_{\theta}^* = e^{i\theta N} \circ e^{i\frac{i}{2}(e^{i\theta} \sin \theta) \Delta_G}. \quad (4.2)
\]

**Proof:** Take \( \Phi = \sum_{n=0}^{\infty} I_n(f_n) \in (\mathcal{S}) \). From (4.1), we see that

\[
e^{i\frac{i}{2}(e^{i\theta} \sin \theta) \Delta_G} \Phi = \sum_{n=0}^{\infty} I_n(\ell_n(\Phi; e^{i\theta} \sin \theta)).
\]

Hence,

\[
e^{i\theta N} \left( e^{i\frac{i}{2}(e^{i\theta} \sin \theta) \Delta_G} \Phi \right) = \sum_{n=0}^{\infty} I_n(e^{i\theta} \ell_n(\Phi; e^{i\theta} \sin \theta)).
\]
Since 
\[ e^{in\theta} \ell_n(\Phi; e^{i\theta} \sin \theta) = h_n(\Phi; \theta), \]
we obtain (4.2).

**LEMMA 2.** For any \( \Phi \in (S) \), we have

\[ U[e^{i\theta \Delta_G} \Phi](\xi) = \int_{S^*} U\Phi(\xi + \sqrt{i\theta}y) \, d\mu(y). \]  
(4.3)

**Remark.** For any \( \Phi \in (S) \), \( \xi \in S \) and \( z_1, z_2 \in \mathbb{C} \), the functional \( U \Phi(z_1 \xi + z_2 \eta) \), \( \eta \in S \), can be extended to a functional \( U \Phi(z_1 \xi + z_2 \eta) \), same symbol \( U \Phi(z_1 \xi + z_2 \eta) \).

**PROOF:** For \( \Phi = \sum_{n=0}^{\infty} I_n(f_n) \in (S) \), the right-hand side of (4.3) has the following expansion:

\[ \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n(u) \int_{S^*} \{ \xi(u_1) + \sqrt{i\theta}x(u_1) \} \cdots \{ \xi(u_n) + \sqrt{i\theta}x(u_n) \} d\mu(x) du \]

\[ = \sum_{n=0}^{\infty} \sum_{\nu=0}^{[n/2]} \frac{n!}{(2\nu)!((n-2\nu)!)} (i\theta)^\nu < \xi^{(n-2\nu)}, \tau^\nu * f_n > = \sum_{m=0}^{\infty} < \xi^m, \ell_m(\Phi; \theta) > . \]

From (4.1), we see that the last series is equal to \( U[e^{i\theta \Delta_G} \Phi](\xi) \).

Define an operator \( J_n \) by

\[ U[J_n \Phi](\xi) = U\Phi \circ j_n(\xi), \quad \Phi \in D_L^{(-p)}, \quad \xi \in S. \]

For all \( n \in \mathbb{N} \) and \( \Phi \in D_L^{(-p)} \), we can easily check \( J_n \Phi \in (S) \). Then we have the following.

**THEOREM 3.** Let \( \Phi \in D_L^{(-p)} \) be a generalized white noise functional with the \( U \)-functional given by \( \psi(F_1, \ldots, F_n) \), where \( \psi \) is an entire function on \( \mathbb{C} \) and \( F_1, \ldots, F_n \in U[N_T] \). We assume the condition

\[ \sum_{k_1, \ldots, k_n=0}^{\infty} \frac{1}{k_1! \cdots k_n!} | \partial_{u_1}^{k_1} \cdots \partial_{u_n}^{k_n} \psi(0, \ldots, 0)|. \]

\[ \sup_N \int_{S^*} \left| ((F_1 \circ j_N)^{k_1} \cdots (F_n \circ j_N)^{k_n}) (ie^{i\alpha_N(t)} \xi + \sqrt{i e^{i\alpha_N(t)} \sin \alpha_N(t)} x) \right| d\mu(x) < \infty \]

holds for all \( t > 0 \) and \( \xi \in S \), where \( \alpha_N(t) = t(\theta_N^T)^2 \). Then

\[ \lim_{N \to \infty} U[F_{\alpha_N(t)} J_N \Phi](\xi) = U[G_{it} \Phi](\xi), \quad \xi \in S. \]  
(4.4)

**PROOF:** From Lemma 2, we have

\[ U[e^{i\alpha_N(t) \sin \alpha_N(t)} \Delta_G J_N \Phi](\xi) = \int_{S^*} U[J_N \Phi](\xi + \sqrt{i e^{i\alpha_N(t) \sin \alpha_N(t)} y}) d\mu(y). \]
This functional is expressed in the form given by

$$\sum_{\ell=0}^{\infty} \langle \xi^{\otimes \ell}, f_{N,\ell} \rangle,$$

where $f_{N,\ell} \in S_{C}^{\otimes \ell}$. Hence, from Lemma 1, we get

$$U[F_{\alpha_{N}(t)}^{*}J_{N}\Phi](\xi) = \sum_{\ell=0}^{\infty} e^{i\alpha_{N}(t)\ell} \langle \xi^{\otimes \ell}, f_{N,\ell} \rangle.$$

From the condition of this theorem and the Lebesgue convergence theorem, we can calculate as follows:

$$\lim_{N \to \infty} U[F_{\alpha_{N}(t)}^{*}J_{N}\Phi](\xi) = \lim_{N \to \infty} U[e^{i\alpha_{N}(t)\sin \alpha_{N}(t)\Delta_{G}}J_{N}\Phi](e^{i\alpha_{N}(t)}\xi)$$

$$= \sum_{k_{1},\ldots,k_{n}=0}^{\infty} \frac{1}{k_{1}!\cdots k_{n}!} \partial_{u_{1}^{1}}^{k_{1}} \cdots \partial_{u_{n}^{n}}^{k_{n}} \psi(0,\ldots,0).$$

$$\lim_{N \to \infty} \int_{S^{*}} ((F_{1} \circ j_{N})^{k_{1}} \cdots (F_{n} \circ j_{N})^{k_{n}})(ie^{i\alpha_{N}(t)}\xi + \sqrt{ie^{i\alpha_{N}(t)}\sin \alpha_{N}(t)}x) d\mu(x).$$

By the direct calculations, it is easily checked that

$$\lim_{N \to \infty} \int_{S^{*}} ((F_{1} \circ j_{N})^{k_{1}} \cdots (F_{n} \circ j_{N})^{k_{n}})(ie^{i\alpha_{N}(t)}\xi + \sqrt{ie^{i\alpha_{N}(t)}\sin \alpha_{N}(t)}x) d\mu(x)$$

$$= U[g_{it}U^{-1}(F_{1}^{k_{1}} \cdots F_{n}^{k_{n}})](\xi) = U[g_{it}U^{-1}F_{1}^{k_{1}} \cdots U[g_{it}U^{-1}F_{n}^{k_{n}}](\xi)^{k_{n}}.$$

Consequently, we obtain (4.4).


