§ 1 Mathematical Description of CDS and QDS

Let fix the notations used throughout this paper. Let $\mu$ be a probability measure on a measurable space $(\Omega, \mathcal{F})$, $P(\Omega)$ be the set of all probability measures on $\Omega$ and $M(\Omega)$ be the set of all measurable functions on $\Omega$. We denote the set of all bounded linear operators on a Hilbert space $\mathcal{H}$ by $B(\mathcal{H})$, and the set of all density operators on $\mathcal{H}$ by $\mathfrak{S}(\mathcal{H})$. Moreover, let $\mathfrak{S}(\mathcal{A})$ be the set of all states on $\mathcal{A}$ (C*-algebra or von Neumann algebra). Therefore the descriptions of classical dynamical systems, quantum dynamical systems and general quantum dynamical systems are given in the following Table:

<table>
<thead>
<tr>
<th>CDS</th>
<th>QDS</th>
<th>GQDS</th>
</tr>
</thead>
<tbody>
<tr>
<td>obs.</td>
<td>real r.v.f in $M(\Omega)$</td>
<td>Hermitian op. $A$ on $\mathcal{H}$ (s.a. op. in $B(\mathcal{H})$)</td>
</tr>
<tr>
<td>state</td>
<td>prob. meas. $\mu \in P(\Omega)$</td>
<td>density op. $\rho$ on $\mathcal{H}$</td>
</tr>
<tr>
<td>expec.</td>
<td>$\int f d\omega$</td>
<td>$tr \rho A$</td>
</tr>
<tr>
<td>-tation</td>
<td></td>
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Table. 1.1 Description of CDS, QDS and GQDS

§ 2 Classical Entropy

2.1 Discrete Case (Shannon's Theory)

A state in a discrete classical system is given by a probability distribution such that
The entropy of a state \( p = \{p_i\} \in \Delta_n \) is
\[
S(p) = -\sum_i p_i \log p_i
\]
The relative uncertainty (relative entropy) is defined by Kullback-Leibler as
\[
S(p, q) = \begin{cases} 
\sum_i p_i \log \frac{p_i}{q_i} & (p << q) \\
\infty & (p \not\ll q)
\end{cases}
\]
for any \( p, q \in \Delta_n \). Once a state \( p \) is changed through a channel \( \Lambda^* \), the information transmitted from an initial state \( p \) to a final state \( q = \Lambda^* p \) is described by the mutual entropy defined by
\[
I(p; \Lambda) = S(r, p \otimes q) = \sum_{ij} r_{ij} \log \frac{r_{ij}}{p_i q_j}
\]
where \( \Lambda^*: \Delta_n \rightarrow \Delta_n; \ q = \Lambda^* p \) is a channel (e.g., \( \Lambda = (p(j|i)) \) transition matrix), \( r_{ij} = p(j|i) p_i \) and \( p \otimes q = \{p_i q_j\} \). The fundamental inequality of Shannon is
\[
0 \leq I(p; \Lambda^*) \leq \min\{S(p), S(q)\}
\]
According to this inequality, the ratio
\[
r(p; \Lambda^*) = \frac{I(p; \Lambda^*)}{S(p)}
\]
represents the efficiency of the channel transmission

2.2 Continuous Case

In classical continuous systems, a state is described by a probability measure \( \mu \). Let \( (\Omega, \mathcal{F}, P(\Omega)) \) be an input probability space and \( (\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}(\overline{\Omega})) \) be an output probability space. A channel is a map \( \Lambda^* \) from \( P(\Omega) \) to \( P(\overline{\Omega}) \), in particular, \( \Lambda^* \) is a Markov type if it is given by
\[
\Lambda^* \varphi(Q) = \int_{\Omega} \lambda(x, Q) \varphi(dx), \varphi \in P(\Omega), Q \in \overline{\mathcal{F}}
\]
where \( \lambda: \Omega \times \overline{\mathcal{F}} \rightarrow R^+ \) with (i) \( \lambda(x, \bullet) \in P(\overline{\Omega}) \), (ii) \( \lambda(\bullet, Q) \in M(\Omega) \). In continuous case, the
entropies are defined as follows: Let $F(\Omega)$ be the set of all finite partitions $\{A_k\}$ of $\Omega$. For any $\varphi \in P(\Omega)$, the entropy is defined by

$$S(\varphi) = \sup \left\{ -\sum_k \varphi(A_k) \log \varphi(A_k); \{A_k\} \in F(\Omega) \right\},$$

which is often infinite. For any $\varphi, \psi \in P(\Omega)$, the relative entropy is given by

$$S(\varphi, \psi) = \sup \left\{ \sum_k \varphi(A_k) \log \frac{\varphi(A_k)}{\psi(A_k)}; \{A_k\} \in F(\Omega) \right\}$$

$$= \int \log \left( \frac{d\varphi}{d\psi} \right) d\psi \quad (\varphi \ll \psi)$$

Let $\Phi, \Phi_0$ be two compound states (measures) defined as follows:

$$\Phi(Q_1, Q_2) = \int \lambda(x, Q_2) \rho(dx), Q_1 \in \mathfrak{S}, Q_2 \in \mathfrak{F}$$

$$\Phi_0(Q_1, Q_2) = (\varphi \otimes \Lambda^* \varphi)(Q_1, Q_2) = \varphi(Q_1) \Lambda^* \varphi(Q_2)$$

For $\varphi \in P(\Omega)$ and a channel $\Lambda^*$, the mutual entropy is given by

$$I(\varphi; \Lambda^*) = S(\Phi, \Phi_0).$$

§ 3 Quantum Entropy

3.1 Entropies for density operators

A state in quantum systems is described by a density operator on a Hilbert space $\mathcal{H}$. The entropies are defined as follows: For a state $\rho \in \mathcal{S}(\mathcal{H})$, the entropy [N.1] is given by

$$S(\rho) = -tr \rho \log \rho.$$
Theorem 3.3 For any density operator $\rho \in \mathcal{S}(\mathcal{H})$, the followings hold:

1. Positivity: $S(\rho) \geq 0$.

2. Symmetry: Let $\rho = U^{-1}\rho U$ for an invertible operator $U$. Then
   
   $S(\rho) = S(\rho)$

3. Concavity: $S(\lambda \rho_1 + (1-\lambda)\rho_2) \geq \lambda S(\rho_1) + (1-\lambda)S(\rho_2)$ for any $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{H})$.

4. Additivity: $S(\rho \otimes \rho_2) = S(\rho_1) + S(\rho_2)$ for any $\rho_k \in \mathcal{S}(\mathcal{H})$.

5. Subadditivity: For the reduced states $\rho_1, \rho_2$ of $\rho \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}_2)$,
   
   $S(p) \leq S(\rho_1) + S(p_2)$

6. Lower Semicontinuity: If $\|\rho - p\|_{1} (\equiv tr|p_\rho - \rho|) \to 0$, then
   
   $S(\rho) \leq \lim inf S(\rho_n)$

7. Continuity: Let $\rho_n, \rho$ be elements in $\mathcal{S}(\mathcal{H})$ which satisfy the following conditions:
   
   (i) $\rho_n \to \rho$ weak as $n \to \infty$, (ii) $\rho_n \leq A (\forall n)$ for some compact operator $A$, and
   
   (iii) $-\sum a_i \log a_i < +\infty$ for the eigenvalues $\{a_i\}$ of $A$. Then $S(\rho_n) \to S(\rho)$.

8. Strong Subadditivity: Let $\mathcal{H} = \mathcal{H} \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ and denote the reduced states $tr_{\mathcal{H}_2 \otimes \mathcal{H}_3} \rho$ by $\rho_1, \rho_2, \rho_3$ and $\rho_{12}, \rho_{23}, \rho_{13}$, respectively. Then
   
   $S(\rho) + S(\rho_2) \leq S(\rho_{12}) + S(\rho_{23})$ and $S(\rho_1) + S(\rho_3) \leq S(\rho_{13}) + S(\rho_{23})$.

For two states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, the relative entropy $[U.2, L.1]$ is given by

$$S(\rho, \sigma) = \begin{cases} tr(\rho \log \rho - \log \sigma) & (\rho << \sigma) \\ +\infty & (\rho \not\sim \sigma) \end{cases}$$

where $\rho << \sigma \iff$ for any $A \geq 0$, $tr\sigma A = 0 \Rightarrow tr\rho A = 0$.

Let $\Lambda^*: \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\overline{\mathcal{H}})$ be a channel and set

$$\sigma = \Lambda^* \rho, \theta_{E} = \sum_{k}p_{k}E_{k} \otimes \Lambda E_{k}, \theta_{\rho} = \rho \otimes \Lambda^* \rho.$$ 

The mutual entropy $[O.1]$ is given by

$$I(\rho; \Lambda^*) = \sup\{S(\theta_E, \theta_{\rho}); E = \{E_k\}\} = \sup\{\sum_{k}p_{k}S(\Lambda^* E_k, \Lambda^* \rho); E = \{E_k\}\}$$

for any state $\rho \in \mathcal{S}(\mathcal{H})$ and any channel $\Lambda^*$. When the decomposition of $\rho$ is fixed such that
\[ \rho = \sum_{k} \lambda_{k} \rho_{k} \text{, then} \]

\[ I(\rho ; \Lambda^* ) = \sum_{k} \lambda_{k} S(\Lambda^* \rho_{k}, \Lambda^{*} \rho). \]

where \[ \theta_{\lambda} = \sum_{k} \lambda_{k} \rho_{k} \otimes \Lambda \rho_{k} \]. The fundamental inequality of Shannon type is obtained:

\[ 0 \leq I(\rho ; \Lambda^* ) \leq \min \{ S(\rho), S(\Lambda^* \rho) \}. \]

3.2 Channeling Transformations

A general quantum system containing continuous cases is described by a C*-algebra or a von Neumann algebra. Let \( A \) be a C*-algebra (complex normed algebra with involution \( * \) such that \( \| A \| = \| A^* \| \), \( \| A' A \| = \| A \|^2 \) and complete w.r.t. \( \| \cdot \| \) and \( \mathcal{S}(A) \) be the set of all states on \( A \) (positive continuous linear functionals \( \varphi \) on \( A \) s.t. \( \varphi(I) = 1 \) if \( I \in A \) )

A channel \( \Lambda^* : \mathcal{S}(A) \rightarrow \mathcal{S}(\overline{A}) \) contains several physical transformations as special cases. First give the mathematical definitions of channels.

**Definition**

Let \( (A, \mathcal{S}(A), \alpha) \) be an input system and \( (\overline{A}, \mathcal{S}(\overline{A}), \overline{\alpha}) \) be an output system. Take any \( \varphi, \phi \in \mathcal{S}(A) \).

1. \( \Lambda^* \) is linear if \( \Lambda^*(\lambda \varphi + (1 - \lambda) \phi) = \lambda \Lambda^* \varphi + (1 - \lambda) \Lambda^* \phi \) for any \( \lambda \in [0,1] \).
2. \( \Lambda^* \) is completely positive (C.P.) if \( \Lambda^* \) is linear and its dual \( \Lambda : \overline{A} \rightarrow A \) satisfies

\[ \sum_{i=1}^{n} A_i \Lambda^*(\overline{A}_i^* \overline{A}_i) A_j \geq 0 \]

for any \( n \in \mathbb{N} \) and any \( \overline{A}_i, A_j \in A \).

3. \( \Lambda^* \) is Schwarz type if \( \Lambda(\overline{A}^*) = \Lambda(\overline{A})^* \) and \( \Lambda(\overline{A})^* \Lambda(\overline{A}) \leq \Lambda(\overline{A}^* \overline{A}) \).
4. \( \Lambda^* \) is stationary if \( \Lambda \circ \alpha = \overline{\alpha} \circ \Lambda \) for any \( t \in R \).
5. \( \Lambda^* \) is ergodic if \( \Lambda^* \) is stationary and \( \Lambda^*(exl(\alpha)) \subset exl(\overline{\alpha}) \).
6. \( \Lambda^* \) is orthogonal if any two orthogonal states \( \varphi_{1}, \varphi_{2} \in \mathcal{S}(A) \) (denoted by \( \varphi_{1} \perp \varphi_{2} \)) implies \( \Lambda^* \varphi_{1} \perp \Lambda^* \varphi_{2} \).
7. \( \Lambda^* \) is deterministic if \( \Lambda \) is orthogonal and bijection.
8. For a subset \( S \) of \( \mathcal{S}(A) \), \( \Lambda^* \) is chaotic for \( S \) if \( \Lambda^* \varphi_{1} = \Lambda^* \varphi_{2} \) for any \( \varphi_{1}, \varphi_{2} \in S \).
9. \( \Lambda^* \) is chaotic if \( \Lambda^* \) is chaotic for \( \mathcal{S}(A) \).

Most of channels appeared in physical processes are C.P. channels. Examples of channels are the followings [O.2, D.1]:

\[ \rho = \sum_{k} \lambda_{k} \rho_{k} \text{, then} \]

\[ I(\rho ; \Lambda^* ) = \sum_{k} \lambda_{k} S(\Lambda^* \rho_{k}, \Lambda^{*} \rho). \]

where \[ \theta_{\lambda} = \sum_{k} \lambda_{k} \rho_{k} \otimes \Lambda \rho_{k} \]. The fundamental inequality of Shannon type is obtained:

\[ 0 \leq I(\rho ; \Lambda^* ) \leq \min \{ S(\rho), S(\Lambda^* \rho) \}. \]
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(1) Unitary evolution:

For any density operator \( \rho \in \mathfrak{S}(\mathcal{H}) \)
\[ \rho \rightarrow \Lambda^t \rho = \text{Ad} U(t) \rho = U(t) \rho U(t)^\dagger, \quad t \in \mathbb{R}, \]
where \( U(t) = \exp(itH) \).

(2) Semigroup evolution:

\[ \rho \rightarrow \Lambda^t \rho = V(t) \rho V(t)^\dagger, \quad t \in \mathbb{R}^+, \]
where \((V(t); t \in \mathbb{R}^+)\) is a one parameter semigroup on \( \mathcal{H} \).

(3) Measurement:

When we measure an observable \( A = \sum_n a_n P_n \) (spectral decomposition) in a state \( \rho \), the state \( \rho \) changes to a state \( \Lambda^t \rho \) by this measurement such as
\[ \rho \rightarrow \Lambda^t \rho = \sum_n \rho P_n. \]

(4) Reduction:

If a system \( \Sigma_1 \) interacts with an external system \( \Sigma_2 \) described by another Hilbert space \( \mathcal{K} \) and the initial states of \( \Sigma_1 \) and \( \Sigma_2 \) are \( \rho \) and \( \sigma \), respectively, then the combined state \( \theta_t \) of \( \Sigma_1 \) and \( \Sigma_2 \) at time \( t \) after the interaction between two systems is given by
\[ \theta_t = U(t) (\rho \otimes \sigma) U(t)^\dagger, \]
where \( U(t) = \exp(itH) \) with the total Hamiltonian \( H \) of \( \Sigma_1 \) and \( \Sigma_2 \). A channel is obtained by taking the partial trace w.r.t. \( \mathcal{K} \) such as
\[ \rho \rightarrow \Lambda^t \rho = \text{tr}_\mathcal{K} \theta_t. \]

3.3 Entropies in GQDS

The entropy (uncertainty) of a state \( \varphi \in \mathcal{S} \) seen from the reference system \( \mathcal{S} \), a weak *-compact convex subset of \( \mathfrak{S} \), is given by [O.2,O.3].

Every state \( \varphi \in \mathcal{S} \) has a maximal measure \( \mu \) pseudosupported on \( \text{ex}\mathcal{S} \) such that
\[ \varphi = \int \varphi d\mu. \]

The measure \( \mu \) giving the above decomposition is not unique unless \( \mathcal{S} \) is a Choquet simplex, so that we denote the set of all such measures by \( M_\varphi(\mathcal{S}) \). Put
\[ D_\varphi(\mathcal{S}) = \left\{ M_\varphi(\mathcal{S}) ; \exists \{\mu_k\} \subset R^+ \text{ and } \{\varphi_k\} \subset \text{ex}\mathcal{S} \text{ s.t. } \sum_k \mu_k = 1, \mu = \sum \mu_k \delta(\varphi_k) \right\}. \]
where $\delta(\varphi)$ is the Dirac measure concentrated on $\{\varphi\}$, and put

$$H(\mu) = -\sum_i \mu_i \log \mu_i$$

for a measure $\mu \in D_\varphi(\mathcal{B})$. Then the entropy of a state $\varphi \in \mathcal{B}$ w.r.t. $\mathcal{B}$ is defined by

$$S^\mathcal{B}(\varphi) = \inf \{H(\mu); \mu \in D_\varphi(\mathcal{B})\}$$

if $D_\varphi(\mathcal{B}) = \emptyset$.

The entropy (mixing entropy) of a general state $\varphi$ satisfies the following properties [O.2,O.3].

**Theorem** When $A = B(\mathcal{H})$ and $\alpha_t = \text{Ad}(U_t)$ with an unitary operator $U_t$, for any state $\varphi$ given by $\varphi(\cdot) = tr \rho \cdot$ with a density operator $\rho$, the followings hold:

1. $S(\varphi) = -tr \rho \log \rho$.
2. If $\varphi$ is an $\alpha$-invariant state and every eigenvalue of $\rho$ is non-degenerate, then $S'(\varphi) = S(\varphi)$.
3. If $\varphi \in K(\alpha)$, then $S^K(\varphi) = 0$.

**Theorem** For any $\varphi \in K(\alpha)$,

1. $S^K(\varphi) \leq S'(\varphi)$.
2. $S^K(\varphi) \leq S(\varphi)$.

This $\mathcal{B}$ (or mixing) entropy gives a measure of the uncertainty observed from the reference system $\mathcal{B}$. Similar properties as $S(\rho)$ hold (see [O.3]).

The relative entropy for two general states $\varphi$ and $\psi$ has been introduced by Araki [A.1,A.2] and Uhlmann [U.1] and their relation is considered in [H.1,H.2].

**<Araki's definition>**

Let $\mathcal{M}$ be $\sigma$-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and $\varphi, \psi$ be normal states on $\mathcal{M}$ given by $\varphi(\cdot) = \langle x, \cdot x \rangle$ and $\psi(\cdot) = \langle y, \cdot y \rangle$ with $x, y \in \mathcal{H}$. The operator $S_{\varphi}$ is defined by

$$S_{\varphi}(Ay + z) = s^\mathcal{M}(y)Ax, \ A \in \mathcal{M}, \ s^\mathcal{M}(y)z = 0.$$ 

on the domain $\mathcal{M}y + \{I - s^\mathcal{M}(y)\} \mathcal{H}$, where $s^\mathcal{M}(y)$ is the projection from $\mathcal{H}$ to $\{M y\}^\perp$, the
$\{\Omega y\}^{-}$-support of $y$. Using this $S_{xy}$, the relative modular operator $\Delta_{xy}$ is defined as $\Delta_{xy} = (S_{xy})' S_{xy}$, with spectral decomposition denoted by $\int \lambda d_{xy}(\lambda)$. Then the relative entropy is given by

$$S(\psi | \varphi) = \begin{cases} \int_0^\infty \log \lambda d_{xy}(\lambda) y & \text{if } \psi \ll \varphi \\ +\infty & \text{otherwise,} \end{cases}$$

where $\psi \ll \varphi$ means that $\varphi(A^*A) = 0$ implies $\psi(A^*A) = 0$ for $A \in \Omega$.

\section*{Uhlmann's definition}

Let $\mathcal{L}$ be a linear space and $p, q$ be seminorms on $\mathcal{L}$, $\alpha$ a positive Hermitian form on $\mathcal{L}$. Put $G = \{\alpha ; |\alpha(x, y)| \leq p(x)q(y), x, y \in \mathcal{L}\}$ and $QM(p, q) = \sup \{\alpha(x, x)^{1/2}; \alpha \in G\}$. There exists a quadratical interpolation $t \in [0, 1] \rightarrow p_i$ from $p$ to $q$ ($p_i \equiv QI_l(p, q)$) such that

1. $p_i$ cont.
2. $t = \frac{1}{2} (t_1 + t_2) \Rightarrow p_i = QM(p_{t_1}, p_{t_2})$
3. $p_{1/2} = QM(p, q)$
4. $p_1 = QM(p, p_i)$
5. $p_{t/2} = QM(p_{t/2}, q)$

Let $\mathcal{L} = A$ and for any states $\varphi, \psi \in \mathcal{S}(A)$

$$p(A) = \varphi(A^*A)^{1/2}$$
$$q(A) = \psi(A^*A)^{1/2}$$

Then the relative entropy for $\varphi$ and $\psi$ is given by

$$S(\varphi | \psi) = -\lim_{t \rightarrow \infty} \inf \frac{1}{t} \{QI_l(p, q)^2(t) - p^2(t)\}$$

For $\varphi \in \mathcal{B}(A) \subset \mathcal{S}(A)$, $\Lambda : \mathcal{S}(A) \rightarrow \mathcal{S}(\overline{A})$, let us define the compound states by

$$\Phi_{\mu}^{\psi} = \int_{\mathcal{A}} \omega \otimes \Lambda \omega d\mu \quad \text{and}$$

$$\Phi_{\psi} = \varphi \otimes \Lambda \varphi$$

The mutual entropy w.r.t. $\mathcal{B}$ and $\mu$ is

$$I_{\mu}^{\psi}(\varphi; \Lambda^*) = S(\Phi_{\mu}^{\psi} | \Phi_{\psi})$$
and the mutual entropy w.r.t. $\mathcal{B}$ is defined as $[O.3]$

$$I^\ell (\varphi; \Lambda^*) = \lim_{\epsilon \to 0} \inf \left\{ I_\mu^\ell (\varphi; \Lambda^*) ; \mu \in F_\varphi^\epsilon (S) \right\}$$

$$= \sup \left\{ \sum_k \mu_k S(\Lambda^* \omega_k | \Lambda^* \varphi) ; \varphi = \sum_k \mu_k \varphi_k \right\}$$

where

$$F_\varphi^\epsilon (S) = \left\{ \mu \in D_\varphi (S); S^\ell (\varphi) \leq H(\mu) \leq S^\ell (\varphi) + \epsilon < +\infty \right\}$$

$$D_\varphi (S) = \left\{ \mu \in M_\varphi (S); \exists \{ \mu_k \} \subset \mathbb{R}^+ \text{ s.t.} \mu = \sum_k \mu_k \delta (\varphi_k), \varphi_k \in \text{ex} S, \sum_k \mu_k = 1 \right\}$$

when a state $\varphi$ is expressed as $\varphi = \sum_k \mu_k \omega_k$ (fixed), the mutual entropy is given by

$$I^\ell (\varphi; \Lambda^*) = \sum_k \mu_k S(\Lambda^* \omega_k, \Lambda^* \varphi)$$

This entropy and $\mathcal{B}$-entropy contains Connes-Thirring-Narnhofer entropy as a special case $[M.1]$. The inequality is satisfied for almost all physical cases.

$$0 \leq I^\ell (\varphi; \Lambda^*) \leq S^\ell (\varphi)$$

The fundamental properties of the relative entropy and the mutual entropy are the followings $[A.1,A.2, U.1, H.1, O.3, O.4]$. Theorem

1. Positivity: $S(\varphi | \psi) \geq 0$.
2. Joint Convexity: $S(\lambda \psi_1 + (1-\lambda) \psi_2 | \lambda \varphi_1 + (1-\lambda) \varphi_2) \leq \lambda S(\psi_1 | \varphi_1) + (1-\lambda) S(\psi_2 | \varphi_2)$.
3. Additivity: $S(\psi_1 \otimes \psi_2 | \varphi_1 \otimes \varphi_2) = S(\psi_1 | \varphi_1) + S(\psi_2 | \varphi_2)$.
4. Lower Semicontinuity: If $\lim_{n \to \infty} \| \psi_n - \psi \| = 0$ and $\lim_{n \to \infty} \| \varphi_n - \varphi \| = 0$, then $S(\psi | \varphi) \leq \liminf S(\psi_n | \varphi_n)$. Moreover, if there exists a positive number $\lambda$ satisfying $\psi_n \leq \lambda \varphi_n$, then $\lim_{n \to \infty} S(\psi_n | \varphi_n) = S(\psi | \varphi)$.
5. Monotonicity: For a channel $\Lambda$ from $\mathfrak{S}$ to $\overline{\mathfrak{S}}$, $S(\Lambda^* \psi | \Lambda^* \varphi) \leq S(\psi | \varphi)$.
6. Lower Bound: $\| \psi - \varphi \|^2 \leq 2S(\psi | \varphi)$.
Theorem [O.3]

1. If $\Lambda$ is deterministic, then $I(\varphi; \Lambda^*) = S(\varphi)$.
2. If $\Lambda$ is chaotic, then $I(\varphi; \Lambda^*) = 0$.
3. For a state $\varphi = \text{tr}_p \cdot$, if $\Lambda^*$ is ergodic and the state is stationary for a time evolution $\alpha_t = \text{Ad}U_t$, and if every eigenvalue of $\rho$ is nonzero and nondegenerate, then $I(\rho; \Lambda^*) = S(\Lambda^\rho)$.

This mutual entropy is extensively used for analysing optical communication processes [B.1].

The CNT entropy $H_\phi(\mathfrak{N})$ of $\mathfrak{C}$-subalgebra $\mathfrak{N} \subset \mathcal{A}$ is defined by [C.1].

$$H_\phi(\mathfrak{N}) = \sup_{\varphi = \sum \mu \varphi_j} \sum \mu S(\varphi_j \mathfrak{N} \varphi \mathfrak{N})$$

where the supremum is taken over all finite decomposition $\varphi = \sum \mu \varphi_j$ of $\varphi$. This entropy is a mutual entropy when a channel is the restriction to subalgebra. There are some relations between the mixing entropy $S^\alpha(\varphi)$ and the CNT entropy.

Theorem [M.1]

1. For any state $\varphi$ on a unital $\mathfrak{C}$-algebra $\mathcal{A}$,

$$S^\otimes(\varphi) = H_\phi(\mathcal{A})$$

2. Let $(\mathfrak{M}, G \alpha)$ be a $G$-finite $W^*$-dynamical system, $\varphi$ be a $G$-invariant normal state of $\mathfrak{M}$, then

$$S^\alpha(\varphi) = H_\phi(\mathfrak{M}^\alpha)$$

3. Let $\mathcal{A}$ be the $\mathfrak{C}$-algebra $\mathcal{C}(\mathcal{H})$ of all compact operators on a Hilbert space $\mathcal{H}$, and $G$ be a group, $\alpha$ be a $^*$-automorphic action of $G$-invariant density operator. Then

$$S^\alpha(\rho) = H_\rho(\mathcal{A}^\alpha)$$

The pseudo-mutual $\mathfrak{S}$-entropy $J^\mathfrak{S}(\varphi; \Lambda^*)$ is given by

$$J^\mathfrak{S}(\varphi; \Lambda^*) = \sup\left\{ \sum \mu S(\Lambda^\varphi_1 \Lambda^\varphi_2); \varphi = \sum \mu \varphi_j, \varphi_j \in \mathfrak{S} \right\}.$$
Theorem [M.1]

(1) \[ 0 \leq I^\delta(\varphi; \Lambda^\varphi) \leq J^\delta(\varphi; \Lambda^\varphi) \leq \min\{H^\delta(\varphi), H^{\Lambda^\varphi}(\Lambda^\varphi)\}. \]

(2) Let \( \Lambda^\varphi \) be a G-stationary channel from \( \mathcal{A} \) to \( \overline{\mathcal{A}} \) and \( G \) be compact. Then
\[ 0 \leq I^{(\alpha)}(\varphi; \Lambda^\varphi) \leq \min\{S^{J(\alpha)}(\varphi), S^{J(\overline{\alpha})}(\Lambda^\varphi)\}. \]

References


