数理解析研究所講究録
第 874 巻 1994 年 168-179

Foundation of Quantum Entropy

Masanori Ohya

Department of Information Sciences
Science University of Tokyo
Noda City, Chiba 278, Japan

§1 Mathematical Description of CDS and QDS

Let fix the notations used throughout this paper. Let μ be a probability measure on a measureble space $(\Omega, \mathfrak{F})$, $P(\Omega)$ be the set of all probability measures on Ω and $M(\Omega)$ be the set of all measurable functions on Ω. We denote the set of all bounded linear operators on a Hilbert space $\mathcal{H}$ by $B(\mathcal{H})$, and the set of all density operators on $\mathcal{H}$ by $\mathfrak{S}(\mathcal{H})$. Moreover, let $\mathfrak{S}(\mathcal{A})$ be the set of all states on $\mathcal{A}$ (C*-algebra or von Neumann algebra). Therefore the descriptions of classical dynamical systems, quantum dynamical systems and general quantum dynamical systems are given in the following Table:

	CDS	QDS	GQDS
obs.	real r.v.f in $M(\Omega)$	Hermitian op. A on $\mathcal{H}$ (s.a. op. in $B(\mathcal{H})$)	self-adjoint A in C*-algebra $\mathcal{A}$
state	prob. meas. $\mu \in P(\Omega)$	density op. ρ on $\mathcal{H}$	p.l.fnal $\varphi \in \mathfrak{S}$ with $\varphi(I)=1$
expec -tation	$\int_\Omega f d\omega$	$tr\rho A$	$\varphi(A)$

Table. 1.1 Description of CDS, QDS and GQDS

§2 Classical Entropy

2.1 Discrete Case (Shannon's Theory)

A state in a discrete classical system is given by a probability distribution such that

$$\Delta_n = \left\{ p = \{p_i\}_{i=1}^n ; \sum_i p_i = 1, p_i \geq 0 \right\}$$

The entropy of a state $p = \{p_i\} \in \Delta_n$ is

$$S(p) = -\sum_i p_i \log p_i$$

The relative uncertainty (relative entropy) is defined by Kullback-Leibler as

$$S(p,q) = \begin{cases} \sum_i p_i \log \frac{p_i}{q_i} & (p \ll q) \\ \infty & (p \not\ll q) \end{cases}$$

for any $p,q \in \Delta_n$. Onece a state p is changed through a channel Λ^*, the information transmitted from a initial state p to a final state $q \equiv \Lambda^* p$ is described by the mutual entropy defined by

$$I(p; \Lambda^*) = S(r, p \otimes q) = \sum_{ij} r_{ij} \log \frac{r_{ij}}{p_i q_j}$$

where $\Lambda^* : \Delta_n \to \Delta_m$; $q = \Lambda^* p$ is a channel (e.g., $\Lambda = (p(j \mid i))$ transition matrix), $r_{ij} = p(j \mid i) p_i$ and $p \otimes q = \{p_i q_j\}$. The fundamental inequality of Shannon is

$$0 \leq I(p; \Lambda^*) \leq \min\{S(p), S(q)\}$$

According to this inequality, the ratio

$$r(p; \Lambda^*) = \frac{I(p; \Lambda^*)}{S(p)}$$

represents the efficiency of the channel transmission

2.2 Continuous Case

In classical continuous systems, a state is described by a probability measure μ. Let $(\Omega, \mathfrak{F}, P(\Omega))$ be an input probability space and $(\overline{\Omega}, \overline{\mathfrak{F}}, P(\overline{\Omega}))$ be an output probability space. A channel is a map Λ^* from $P(\Omega)$ to $P(\overline{\Omega})$, in particular, Λ^* is a Markov type if it is given by

$$\Lambda^* \varphi(Q) = \int_\Omega \lambda(x, Q) \varphi(dx), \ \varphi \in P(\Omega), Q \in \overline{\mathfrak{F}}$$

where $\lambda : \Omega \times \overline{\mathfrak{F}} \to R^+$ with (i) $\lambda(x, \bullet) \in P(\overline{\Omega})$, (ii) $\lambda(\bullet, Q) \in M(\Omega)$. In continuous case, the entropies are defined as follows: Let $F(\Omega)$ be the set of all finite partitions $\{A_k\}$ of Ω. For any $\varphi \in P(\Omega)$, the entropy is defined by

$$S(\varphi) = \sup\left\{-\sum_k \varphi(A_k)\log\varphi(A_k);\ \{A_k\} \in F(\Omega)\right\},$$

which is often infinite. For any $\varphi, \psi \in P(\Omega)$, the relative entropy is given by

$$S(\varphi,\psi) = \sup\left\{\sum_k \varphi(A_k)\log\frac{\varphi(A_k)}{\psi(A_k)};\ \{A_k\} \in F(\Omega)\right\}$$

$$= \begin{cases} \int_\Omega \log\left(\dfrac{d\varphi}{d\psi}\right)d\psi & (\varphi \ll \psi) \\ +\infty & (\varphi \not\ll \psi) \end{cases}$$

Let Φ, Φ_0 be two compound states (measures) defined as follows :

$$\Phi(Q_1,Q_2) = \int_\Omega \lambda(x,Q_2)\varphi(dx),\ Q_1 \in \mathfrak{F},\ Q_2 \in \overline{\mathfrak{F}}$$

$$\Phi_0(Q_1,Q_2) = (\varphi \otimes \Lambda^*\varphi)(Q_1,Q_2) = \varphi(Q_1)\Lambda^*\varphi(Q_2)$$

For $\varphi \in P(\Omega)$ and a channel Λ^*, the mutual entropy is given by

$$I(\varphi;\Lambda^*) = S(\Phi,\Phi_0).$$

§3 Quantum Entropy

3.1 Entropies for density operators

A state in quantum systems is described by a density operator on a Hilbert space $\mathcal{H}$. The entropies are defined as follows: For a state $\rho \in \mathfrak{S}(\mathcal{H})$, the entropy [N.1] is given by

$$S(\rho) = -tr\rho\log\rho.$$

If $\rho = \sum_k p_k E_k$ (Schatten decomposition, $\dim E_n = 1$), then

$$S(\rho) = -\sum_k p_k \log p_k.$$

Let us summarize the properties of the entropy $S(\rho)$.

Theorem 3.3 For any density operator $\rho \in \mathfrak{S}(\mathcal{H})$, the followings hold:

(1) Positivity : $S(\rho) \geq 0$.

(2) Symmetry : Let $\rho' = U^{-1}\rho U$ for an invertible operator U. Then

$$S(\rho') = S(\rho)$$

(3) Concavity : $S(\lambda\rho_1 + (1-\lambda)\rho_2) \geq \lambda S(\rho_1) + (1-\lambda)S(\rho_2)$ for any $\rho_1, \rho_2 \in \mathfrak{S}(\mathcal{H})$.

(4) Additivity : $S(\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2)$ for any $\rho_k \in \mathfrak{S}(\mathcal{H})$.

(5) Subadditivity : For the reduced states ρ_1, ρ_2 of $\rho \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$,

$$S(\rho) \leq S(\rho_1) + S(\rho_2).$$

(6) Lower Semicontinuity : If $\|\rho_n - \rho\|_1 \left(\equiv tr|\rho_n - \rho|\right) \to 0$, then

$$S(\rho) \leq \liminf S(\rho_n).$$

(7) Continuity : Let ρ_n, ρ be elements in $\mathfrak{S}(\mathcal{H})$ which satisfy the following conditions :

(i) $\rho_n \to \rho$ weak as $n \to \infty$, (ii) $\rho_n \leq A$ $(\forall n)$ for some compact operator A, and

(iii) $-\sum_k a_k \log a_k < +\infty$ for the eigenvalues $\{a_k\}$ of A, Then $S(\rho_n) \to S(\rho)$.

(8) Strong Subadditivity : Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ and denote the reduced states $tr_{\mathcal{H}_i \otimes \mathcal{H}_j}\rho$ by ρ_{ij} and ρ_k, respectively. Then

$$S(\rho) + S(\rho_2) \leq S(\rho_{12}) + S(\rho_{23}) \text{ and } S(\rho_1) + S(\rho_2) \leq S(\rho_{13}) + S(\rho_{23}).$$

For two states $\rho, \sigma \in \mathfrak{S}(\mathcal{H})$, the relative entropy [U.2, L.1] is given by

$$S(\rho, \sigma) = \begin{cases} tr\rho(\log\rho - \log\sigma) & (\rho << \sigma) \\ +\infty & (\rho \not<\!\!< \sigma) \end{cases}$$

where $\rho << \sigma \Leftrightarrow$ for any $A \geq 0$, $tr\sigma A = 0 \Rightarrow tr\rho A = 0$.

Let $\Lambda^* : \mathfrak{S}(\mathcal{H}) \to \mathfrak{S}(\overline{\mathcal{H}})$ be a channel and set

$$\sigma = \Lambda^*\rho,\ \theta_E = \sum_k p_k E_k \otimes \Lambda^* E_k,\ \theta_0 = \rho \otimes \Lambda^*\rho.$$

The mutual entropy [O.1] is given by

$$\begin{aligned} I(\rho; \Lambda^*) &= \sup\{S(\theta_E, \theta_0);\ E = \{E_k\}\} \\ &= \sup\left\{\sum_k p_k S(\Lambda^* E_k, \Lambda^*\rho);\ E = \{E_k\}\right\} \end{aligned}$$

for any state $\rho \in \mathfrak{S}(\mathcal{H})$ and any channel Λ^*. When the decomposition of ρ is fixed such that

$\rho = \sum_k \lambda_k \rho_k$, then

$$I(\rho;\Lambda^*) = \sum_k \lambda_k S(\Lambda^* \rho_k, \Lambda^* \rho).$$

where $\theta_\lambda = \sum_k \lambda_k \rho_k \otimes \Lambda^* \rho_k$. The fundamental inequality of Shannon type is obtained:

$$0 \le I(\rho;\Lambda^*) \le \min\{S(\rho), S(\Lambda^* \rho)\}.$$

3.2 Channeling Transformations

A general quantum system containing continuous cases is described by a C*-algebra or a von Neumann algebra. Let $\mathcal{A}$ be a C*-algebra (complex normed algebra with involution * such that $\|A\| = \|A^*\|$, $\|A^*A\| = \|A\|^2$ and complete w.r.t. $\|\cdot\|$) and $\mathfrak{S}(\mathcal{A})$ be the set of all states on $\mathcal{A}$ (positive continuous linear functionals φ on $\mathcal{A}$ s.t. $\varphi(I) = 1$ if $I \in \mathcal{A}$)

A cahnnel $\Lambda^* : \mathfrak{S}(\mathcal{A}) \to \mathfrak{S}(\overline{\mathcal{A}})$ contains several physical transformations as special cases. First give the mathematical definitions of channels.

Definition

Let $(\mathcal{A}, \mathfrak{S}(\mathcal{A}), \alpha)$ be an input system and $(\overline{\mathcal{A}}, \mathfrak{S}(\overline{\mathcal{A}}), \overline{\alpha})$ be an output system. Take any $\varphi, \phi \in \mathfrak{S}(\mathcal{A})$.

(1) Λ^* is linear if $\Lambda^*(\lambda\varphi + (1-\lambda)\phi) = \lambda\Lambda^*\varphi + (1-\lambda)\Lambda^*\phi$ for any $\lambda \in [0,1]$.

(2) Λ^* is completely positive (C.P.) if Λ^* is linear and its dual $\Lambda : \overline{\mathcal{A}} \to \mathcal{A}$ satisfies

$$\sum_{i,j=1}^{n} A_i^* \Lambda(\overline{A}_i^* \overline{A}_j) A_j \ge 0$$

for any $n \in N$ and any $\overline{A}_i \in \overline{\mathcal{A}}$, $A_i \in \mathcal{A}$.

(3) Λ^* is Schwarz type if $\Lambda(\overline{A}^*) = \Lambda(\overline{A})^*$ and $\Lambda(\overline{A})^* \Lambda(\overline{A}) \le \Lambda(\overline{A}^* \overline{A})$.

(4) Λ^* is stationary if $\Lambda \circ \alpha_t = \overline{\alpha}_t \circ \Lambda$ for any $t \in R$.

(5) Λ^* is ergodic if Λ^* is stationary and $\Lambda^*(exI(\alpha)) \subset exI(\overline{\alpha})$.

(6) Λ^* is orthogonal if any two orthogonal states $\varphi_1, \varphi_2 \in \mathfrak{S}(\mathcal{A})$ (denoted by $\varphi_1 \perp \varphi_2$) implies $\Lambda^*\varphi_1 \perp \Lambda^*\varphi_2$.

(7) Λ^* is deterministic if Λ is orthogonal and bijection.

(8) For a subset S of $\mathfrak{S}(\mathcal{A})$, Λ^* is chaotic for S if $\Lambda^*\varphi_1 = \Lambda^*\varphi_2$ for any $\varphi_1, \varphi_2 \in S$.

(9) Λ^* is chaotic if Λ^* is chaotic for $\mathfrak{S}(\mathcal{A})$.

Most of channels appeared in physical processes are C.P. channels. Examples of channels are the followings [O.2, D.1]:

(1) Unitary evolution :

For any density operator $\rho \in \mathfrak{S}(\mathcal{H})$

$\rho \to \Lambda_t^* \rho = AdU_t(\rho) \equiv U_t^* \rho U_t, t \in R, U_t = \exp(itH)$

(2) Semigroup evolution :

$\rho \to \Lambda_t^* \rho = V_t^* \rho V_t, t \in R^+$, where $(V_t; t \in R^+)$ is a one parameter semigroup on $\mathcal{H}$

(3) Measurement :

When we measure an obserbable $A = \sum_n a_n P_n$ (spectral decomposition) in a state ρ, the state ρ changes to a state $\Lambda^* \rho$ by this measurement such as

$$\rho \to \Lambda^* \rho = \sum_n P_n \rho P_n$$

(4) Reduction :

If a system Σ_1 interacts with an external system Σ_2 described by another Hilbert space $\mathcal{K}$ and the initial states of Σ_1 and Σ_2 are ρ and σ, respectively, then the combined state θ_t of Σ_1 and Σ_2 at time t after the interaction between two systems is given by

$$\theta_t = U_t^* (\rho \otimes \sigma) U_t,$$

where $U_t = \exp(itH)$ with the total Hamiltonian H of Σ_1 and Σ_2. A channel is obtained by taking the partial trace w.r.t. $\mathcal{K}$ such as

$$\rho \to \Lambda_t^* \rho = tr_{\mathcal{K}} \theta_t.$$

3.3 Entropies in GQDS

The entropy (uncertainty) of a state $\varphi \in \mathcal{S}$ seen from the reference system $\mathcal{S}$, a weak *-compact convex subset of $\mathfrak{S}$, is given by [O.2,O.3].

Every state $\varphi \in \mathcal{S}$ has a maximal measure μ pseudosupported on $ex\mathcal{S}$ such that

$$\varphi = \int_{\mathcal{S}} \omega d\mu$$

The measure μ giving the above decomposition is not unique unless $\mathcal{S}$ is a Choquet simplex, so that we denote the set of all such measures by $M_\varphi(\mathcal{S})$. Put

$$D_\varphi(\mathcal{S}) = \left\{ M_\varphi(\mathcal{S}) ; \exists \{\mu_k\} \subset R^+ \text{ and } \{\varphi_k\} \subset ex\mathcal{S} \text{ s.t. } \sum_k \mu_k = 1, \mu = \sum_k \mu_k \delta(\varphi_k) \right\},$$

where $\delta(\varphi)$ is the Dirac measure concentrated on $\{\varphi\}$, and put

$$H(\mu) = -\sum_k \mu_k \log \mu_k$$

for a measure $\mu \in D_\varphi(\mathcal{S})$. Then the entropy of a state $\varphi \in \mathcal{S}$ w.r.t. $\mathcal{S}$ is defined by

$$S^{\mathcal{S}}(\varphi) = \begin{cases} \inf\{H(\mu) \,;\, \mu \in D_\varphi(\mathcal{S})\} \\ +\infty \qquad if\ D_\varphi(\mathcal{S}) = \phi \end{cases}$$

The entropy (mixing entropy) of a general state φ satisfies the following properties [O.2,O.3].

Theorem When $\mathcal{A} = B(\mathcal{H})$ and $\alpha_t = Ad(U_t)$ with an unitary operator U_t, for any state φ given by $\varphi(\cdot) = tr\rho\ \cdot$ with a density operator ρ, the followings hold:

(1) $S(\varphi) = -tr\rho \log \rho$.

(2) If φ is an α-invariant state and every eigenvalue of ρ is non-degenerate, then $S^I(\varphi) = S(\varphi)$.

(3) If $\varphi \in K(\alpha)$, then $S^K(\varphi) = 0$.

Theorem For any $\varphi \in K(\alpha)$,

(1) $S^K(\varphi) \le S^I(\varphi)$.

(2) $S^K(\varphi) \le S(\varphi)$.

This $\mathcal{S}$ (or mixing) entropy gives a measure of the uncertainty observed from the reference system $\mathcal{S}$. Similar properties as $S(\rho)$ hold (see [O.3]).

The relative entropy for two general states φ and ψ has been introduced by Araki [A.1,A.2] and Uhlmann [U.1] and their relation is considered in [H.1,H.2].

<Araki's definition>

Let $\mathfrak{N}$ be σ-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and φ,ψ be normal states on $\mathfrak{N}$ given by $\varphi(\cdot) = \langle x, \cdot\, x\rangle$ and $\psi(\cdot) = \langle y, \cdot\, y\rangle$ with $x,y \in \mathcal{H}$. The operator $S_{x,y}$ is defined by

$$S_{x,y}(Ay + z) = s^{\mathfrak{N}}(y)A^* x,\ A \in \mathfrak{N},\ \ s^{\mathfrak{N}}(y)z = 0.$$

on the domain $\mathfrak{N}y + (I - s^{\mathfrak{N}}(y))\mathcal{H}$, where $s^{\mathfrak{N}}(y)$ is the projection from $\mathcal{H}$ to $\{\mathfrak{N}'y\}^-$, the

$\{\mathfrak{M}'y\}^-$-support of y. Using this $S_{x,y}$, the relative modular operator $\Delta_{x,y}$ is defined as $\Delta_{x,y} = (S_{x,y})^* S_{x,y}^-$, with spectral decomposition denoted by $\int_0^\infty \lambda de_{x,y}(\lambda)$. Then the relative entropy is given by

$$S(\psi \mid \varphi) = \begin{cases} \int_0^\infty \log \lambda d\langle y, e_{x,y}(\lambda) y\rangle & \text{if} \quad \psi << \varphi \\ +\infty & \text{otherwise,} \end{cases}$$

where $\psi << \varphi$ means that $\varphi(A^*A) = 0$ implies $\psi(A^*A) = 0$ for $A \in \mathfrak{M}$.

< Uhlmann's definition>

Let $\mathcal{L}$ be a linear space and p, q be seminorms on $\mathcal{L}$, α a positive Hermitian form on $\mathcal{L}$. Put $\mathcal{G} \equiv \{\alpha; |\alpha(x,y)| \le p(x)q(y), x, y \in \mathcal{L}\}$ and $QM(p,q) \equiv \sup\{\alpha(x,x)^{1/2}; \alpha \in \mathcal{G}\}$. There exists a quadratical interpolation $t \in [0,1] \to p_t$ from p to q $(p_t \equiv QI_t(p,q))$ such that

(1) p_t cont.
(2) $t = \frac{1}{2}(t_1 + t_2) \Rightarrow p_t = QM(p_{t_1}, p_{t_2})$
(3) $p_{1/2} = QM(p,q)$
(4) $p_{t/2} = QM(p, p_t)$
(5) $p_{\frac{t+1}{2}} = QM(p_t, q)$

Let $\mathcal{L} = \mathcal{A}$ and for any states $\varphi, \psi \in \mathfrak{S}(\mathcal{A})$

$$p(A) = \varphi(AA^*)^{1/2}$$
$$q(A) = \psi(A^*A)^{1/2}$$

Then the relative entropy for φ and ψ is given by

$$S(\varphi|\psi) = -\liminf_{t\to\infty} \frac{1}{t}\{QI_t(p,q)^2(I) - p^2(I)\}$$

For $\varphi \in \mathcal{S}(\mathcal{A}) \subset \mathfrak{S}(\mathcal{A})$, $\Lambda^* : \mathfrak{S}(\mathcal{A}) \to \mathfrak{S}(\overline{\mathcal{A}})$, let us define the compound states by

$$\Phi_\mu^{\mathcal{S}} = \int_{\mathcal{S}} \omega \otimes \Lambda^*\omega \, d\mu \quad \text{and}$$

$$\Phi_0 = \varphi \otimes \Lambda^*\varphi$$

The mutual entropy w.r.t. $\mathcal{S}$ and μ is

$$I_\mu^{\mathcal{S}}(\varphi; \Lambda^*) = S(\Phi_\mu^{\mathcal{S}} \mid \Phi_0)$$

and the mutual entropy w.r.t. $\mathcal{S}$ is defined as [O.3]

$$I^{\mathcal{S}}(\varphi;\Lambda^*) = \lim_{\varepsilon\to 0}\inf\left\{I_\mu^{\mathcal{S}}(\varphi;\Lambda^*);\ \mu\in F_\varphi^\varepsilon(S)\right\}$$
$$= \sup\left\{\sum_k \mu_k S(\Lambda^*\omega_k \mid \Lambda^*\varphi);\ \varphi = \sum_k \mu_k\varphi_k\right\}$$

where

$$F_\varphi^\varepsilon(S) = \begin{cases}\left\{\mu\in D_\varphi(S);\ S^{\mathcal{S}}(\varphi)\le H(\mu)\le S^{\mathcal{S}}(\varphi)+\varepsilon<+\infty\right\} \\ M_\varphi(S) \quad \text{if} \quad S^{\mathcal{S}}(\varphi)=+\infty\end{cases}$$

$$D_\varphi(S) = \left\{\mu\in M_\varphi(S);\ \exists\{\mu_k\}\subset R^+ \text{ s.t. } \mu=\sum_k\mu_k\delta(\varphi_k),\ \varphi_k\in exS,\ \sum_k\mu_k=1\right\}$$

when a state φ is expressed as $\varphi=\sum_k\mu_k\omega_k$ (fixed), the mutual entropy is given by

$$I^{\mathcal{S}}(\varphi\,;\Lambda^*) = \sum_k \mu_k S(\Lambda^*\omega_k,\Lambda^*\varphi)$$

This entropy and $\mathcal{S}$-entropy contains Connes-Thiring-Narnhofer entropy as a special case [M.1].

The inequality is satisfied for almost all physical cases.

$$0\le I^{\mathcal{S}}(\varphi;\Lambda^*)\le S^{\mathcal{S}}(\varphi)$$

The fundamental properties of the relatie entropy and the mutual entropy are the followings [A.1,A.2, U.1, H.1, O.3, O.4].

<u>Theorem</u>

(1) Positivity : $S(\varphi \mid \psi)\ge 0$.

(2) Joint Convexity : $S(\lambda\psi_1+(1-\lambda)\psi_2 \mid \lambda\varphi_1+(1-\lambda)\varphi_2)\le\lambda S(\psi_1\mid\varphi_1)+(1-\lambda)S(\psi_2\mid\varphi_2)$.

(3) Additivity : $S(\psi_1\otimes\psi_2\mid\varphi_1\otimes\varphi_2)=S(\psi_1\mid\varphi_1)+S(\psi_2\mid\varphi_2)$.

(4) Lower Semicontinuity : If $\lim_{n\to\infty}\|\psi_n-\psi\|=0$ and $\lim_{n\to\infty}\|\varphi_n-\varphi\|=0$, then $S(\psi\mid\varphi)\le\liminf_{n\to\infty}S(\psi_n\mid\varphi_n)$. Moreover, if there exists a positive number λ satisfying $\psi_n\le\lambda\varphi_n$, then $\lim_{n\to\infty}S(\psi_n\mid\varphi_n)=S(\psi\mid\varphi)$.

(5) Monotonicity : For a channel Λ^* from $\mathfrak{S}$ to $\overline{\mathfrak{S}}$,
$$S(\Lambda^*\psi\mid\Lambda^*\varphi)\le S(\psi\mid\varphi).$$

(6) Lower Bound : $\|\psi-\varphi\|^2\le 2S(\psi\mid\varphi)$.

Theorem [O.3]

(1) If Λ^* is deterministic, then $I(\varphi\,;\Lambda^*) = S(\varphi)$.

(2) If Λ^* is chaotic, then $I(\varphi\,;\Lambda^*) = 0$

(3) For a state $\varphi = tr\rho\,\bullet$, if Λ^* is ergodic and the state is stationary for a time evolution $\alpha_t = AdU_t$, and if every eigenvalue of ρ is nonzero and nondegenerate, then $I(\rho;\Lambda^*) = S(\Lambda^*\rho)$.

This mutual entropy is extensively used for analysing optical communication processes [B.1].

The CNT entropy $H_\varphi(\mathfrak{N})$ of C*-subalgebra $\mathfrak{N} \subset \mathcal{A}$ is defined by [C.1].

$$H_\varphi(\mathfrak{N}) \equiv \sup_{\varphi=\sum_j \mu_j\varphi_j} \sum_j \mu_j S(\varphi_j\ \mathfrak{N} \mid \varphi\ \mathfrak{N})$$

where the supremum is taken over all finite decomposition $\varphi = \sum_j \mu_j\varphi_j$ of φ. This entropy is a mutual entropy when a channel is the restriction to subalgebra. There are some relations between the mixing entropy $S^{\mathcal{S}}(\varphi)$ and the CNT entropy.

Theorem [M.1]

(1) For any state φ on a unital C*-algebra $\mathcal{A}$,

$$S^{\mathfrak{S}}(\varphi) = H_\varphi(\mathcal{A})$$

(2) Let $(\mathfrak{M}\ , G\,\alpha)$ be a G-finite W*-dynamical system, φ be a G-invariant normal state of $\mathfrak{M}$, then

$$S^{I(\alpha)}(\varphi) = H_\varphi(\mathfrak{M}^{\alpha})$$

(3) Let $\mathcal{A}$ be the C*-algebra $C(\mathcal{H})$ of all compact operators on a Hilbert space $\mathcal{H}$, and G be a group, α be a *-automorphic action of G-invariant density operator. Then

$$S^{I(\alpha)}(\rho) = H_\rho(\mathcal{A}^{\alpha})$$

The pseudo-mutual $\mathcal{S}$-entropy $J^{\mathcal{S}}(\varphi;\Lambda^*)$ is given by

$$J^{\mathcal{S}}(\varphi;\Lambda^*) = \sup\left\{\sum_j \mu_j S(\Lambda^*\varphi_j \mid \Lambda^*\varphi);\ \varphi = \textstyle\sum_j \mu_j\varphi_j, \varphi_j \in \mathcal{S}\right\}.$$

Theorem [M.1]

(1) $0 \le I^{\mathcal{S}}(\varphi;\Lambda^*) \le J^{\mathcal{S}}(\varphi;\Lambda^*) \le \min\{H^{\mathcal{S}}(\varphi), H^{\Lambda^*\mathcal{S}}(\Lambda^*\varphi)\}$.

(2) Let Λ^* be a G-stationary channel from $\mathcal{A}$ to $\overline{\mathcal{A}}$ and G be compact. Then

$$0 \le I^{I(\alpha)}(\varphi;\Lambda^*) \le \min\left\{S^{I(\alpha)}(\varphi), S^{I(\overline{\alpha})}(\Lambda^*\varphi)\right\}.$$

References

[A.1] Araki, H. (1976): *Relative entropy of states of von Neumann algebras*, Publ. RIMS, Kyoto Univ. 11, pp. 809-833.

[A.2] Araki, H. (1977): *Relative entropy for states of von Neumann algebras II*, Publ. RIMS, Kyoto Univ., 13, pp. 173-192.

[B.1] Bendjaballah, C., Hirota, O. and Ohya, M: *Quantum Optical Communication*, to be published in Springer-Verlag.

[C.1] Connes, A. Narnhofer, H., Thirring, W. (1987): *Dynamical entropy of C*-algebras and von Neumann algebras*, Commun. Math. Phys., 112, pp. 691.

[D.1] Davies, E.B. (1976): *Quantum theory of open systems*, Academic Press.

[H.1] Hiai, F., Ohya, M. Tsukada, M. (1981): Sufficiency, KMS condition and relative entropy in von Neumann algebras, Pacific J. Math., 96, pp. 99-109.

[H.2] Hiai, F., Ohya, M. Tsukada, M. (1983): *Sufficiency and relative entropy in *-algebras with applications to quantum systems*, Pacific J. Math. 107, pp. 117--140

[L.1] Lindblad, G. (1975): *Completely positive maps and entropy inequalities*, Commun. Math. Phys., 40, pp. 147-151.

[M.1] Muraki, N., Ohya, M., Petz, D. (1992): *Note on entropy of general quantum systems*, to be published

[N.1] Neumann, J. von (1932): *Die Mathematischen Grundlagen der Quantenmechanik*, Springer-Berlin

[O.1] Ohya, M (1983): *On compound state and mutual information in quantum information theory*, IEEE Trans. Information Theory, 29, pp. 770-777.

[O.2] Ohya, M. (1984): *Entropy transmission in C*-dynamical systems*, J. Math. Anal. Appl., 100, pp. 222-235.

[O.3] Ohya, M. (1989): *Some aspects of quantum information theory and their applications to irreversible processes*, Rep. Math. Phys., 27, pp. 19-47.

[O.4] Ohya, M., Petz, D. (1993): *Quantum Entropy and its Use*, Springer-Verlag.

[U.1] Uhlmann, A. (1977): *Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in interpolation theory*, Commun. Math. Phys., 54, pp. 21-32.

[U.2] Umegaki, H. (1962): *Conditional expectation in an operator algebra, IV, (entropy and information)*, Kodai Math. Sem. Rep., 14, pp. 59-85.