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Entropy in Derived Towers of Subfactors

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Introduction

Jones' index theory [16] for type II_1 subfactors was extended by Kosaki [20] to that for conditional expectations between arbitrary factors. Let $N \subset M$ be an inclusion of factors with finite index. Unless $N' \cap M = \mathbf{C}$, there are many faithful normal conditional expectations $E: M \to N$. But there exists a unique conditional expectations $E_0 : M \to N$ which minimizes $\mathrm{Ind}\, E$ [10, 11, 23]. This E_0 is called the minimal conditional expectation and $[M : N]_0 = \mathrm{Ind}\, E_0$ is the minimal index of $N \subset M$. The notion of minimal index is quite important particularly in type III index theory.

Pimsner and Popa [27] extensively developed the relative entropy $H(M|N)$ for type II_1 factors $N \subset M$ in connection with the Jones index $[M : N]$. They showed the inequality $H(M|N) \le \log[M : N]$ and obtained several characterizations for the equality. Note in particular that $H(M|N) = \log[M : N]$ if and only if $[M : N] = [M : N]_0$ (i.e. the trace-preserving conditional expectation $M \to N$ is minimal). In this case $N \subset M$ is called extremal [31]. The relative entropy $K_\varphi(M|N)$ of this kind was defined in [12, 17] in more general setups and its relation with the minimal index was obtained in [12, 13]. The multiplicative chain rule of minimal index for subfactors was established in [21, 25]. See [18] for the most general form of indical chain rule as well as the additive chain rule of relative entropy.

Choda [5, 6] investigated the Connes-Størmer dynamical entropy $H(\sigma)$ and $H(M|\sigma(M))$ for an endomorphism σ of a finite von Neumann algebra M. In [6, 7], when Γ is the canonical shift introduced by Ocneanu [26] on R generated by the derived tower of factors $N \subset M$, the entropies $H(\Gamma)$ and $H(R|\Gamma(R))$ were related with $[M : N]$ or $[M : N]_0$. In fact, we have $H(\Gamma) \le H(R|\Gamma(R)) \le 2H(\Gamma) \le 2\log[M : N]_0$.

As we already mentioned, many close relations between index theory and entropy theory are known so far. In this paper let us further investigate the canonical shift from entropic point of view. We discuss only the case of II_1 factors. But this is not a true restriction whenever we consider the canonical shift on the derived tower induced from the minimal conditional expectation (see the final section).

First in Section 1, following [29, 31] we recall the standard invariants (i.e. the principal graph and the standard eigenvectors) of an inclusion $N \subset M$ of type II_1 factors. In Section 2, we continue [7] and characterize the equality cases $H(R|\Gamma(R)) = 2H(\Gamma)$ and $H(\Gamma) = \log[M : N]$ in terms of the standard invariants. It is shown that the standard invariant of $N \subset M$ is strongly amenable [31] if and only if $\frac{1}{2}H(R|\Gamma(R)) = H(\Gamma) = \log[M : N]$, and this is the case when $N \subset M$ has subexponential growth.

In Section 3, let $\mathcal{A}$ denote the quasilocal C^*-algebra $\overline{\bigcup_n (M'_{-n} \cap M_n)}$ where $\cdots M_{-2} \subset M_{-1} = N \subset M_0 = M \subset M_1 \subset \cdots$ is the tower of basic constructions. We regard $(\mathcal{A}, \Gamma)$ as a generalization of one-dimensional quantum spin system. Setting a suitable interaction, we discuss the variational principle, the Gibbs condition, and the KMS condition with respect to the generated time evolution.

Contents of Sections 2 and 3 are somewhat independent, and full details and further development will be separately presented elsewhere.

1. Preliminaries

Let $N \subset M$ be an inclusion of II_1 factors with $[M : N] < +\infty$. Let

$$\cdots \subset M_{-3} \subset M_{-2} \subset M_{-1} = N \subset M_0 = M \subset M_1 \subset M_2 \subset \cdots \tag{1.1}$$

be the Jones tower of tunnel and basic constructions [16]. The derived tower of relative commutants for $N \subset M$ is

$$\mathbf{C} = M' \cap M \subset M' \cap M_1 \subset M' \cap M_2 \subset \cdots, \tag{1.2}$$

which is an increasing sequence of finite dimensional algebras. (The derived tower is usually defined as $\{N' \cap M_n\}$, but we prefer the dual one $\{M' \cap M_n\}$ in accordance with [29, 31].)

Let τ denote the unique normalized trace, i.e. the λ-Markov trace on $\bigcup_n M_n$ where $\lambda = [M : N]^{-1}$. We denote by $\bar{R}$ the type II_1 von Neumann algebra generated by $\bigcup_n (M'_{-n} \cap M_n)$ via the GNS representation with respect to τ. The normal trace on $\bar{R}$ extending τ is denoted by the same τ. Define the von Neumann subalgebras $R = R_0 \supset R_1 \supset R_2 \supset \cdots$ of $\bar{R}$ by $R_j = (\bigcup_n (M'_j \cap M_n))''$. On the other hand, the core (or the standard part) [30, 31] of $N \subset M$ is $(\bigcup_n (M'_{-n} \cap N))'' \subset (\bigcup_n (M'_{-n} \cap M))''$.

The mirrorings and the canonical shift on the derived tower were introduced in [26]. Since the basic construction of $M \subset M_n$ is M_{2n} [28], the mirroring (antiautomorphism) γ_n of $M' \cap M_{2n}$ is defined by

$$\gamma_n(x) = J_n x^* J_n, \qquad x \in M' \cap M_{2n},$$

where J_n is the modular conjugation on $L^2(M_n, \tau)$. Then we have $\gamma_{n+1} \circ \gamma_n = \gamma_n \circ \gamma_{n-1}$ on $M' \cap M_{2n-2}$ (see [7] for details). Hence we can define the canonical shift Γ on $\bigcup_n (M' \cap M_n)$ by

$$\Gamma(x) = \gamma_{n+1}(\gamma_n(x)), \qquad x \in M' \cap M_{2n},$$

which is an endomorphism of $\bigcup_n (M' \cap M_n)$. Since $\tau \circ \Gamma = \tau$, Γ can extend to an endomorphism of R. Then Γ is a 2-shift on the tower (1.2) in the sense of [6] and $\Gamma(R) = R_2$. Starting from $M_{-k} \subset M_{-k+1}$, we can extend Γ to the canonical shift on $(\bigcup_n (M'_{-k} \cap M_n))''$ for any $k \geq 1$. Thus Γ extends to an automorphism of $\bar{R}$, which is denoted by the same Γ. We may call Γ on R and $\bar{R}$ the unilateral and bilateral canonical shifts, respectively.

Following [29, 31] let us now introduce the standard matrix (or the principal graph) and the standard eigenvectors of $N \subset M$. The standard matrix $\Lambda = \Lambda_{N,M} = [a_{kl}]_{k \in K, l \in L}$ is defined so that $[a_{kl}]_{k \in K_n, l \in L_n}$ is the inclusion matrix of $M' \cap M_{2n} \subset M' \cap M_{2n+1}$ and $[a_{kl}]^t_{k \in K_{n+1}, l \in L_n}$ is that of $M' \cap M_{2n+1} \subset M' \cap M_{2n+2}$, where $K_0 = \{k_0 = *\} \subset K_1 \subset K_2 \subset \cdots$, $K = \bigcup_n K_n$, and $L_0 \subset L_1 \subset L_2 \subset \cdots$, $L = \bigcup_n L_n$. We denote by $\vec{d_n} = (d_{n,k})_{k \in K_n}$ and $\vec{r_n} = (r_{n,k})_{k \in K_n}$ the dimension vector and the trace vector of $M' \cap M_{2n}$, and by $\vec{d'}_n = (d'_{n,l})_{l \in L_n}$ and $\vec{r'}_n = (r'_{n,l})_{l \in L_n}$ those of $M' \cap M_{2n+1}$. We have for $n \geq 0$

$$d'_{n,l} = \sum_{k \in K_n} d_{n,k} a_{k,l}, \qquad l \in L_n, \tag{1.3}$$

$$r_{n,k} = \sum_{l \in L_n} a_{kl} r'_{n,l}, \qquad k \in K_n, \tag{1.4}$$

$$r_{n+1,k} = \lambda r_{n,k}, \qquad k \in K_n,$$

$$r'_{n+1,l} = \lambda r'_{n,l}, \qquad l \in L_n.$$

The standard eigenvectors $\vec{s} = (s_k)_{k\in K}$ and $\vec{t} = (t_l)_{l\in L}$ are defined so that $\vec{r}_n = (\lambda^n s_k)_{k\in K_n}$ and $\vec{r'}_n = (\lambda^n t_l)_{l\in L_n}$ for all $n \geq 0$. Then

$$\Lambda\vec{t} = \vec{s}, \qquad \Lambda^t\vec{s} = \lambda^{-1}\vec{t},$$

$$\Lambda\Lambda^t\vec{s} = \lambda^{-1}\vec{s}, \qquad \Lambda^t\Lambda\vec{t} = \lambda^{-1}\vec{t}. \tag{1.5}$$

It follows from (1.5) that

$$\sum_l a_{kl}^2 \leq \lambda^{-1}, \qquad \sum_k a_{kl}^2 \leq \lambda^{-1}, \tag{1.6}$$

Furthermore we denote by $(f_{n,k})_{k\in K_n}$ and $(f'_{n,l})_{l\in L_n}$ the sets of minimal central projections of $M' \cap M_{2n}$ and $M' \cap M_{2n+1}$, respectively, so that $\tau(f_{n,k}) = d_{n,k}r_{n,k}$ and $\tau(f'_{n,l}) = d'_{n,l}r'_{n,l}$.

When $N \subset M$ is extremal, we have [31]

$$s_k = [p_k M_{2n} p_k : M p_k]^{1/2}, \qquad k \in K_n,$$

$$t_l = \lambda^{1/2}[p'_l M_{2n+1} p'_l : M p'_l]^{1/2}, \qquad l \in L_n,$$

where p_k and p'_l are minimal projections in the kth summand of $M' \cap M_{2n}$ and in the lth summand of $M' \cap M_{2n+1}$, respectively. These imply that $s_k \geq 1$ and $t_l \geq \lambda^{1/2}$ for extremal $N \subset M$.

2. Entropy of canonical shifts and strong amenability

Let $H(\Gamma)$ be the dynamical entropy of Γ with respect to τ [8]. By [6, Theorem 14] we have

$$H(\Gamma) = \lim_{n\to\infty} \frac{1}{n} H(M' \cap M_{2n}) = \lim_{n\to\infty} \frac{2}{n} H(M' \cap M_n), \tag{2.1}$$

where $H(M' \cap M_n)$ denotes the von Neumann entropy of $\tau|M' \cap M_n$. For $j \geq 1$ let $H(R|R_j)$ be the relative entropy of R relative to R_j [8, 27], which is given as

$$H(R|R_j) = \lim_{n\to\infty} H(M' \cap M_n | M'_j \cap M_n)$$

by [27, Proposition 3.4]. The following relations were shown in [6, Theorem 14] under the assumption of $N \subset M$ being extremal:

$$H(\Gamma) \leq H(R|R_2) \leq 2H(\Gamma) \leq 2\log[M : N].$$

But these hold without the extremality assumption; in fact we have [7]

$$H(\Gamma) \leq H(R|R_2) \leq 2H(\Gamma) \leq 2\log[M:N]_0. \tag{2.2}$$

In the following let us completely characterize, in terms of the standard invariants, when the equalities $H(R|R_2) = 2H(\Gamma)$ and $H(\Gamma) = \log[M:N]$ occur, respectively.

Let $Z(M' \cap M_n)$ be the center of $M' \cap M_n$. Then

$$H(Z(M' \cap M_{2n})) = -\sum_{k \in K_n} d_{n,k} r_{n,k} \log d_{n,k} r_{n,k},$$

$$H(Z(M' \cap M_{2n+1})) = -\sum_{l \in L_n} d'_{n,l} r'_{n,l} \log d'_{n,l} r'_{n,l}.$$

Since by (1.3), (1.4), and (1.6)

$$\sum_{k \in K_n} d_{n,k} a_{k,l} r'_{n,l} = d'_{n,l} r'_{n,l}, \qquad \sum_{l \in L_n} d_{n,k} a_{k,l} r'_{n,l} = d_{n,k} r_{n,k},$$

$$\#\{k \in K : a_{kl} \neq 0\} \leq \lambda^{-1}, \qquad \#\{l \in L : a_{kl} \neq 0\} \leq \lambda^{-1},$$

it is easy to check that

$$|H(Z(M' \cap M_{2n})) - H(Z(M' \cap M_{2n+1}))| \leq \log \lambda^{-1}. \tag{2.3}$$

Setting

$$I(M' \cap M_{2n}) = \sum_{k \in K_n} d_{n,k} r_{n,k} \log \frac{d_{n,k}}{r_{n,k}},$$

we immediately have

$$I(M' \cap M_{2n}) = 2H(M' \cap M_{2n}) - H(Z(M' \cap M_{2n})). \tag{2.4}$$

Furthermore we can show that

$$H(R|R_2) = \lim_{n \to \infty} \{I(M' \cap M_{2n}) - I(M' \cap M_{2n-2})\}. \tag{2.5}$$

Now (2.1) and (2.3)–(2.5) altogether imply the following:

Theorem 2.1. *The limit* $\lim_{n\to\infty} \frac{1}{n} H(Z(M' \cap M_n))$ *exists and*

$$\frac{1}{2} H(R|R_2) + \lim_{n \to \infty} \frac{1}{n} H(Z(M' \cap M_n)) = H(\Gamma).$$

Hence $H(R|R_2) = 2H(\Gamma)$ *if and only if* $\lim_{n\to\infty} \frac{1}{n} H(Z(M' \cap M_n)) = 0$.

We say that the principal graph of $N \subset M$ has subexponential growth if

$$\lim_{n \to \infty} \frac{1}{n} \log |K_n| = 0,$$

or equivalently

$$\lim_{n \to \infty} \frac{1}{n} \log |L_n| = 0,$$

where $|K_n|$ denotes the cardinal number of K_n.

Corollary 2.2. *If the principal graph of* $N \subset M$ *has subexponential growth, then* $H(R|R_2) = 2H(\Gamma)$.

Since

$$\frac{1}{n}H(M' \cap M_{2n}) = \log \lambda^{-1} - \frac{1}{n}\sum_{k \in K_n} \tau(f_{n,k}) \log s_k,$$

$$\frac{1}{n}H(M' \cap M_{2n+1}) = \log \lambda^{-1} - \frac{1}{n}\sum_{l \in L_n} \tau(f'_{n,l}) \log t_l,$$

we have:

Theorem 2.3. *The following equal limits*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k \in K_n} \tau(f_{n,k}) \log s_k = \lim_{n\to\infty} \frac{1}{n} \sum_{l \in L_n} \tau(f'_{n,l}) \log t_l$$

exist and

$$H(\Gamma) + \lim_{n\to\infty} \frac{1}{n} \sum_{k \in K_n} \tau(f_{n,k}) \log s_k = \log[M : N].$$

Hence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k \in K_n} \tau(f_{n,k}) \log s_k \geq 0,$$

and $H(\Gamma) = \log[M : N]$ *if and only if* $\lim_{n\to\infty} \sum_{k \in K_n} \tau(f_{n,k}) \log s_k = 0$.

We say that the standard eigenvector of $N \subset M$ has subexponential growth if

$$\lim_{n\to\infty} \frac{1}{n} \log\Big(\max_{k \in K_n} s_k\Big) = 0,$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{n} \log\Big(\max_{l \in L_n} t_l\Big) = 0.$$

Corollary 2.4. *If the standard eigenvector of* $N \subset M$ *has subexponential growth, then* $H(\Gamma) = \log[M : N]$ *and hence* $N \subset M$ *is extremal.*

In particular, when $\vec{s}$ is bounded, we have the above conclusion. So Corollary 2.4 improves [31, Corollary 1.3.6(ii)].

Since

$$\log \lambda^{-1} - \frac{1}{n}\sum_{k \in K_n} \tau(f_{n,k}) \log d_{n,k} = \frac{1}{n}H(Z(M' \cap M_{2n})) + \frac{1}{n}\sum_{k \in K_n} \tau(f_{n,k}) \log s_k,$$

$$\log \lambda^{-1} - \frac{1}{n}\sum_{l \in L_n} \tau(f'_{n,l}) \log d'_{n,l} = \frac{1}{n}H(Z(M' \cap M_{2n+1})) + \frac{1}{n}\sum_{l \in L_n} \tau(f'_{n,l}) \log t_l,$$

the next theorem can be shown from Theorems 2.1 and 2.3.

Theorem 2.5. *The following equal limits*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k\in K_n}\tau(f_{n,k})\log d_{n,k}=\lim_{n\to\infty}\frac{1}{n}\sum_{l\in L_n}\tau(f'_{n,l})\log d'_{n,l}$$

exist and

$$H(\Gamma)+\lim_{n\to\infty}\frac{1}{n}\sum_{k\in K_n}\tau(f_{n,k})\log d_{n,k}=H(R|R_2).$$

Moreover the following conditions are equivalent:
(i) $H(R|R_2)=2\log[M:N]$;
(ii) $H(R|R_1)=H(R_1|R_2)=\log[M:N]$;
(iii) $\frac{1}{2}H(R|R_2)=H(\Gamma)=\log[M:N]$;
(iv) $\lim_{n\to\infty}\frac{1}{n}\sum_{k\in K_n}\tau(f_{n,k})\log d_{n,k}=\log[M:N]$;
(v) $\lim_{n\to\infty}\frac{1}{n}\sum_{l\in L_n}\tau(f'_{n,l})\log d'_{n,l}=\log[M:N]$.

It is seen that the above equivalent conditions hold if and only if $N\subset M$ is extremal and

$$H(M|N)=\lim_{n\to\infty}H(M'_{-n}\cap M|M'_{-n}\cap N).$$

This last condition is one of the caracterizations in [31, Theorem 5.3.1] for extremal $N\subset M$ whose standard invariant is strongly amenable. When M is hyperfinite, this implies [31, Theorem 4.1.2] that $N\subset M$ itself is strongly amenable, equivalently $N\subset M$ has the generating property. Thus we obtain the combinatorial characterization (iv) or (v) above for strongly amenable extremal II_1 inclusions $N\subset M$. Also it is easy to check that $||\Lambda||^2=[M:N]$ follows from (iv).

When both the principal graph and the standard eigenvector of $N\subset M$ have subexponential growth, $N\subset M$ is said to have subexponential growth. This is equivalent to the condition [31]

$$\lim_{n\to\infty}\Big(\sum_{k\in K_n}s_k\Big)^{1/n}=1,$$

because

$$\sum_{k\in K_n}s_k\le|K_n|\max_{k\in K_n}s_k,\qquad \max_{k\in K_n}s_k\le\sum_{k\in K_n}s_k,\qquad |K_n|\le\sum_{k\in K_n}s_k.$$

So we have the following which affirmatively solves [31, Problem 5.4.6].

Corollary 2.6. *If $N\subset M$ has subexponential growth, then $N\subset M$ is extremal and the standard invariant of $N\subset M$ is strongly amenable.*

We end this section with some examples.

Example 2.7. Let N be the Jones subfactor [16] of the hyperfinite II_1 factor M with $[M:N]=\lambda^{-1}$. Then $M'\cap M_n=\mathrm{Alg}\{1,e_2,\ldots,e_n\}$ and $R=\{e_n:n\ge 2\}''$ where e_n are the Jones projections. Let θ_λ be the shift on R given by $\theta_\lambda(e_n)=e_{n+1}$. Then $\Gamma=\theta_\lambda^2$ and hence $H(\Gamma) = 2H(\theta_\lambda)$. Note that $N \subset M$ has finite depth if $\lambda > 1/4$, it has graph A_∞ and $\vec{s} = (1, 3, 5, \dots)$ if $\lambda = 1/4$, and it has graph $A_{\infty,\infty}$ but is not exremal with $[M : N]_0 = 4$ if $\lambda < 1/4$. It is known (see [7, Example 6.1] for instance) that when $\lambda \geq 1/4$

$$\frac{1}{2}H(R|R_2) = 2H(\theta_\lambda) = H(M|N) = \log \lambda^{-1},$$

and when $\lambda < 1/4$

$$\frac{1}{2}H(R|R_2) = 2H(\theta_\lambda) = H(M|N) = 2\eta(t) + 2\eta(1-t) < \log 4,$$

where $t(1-t) = \lambda$, $t > 0$, and $\eta(t) = -t \log t$. The computation of $H(\theta_\lambda)$ was done in [27] and [5, 32].

Example 2.8. Consider the following inclusions:

$$N = \left\{ \bigoplus_{i=0}^{m} \alpha_i(x) : x \in A \right\} \subset M = A \otimes M_{m+1}(\mathbf{C}),$$

where A is a type II_1 factor and $\theta_0 = \mathrm{id}, \theta_1, \dots, \theta_m \in \operatorname{Aut} A$. Then $N \subset M$ is extremal and $[M : N] = (m+1)^2$. The derived tower of $N \subset M$ was presented in [3, 31]. Let $G \subset \operatorname{Aut} A / \operatorname{Int} A$ be the group generated by $[\theta_i] = \theta_i / \operatorname{Int} A$, $0 \leq i \leq m$. Then the standard invariants of $N \subset M$ are parametrized by the elements of G and the growth of the principal graph is the same as that of G with a generating set $\{[\theta_i], [\theta_i^{-1}] : 0 \leq i \leq m\}$ [22]. Moreover $s_g = 1$ and $t_h = 1/(m+1)$ for all $g, h \in G$. Define the initial distribution μ on G by $\mu(g) = d_{1,g}/(m+1)^2$ for $g \in K_1$ and $\mu(g) = 0$ for $g \in G \setminus K_1$. Let $h(G, \mu)$ be the entropy of (G, μ) given by $h(G, \mu) = \lim_{n \to \infty} \frac{1}{n} H(\mu^n)$ where μ^n denotes the nth convolution of μ [22]. Then we have

$$\frac{1}{2}H(R|R_2) + h(G, \mu) = H(\Gamma) = \log[M : N],$$

$$\lim_{n \to \infty} \{H(M' \cap M_{2n+2}) - H(M' \cap M_{2n})\} = \log[M : N].$$

In particular, let $N \subset M$ be determined by $2m+1$ automorphisms $\theta_0 = \mathrm{id}, \theta_i, \theta_i^{-1} \in \operatorname{Aut} A$, $1 \leq i \leq m$, and define μ_0 on G by $\mu_0(h) = d'_{0,h}/(2m+1)^2$ for $h \in L_0$ and $\mu_0(h) = 0$ for $h \in G \setminus L_0$. Then

$$H(R|R_1) + h(G, \mu_0) = \log[M : N],$$

which was proved in [3, Theorem 1.9]. Also

$$\lim_{n \to \infty} \{H(M' \cap M_{n+1}) - H(M' \cap M_n)\} = \log[M : N].$$

When G becomes the free group F_m on m generators, [3, Proposition 2.11] says that

$$h(F_m, \mu_0) = \frac{2m-2}{2m+1} \log(2m-1).$$

Since

$$H(R|R_2) - H(\Gamma) = 2\{\log(2m+1) - h(F_m, \mu_0)\}$$

tends to 0 as $m \to \infty$, $H(R|R_2)$ can be arbitrarily close to $H(\Gamma)$ in (2.2).

3. Quantum systems arising from subfactors

As before let $N \subset M$ be an inclusion of II_1 factors with $\lambda^{-1} = [M : N] < +\infty$ and the Jones tower (1.1). Set $\mathcal{A}_{(i,j]} = M_i' \cap M_j$ for $i < j$ and in particular $\mathcal{A}_n = \mathcal{A}_{(0,n]}$ $(= M' \cap M_n)$ for $n \geq 1$. Let $\mathcal{A}$ be a quasilocal C^*-algebra defined as the C^*-completion of $\bigcup_{n=1}^{\infty} \mathcal{A}_{(-n,n]}$. In fact, we may define $\mathcal{A}$ as the norm closure of $\bigcup_{n=1}^{\infty} \mathcal{A}_{(-n,n]}$ in $\bar{R}$, and set the canonical trace τ and the canonical shift Γ on $\mathcal{A}$ as the restrictions of τ and Γ on $\bar{R}$ (see Section 1). Then Γ is an automorphism of $\mathcal{A}$ and satisfies $\tau \circ \Gamma = \tau$ and $\Gamma(\mathcal{A}_{(i,j]}) = \mathcal{A}_{(i+2,j+2]}$. Thus a quantum system $(\mathcal{A}, \tau, \Gamma)$ is obtained from $N \subset M$, which generalizes one-dimensional quantum spin systems. The aim of this section is to develop quantum statistical mechanics on this system.

Let $\mathcal{S}_\Gamma(\mathcal{A})$ denote the set of all Γ-invariant states on $\mathcal{A}$. Since $(\mathcal{A}, \Gamma)$ is asymptotically abelian in the norm sense, i.e.

$$\lim_{|n|\to\infty} \| [a, \Gamma^n(b)] \| = 0$$

for all $a, b \in \mathcal{A}$, we know [4, 4.3.11] that $\mathcal{S}_\Gamma(\mathcal{A})$ forms a simplex. Put $\tau_n = \tau|\mathcal{A}_n$ and $\omega_n = \omega|\mathcal{A}_n$ for $\omega \in \mathcal{S}_\Gamma(\mathcal{A})$ and $n \geq 1$. Let $S(\omega_n, \tau_n)$ be the relative entropy of ω_n with respect to τ_n, which is written as

$$S(\omega_n, \tau_n) = \omega\Big(\log \frac{d\omega_n}{d\tau_n}\Big).$$

We then have the monotonicity $S(\omega_n, \tau_n) \leq S(\omega_{n+1}, \tau_{n+1})$ and the superadditivity

$$S(\omega_{2m+2n}, \tau_{2m+2n}) \geq S(\omega_{2m}, \tau_{2m}) + S(\omega_{2n}, \tau_{2n}).$$

These imply the following:

Proposition 3.1. *For every $\omega \in \mathcal{S}_\Gamma(\mathcal{A})$, the limit $\lim_{n\to\infty} \frac{1}{n} S(\omega_n, \tau_n)$ exists and*

$$\lim_{n\to\infty} \frac{1}{n} S(\omega_n, \tau_n) = \sup_{n\geq 1} \frac{1}{2n} S(\omega_{2n}, \tau_{2n}).$$

For $\omega \in \mathcal{S}_\Gamma(\mathcal{A})$ define the mean relative entropy

$$S_{\mathrm{M}}(\omega, \tau) = \lim_{n\to\infty} \frac{1}{n} S(\omega_n, \tau_n),$$

and the mean entropy $s(\omega)$ of ω (with respect to τ) by

$$s(\omega) = \log \lambda^{-1/2} - S_{\mathrm{M}}(\omega, \tau).$$

Note that $s(\omega) \leq \log \lambda^{-1/2} = s(\tau)$, and $s(\omega) = \log \lambda^{-1/2}$ if and only if $\omega = \tau$. In fact, if $s(\omega) = \log \lambda^{-1/2}$ then $S(\omega_{2n}, \tau_{2n}) = 0$ for all $n \geq 1$ by Proposition 3.1, which implies $\omega = \tau$.

The next proposition shows that the mean entropy defined above is identical to the usual one under the subexponential growth of standard eigenvector.

Proposition 3.2. *If the standard eigenvector of $N \subset M$ has subexponential growth (in particular, if $N \subset M$ has finite depth), then for every $\omega \in \mathcal{S}_\Gamma(\mathcal{A})$*

$$s(\omega) = \lim_{n\to\infty} \frac{1}{n} S(\omega_n),$$

where $S(\omega_n)$ is the von Neumann entropy of ω_n.

It can be easily shown that $s(\omega)$ is affine and weakly* upper semicontinuous on $\mathcal{S}_\Gamma(\mathcal{A})$.

In the following the symbol X means a finite interval $(i, j]$ $(i < j)$ in $\mathbf{Z}$ and let $|X| = j - i$. We say that Φ is an interaction if a selfadjoint element $\Phi(X)$ in $\mathcal{A}_X$ is given for each X. Here the interaction energy $\Phi(X)$ is given only for finite intervals in $\mathbf{Z}$, while it is for all finite subsets in the case of usual quantum spin systems. But this restriction is not essential (interactions in quantum spin systems can be redefined into this form). We always assume the following:

(1) Φ is translation-invariant: for any X and $n \in \mathbf{Z}$

$$\Gamma^n(\Phi(X)) = \Phi(X + 2n),$$

(2) Φ has relatively short range:

$$\sum_{X \ni 0} \frac{\|\Phi(X)\|}{|X|} < +\infty.$$

Note that (1) and (2) imply $\sum_{X\ni -1} \|\Phi(X)\|/|X| < +\infty$ as well. Let $\mathcal{B}$ denote the set of all interactions satisfying (1) and (2) above. Define the norm $|||\Phi|||$ of $\Phi \in \mathcal{B}$ by

$$|||\Phi||| = \frac{1}{2}\left(\sum_{X\ni -1} \frac{\|\Phi(X)\|}{|X|} + \sum_{X\ni 0} \frac{\|\Phi(X)\|}{|X|} \right).$$

Given $\Phi \in \mathcal{B}$ and an interval $\Lambda \subset \mathbf{Z}$, the local Hamiltonian $H(\Lambda)$ is given as

$$H(\Lambda) = \sum_{X \subset \Lambda} \Phi(X).$$

For simplicity we write $H_n = H((0, n])$. Furthermore define $A_\Phi \in \mathcal{A}$ by

$$A_\Phi = \frac{1}{2}\left(\sum_{X\ni -1} \frac{\Phi(X)}{|X|} + \sum_{X\ni 0} \frac{\Phi(X)}{|X|} \right).$$

Obviously $\|A_\Phi\| \le |||\Phi|||$.

Proposition 3.3. *For every $\Phi \in \mathcal{B}$ and $\omega \in \mathcal{S}_\Gamma(\mathcal{A})$, the limit $\lim_{n\to\infty} \frac{1}{n}\omega(H_n)$ exists and*

$$\lim_{n\to\infty} \frac{1}{n}\omega(H_n) = \omega(A_\Phi).$$

Theorem 3.4. *For every $\Phi \in \mathcal{B}$, the limit $\lim_{n\to\infty} \frac{1}{n} \log \tau(e^{-H_n})$ exists and*

$$\log \lambda^{-1/2} + \lim_{n\to\infty} \frac{1}{n} \log \tau(e^{-H_n}) = \sup_{\omega \in \mathcal{S}_\Gamma(\mathcal{A})} \{s(\omega) - \omega(A_\Phi)\}.$$

Define the thermodynamic free energy (or the pressure) $p(\Phi)$ of $\Phi \in \mathcal{B}$ by

$$p(\Phi) = \log \lambda^{-1/2} + \lim_{n\to\infty} \frac{1}{n} \log \tau(e^{-H_n}).$$

The above theorem gives the variational equality:

$$p(\Phi) = \sup_{\omega \in \mathcal{S}_\Gamma(\mathcal{A})} \{s(\omega) - \omega(A_\Phi)\}.$$

Since $\omega \mapsto s(\omega) - \omega(A_\Phi)$ is weakly* upper semicontinuous and affine, it follows that

$$\mathcal{S}_\Phi(\mathcal{A}) = \{\omega \in \mathcal{S}_\Gamma(\mathcal{A}) : p(\Phi) = s(\omega) - \omega(A_\Phi)\}$$

is nonempty and becomes a face of $\mathcal{S}_\Gamma(\mathcal{A})$. So $\mathcal{S}_\Phi(\mathcal{A})$ is a simplex. When $\omega \in \mathcal{S}_\Phi(\mathcal{A})$, we say that ω satisfies the variational principle (or it is thermodynamically stable) with respect to Φ. From the above variational equality we can easily show as [4, 6.2.40] that $p(\Phi)$ is convex in $\Phi \in \mathcal{B}$ and

$$|p(\Phi) - p(\Psi)| \le |||\Phi - \Psi|||, \qquad \Phi, \Psi \in \mathcal{B}.$$

The next proposition says that when $N \subset M$ has finite depth, the above $p(\Phi)$ is identical to the usual one defined by using Tr_n instead of τ_n. Here Tr_n denotes the canonical trace on $\mathcal{A}_n$ in the sense that $\mathrm{Tr}_n(e) = 1$ for any minimal projections $e \in \mathcal{A}_n$.

Proposition 3.5. *If $N \subset M$ has finite depth, then for every $\Phi \in \mathcal{B}$*

$$p(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Tr}_n(e^{-H_n}).$$

We use the notion of the inner perturbation of a state on $\mathcal{A}$ to introduce the Gibbs condition. Let ω be a state on $\mathcal{A}$ and $h = h^* \in \mathcal{A}$. Note that $\psi \mapsto S(\psi, \omega) + \psi(h)$ is weakly* lower semicontinuous and strictly convex on the state space of $\mathcal{A}$. So the perturbed state $[\omega^h]$ is defined as the unique minimizer of this functional. We know (see [2, 9]) that for selfadjoint $h, k \in \mathcal{A}$ the chain rule $[[\omega^h]^k] = [\omega^{h+k}]$ holds and

$$S([\omega^h], [\omega^k]) \le 2||h - k||. \tag{3.1}$$

Given $\Phi \in \mathcal{B}$ and $n \ge 1$, the surface energy W_n is defined as

$$W_n = \sum \{\Phi(X) : X \cap (0, n] \ne \emptyset,\ X \cap (0, n]^c \ne \emptyset\},$$

whenever the sum in the right-hand side converges in norm. The canonical state (or the local Gibbs state) φ_n^c on $\mathcal{A}_n$ is defined by

$$\varphi_n^c(a) = \frac{\tau(e^{-H_n}a)}{\tau(e^{-H_n})}, \qquad a \in \mathcal{A}_n.$$

Definition 3.6. Let $\Phi \in \mathcal{B}$ and assume that W_n is defined for any $n \geq 1$. Let $\omega \in \mathcal{S}_\Gamma(\mathcal{A})$ and π_ω be the GNS representation of $\mathcal{A}$ with the cyclic vector Ω_ω.

(1) We say that ω satisfies the Gibbs condition with respect to Φ if Ω_ω is separating for $\pi_\omega(\mathcal{A})''$ and the following holds for any $n \geq 1$:

$$[\omega^{-W_n}](ab) = \varphi_n^c(a)\psi_n(b), \qquad a \in \mathcal{A}_n,\ b \in \mathcal{A}_{(0,n]^c},$$

where $\mathcal{A}_{(0,n]^c}$ is the C^*-subalgebra of $\mathcal{A}$ generated by $\mathcal{A}_{(-m,0]}$ and $\mathcal{A}_{(n,n+m]}$ $(m \geq 1)$, and ψ_n is some state on $\mathcal{A}_{(0,n]^c}$.

(2) Also ω is said to satisfy the Gibbs condition in the weak sense with respect to Φ if $[\omega^{-W_n}]|\mathcal{A}_n = \varphi_n^c$ holds for any $n \geq 1$.

The monotonicity of relative entropy and (3.1) show the following:

Proposition 3.7. *Let $\Phi \in \mathcal{B}$ be such that W_n is defined for every $n \geq 1$ and $\frac{1}{n}\|W_n\| \to 0$. If $\omega \in \mathcal{S}_\Gamma(\mathcal{A})$ satisfies the Gibbs condition in the weak sense with respect to Φ, then it satisfies the variational principle with respect to Φ.*

The following example gives Gibbs states of a special type, which generalize translation-invariant product states in quantum spin systems.

Example 3.8. Suppose that $N' \cap M \neq \mathbf{C}$. For each (not necessarily trace-preserving) conditional expectation $E : M \to N$ let $E_n : M_n \to M_{n-1}$ $(n \in \mathbf{Z})$ be the iterated conditional expectations from E [20]. We can define a state ω on $\mathcal{A}$ by

$$\omega|\mathcal{A}_{(i,j]} = E_{i+1} \circ E_{i+2} \circ \cdots \circ E_j|\mathcal{A}_{(i,j]},$$

which is Γ-invariant because the canonical shift as well as the Jones tower is defined apart from the choice of conditional expectation (see [7]). Put

$$h_n = \frac{d(E_n|\mathcal{A}_{(n-1,n]})}{d(\tau|\mathcal{A}_{(n-1,n]})}, \qquad n \in \mathbf{Z}.$$

We then have $\Gamma^n(h_{-1}) = h_{2n-1}$, $\Gamma^n(h_0) = h_{2n}$, and

$$\frac{d(\omega|\mathcal{A}_n)}{d(\tau|\mathcal{A}_n)} = h_1 h_2 \cdots h_n.$$

Now define an interaction Φ by $\Phi((n-1,n]) = -\log h_n$ $(n \in \mathbf{Z})$ and $\Phi(X) = 0$ for other X. Then it is immediate that ω satisfies the Gibbs condition with respect to Φ where $\varphi_n^c = \omega|\mathcal{A}_n$, $e^{-H_n} = h_1 \cdots h_n$, and $W_n = 0$.

We say that an interaction Φ has finite range if there exists $N \geq 1$ such that $\Phi(X) = 0$ whenever $|X| > N$. A weaker condition of finite body is also used in the case of quantum spin systems, but there is no difference in our setup where $\Phi(X)$ are restricted to finite intervals X. We set another useful condition of Φ as follows: for some $r > 0$

$$||\Phi||_r = \sum_{X \ni 0} e^{r|X|}||\Phi(X)|| < +\infty. \tag{3.2}$$

This is satisfied if Φ has finite range. Also note that (3.2) implies the assumption of Proposition 3.7.

The next proposition shows the existence of the time evolution associated with an interaction Φ.

Proposition 3.9. *Assume either that $\Phi \in \mathcal{B}$ satisfies (3.2) for some $r > 0$ or that $\sum_{X \ni 0} ||\Phi(X)|| < +\infty$ and $\sup_{n \geq 1} ||W_n|| < +\infty$ (i.e. the surface energies are uniformly bounded). Then there exists a strongly continuous one-parameter automorphism group α_t^Φ ($t \in \mathbf{R}$) such that*

$$\lim_{m,n \to \infty} ||\alpha_t^\Phi(a) - e^{itH((-m,n])} a e^{-itH((-m,n])}|| = 0, \qquad a \in \mathcal{A},\ t \in \mathbf{R}.$$

The proof is the same as in the case of quantum spin systems (see [19], [4, 6.2.4, 6.2.6]). Here note that (3.2) implies $\sum_{X \ni -1} e^{r|X|}||\Phi(X)|| < +\infty$ as well from the Γ-invariance of Φ. We have $\alpha_t^\Phi \circ \Gamma = \Gamma \circ \alpha_t^\Phi$.

Now we consider the KMS condition for a state on $\mathcal{A}$ with respect α^Φ. Our main result is stated as follows.

Theorem 3.10. *Assume that $\mathcal{A}$ is induced by $N \subset M$ having finite depth and Φ satisfies the same as Proposition 3.9. Then the following conditions for $\omega \in \mathcal{S}_\Gamma(\mathcal{A})$ are equivalent:*

(i) ω satisfies the KMS condition with respect to α^Φ;

(ii) ω satisfies the Gibbs condition with respect to Φ;

(iii) ω satisfies the Gibbs condition in the weak sense with respect to Φ;

(iv) ω satisfies the variational principle with respect to Φ.

As in the case of quantum spin systems, we have the uniqueness of α^Φ-KMS states when Φ has uniformly bounded surface energies. The proof can be done in the same way as [1] by using Theorem 3.10 and a method of relative entropy (in particular (3.1)).

Theorem 3.11. *Assume that $N \subset M$ has finite depth and Φ satisfies $\sum_{X \ni 0} ||\Phi(X)|| < +\infty$ and $\sup_{n \geq 1} ||W_n|| < +\infty$ (this is the case if Φ has finite range.) Then there exists a unique KMS state with respect to α^Φ.*

Note that the KMS condition (i) of Theorem 3.10 does not depend on the canonical trace τ, while other (ii)–(iv) do so. In particular when $\Phi = 0$, (i) means that ω is tracial, while each of (ii)–(iv) is nothing but $\omega = \tau$. It is seen that when $N \subset M$ has graph A_∞ or $A_{\infty,\infty}$, there are uncountable many extremal tracial states on $\mathcal{A}$. So we know that the assumption of $N \subset M$ having finite depth is essential in Theorems 3.10 and 3.11. Indeed in the proof of Theorem 3.10 we use the fact that τ is a unique tracial state on $\mathcal{A}$

if $N \subset M$ has finite depth. However we have (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) without the finite depth assumption.

4. Remarks on the type III case

Let $N \subset M$ be an inclusion of arbitrary factors with finite index, and $E_0 : M \to N$ be the minimal conditional expectation. Iterating upward and downward on the Jones tower from E_0 we set conditional expectations $E_n : M_n \to M_{n-1}$ ($n \in \mathbf{Z}$). We have the faithful trace ϕ on $\bigcup_n (M'_{-n} \cap M_n)$ defined by

$$\phi | M'_{-n} \cap M_n = E_{-n+1} \circ \cdots \circ E_{n-1} \circ E_n | M'_{-n} \cap M_n, \qquad n \geq 1.$$

(The traciality of ϕ follows from [21].) Then all the material (e.g. $R, \Gamma, \Lambda, \vec{s}, \vec{t}$) in Section 1 can be analogously obtained with use of the trace ϕ in place of τ.

Let $N \subset M$ be infinite factors and J_M, J_N be the modular conjugations associated with a common cyclic and separating vector for M, N. Then Longo's canonical endomorphism is $\gamma = \mathrm{Ad}(J_N J_M) : M \to N$, and the Jones tower is identified as follows [24]: $M_{2n} = \gamma^n(M)$ and $M_{2n-1} = \gamma^n(N)$ for $n \in \mathbf{Z}$. Moreover Γ on R is given by $\Gamma = \gamma^{-1} | R = \mathrm{Ad}(J_M J_N) | R$. The sector theory plays a crucial role in theory of type III subfactors (see [24], [14]).

When we consider (R, Γ) or $(\mathcal{A}, \Gamma)$ obtained from a type III inclusion $N \subset M$, all discussions in Sections 2 and 3 are valid with the trace ϕ coming from the minimal conditional expectation in place of τ. But the next proposition says that as far as we consider the canonical shift, all things can be reduced to the type II_1 case. This is the reason why we restrict to type II_1 inclusions in this paper.

Proposition 4.1. *Let $N \subset M$ be any inclusion of facators with finite index and define the standard invariants $\Lambda = [a_{kl}]_{k \in K, l \in L}$ and the standard vectors $\vec{s}, \vec{t}$ via the minimal conditional expectation. Then there exists an extremal inclusion of type II_1 factors $B \subset A$ with $[A : B] = [M : N]_0$ such that $\Lambda, \vec{s}, \vec{t}$ are the standard invariants of $B \subset A$. Furthermore the canonical shift (R, Γ) for $N \subset M$ coincides with that of $B \subset A$.*

This can be proved by using [15], which was suggested by M. Izumi.

References

[1] H. Araki, On uniqueness of KMS states of one-dimensional quantum lattice systems, *Comm. Math. Phys.* **44** (1975), 1–7.

[2] H. Araki, Relative entropy for states of von Neumann algebras II, *Publ. Res. Inst. Math. Sci.* **13** (1977), 173–192.

[3] D. Bisch, Entropy of groups and subfactors, *J. Funct. Anal.* **103** (1992), 190–208.

[4] O. Bratteli and D. W. Robinson, *Opetrator Algebras and Quantum Statistical Mechanics II*, Springer, New York-Berlin-Heidelberg, 1981.

[5] M. Choda, Entropy for *-endomorphisms and relative entropy for subalgebras, *J. Operator Theory* **25** (1991), 125–140.

[6] M. Choda, Entropy for canonical shifts, *Trans. Amer. Math. Soc.* **334** (1992), 827–849.

[7] M. Choda and F. Hiai, Entropy for canonical shifts. II, *Publ. Res. Inst. Math. Sci.* **27** (1991), 461–489.

[8] A. Connes and E. Størmer, Entropy for automorphisms of II_1 von Neumann algebras, *Acta. Math.* **134** (1975), 289–306.

[9] M. J. Donald, Relative hamiltonians which are not bounded from above, *J. Funct. Anal.* **91** (1990), 143–173.

[10] J. F. Havet, Esperance conditionnelle minimale, *J. Operator Theory* **24** (1990), 33–55.

[11] F. Hiai, Minimizing indices of conditional expectations onto a subfactor, *Publ. Res. Inst. Math. Sci.* **24** (1988), 673–678.

[12] F. Hiai, Minimum index for subfactors and entropy, *J. Operator Theory* **24** (1990), 301–336.

[13] F. Hiai, Minimum index for subfactors and entropy. II, *J. Math. Soc. Japan* **43** (1991), 347–379.

[14] M. Izumi, Application of fusion rules to classification of subfactors, *Publ. Res. Inst. Math. Sci.* **27** (1991), 953–994.

[15] M. Izumi, On type II and type III principal graphs of subfactors, preprint.

[16] V. F. R. Jones, Index for subfactors, *Invent. Math.* **72** (1983), 1–25.

[17] S. Kawakami, Some remarks on index and entropy for von Neumann algebras, *Proc. Japan Acad.* **65A** (1989), 323–325.

[18] S. Kawakami, Chain rule of indical derivative, in *Studies on group actions on operator algebras and index and entropy*, Kokyuroku, Vol. 839, Res. Inst. Math. Sci., Kyoto Univ., 1993, pp. 88–95 (in Japanese).

[19] A. Kishimoto, Dissipations and derivations, *Comm. Math. Phys.* **47** (1976), 25–32.

[20] H. Kosaki, Extension of Jones' theory on index to arbitrary factors, *J. Funct. Anal.* **66** (1986), 123–140.

[21] H. Kosaki and R. Longo, A remark on the minimal index of subfactors, *J. Funct. Anal.* **107** (1992), 458–470.

[22] V. A. Kaimanovich and A. M. Vershik, Random walks on discrete groups: boundary and entropy, *Ann. Prob.* **11** (1983), 457–490.

[23] R. Longo, Index of subfactors and statistics of quantum fields. I, *Comm. Math. Phys.* **126** (1989), 217–247.

[24] R. Longo, Index of subfactors and statistics of quantum fields. II, *Comm. Math. Phys.* **130** (1990), 285–309.

[25] R. Longo, Minimal index and braided subfactors, *J. Funct. Anal.* **109** (1992), 98–112.

[26] A. Ocneanu, Quantized groups, string algebras and Galois theory for algebras, preprint.

[27] M. Pimsner and S. Popa, Entropy and index for subfactors, *Ann. Sci. École Norm. Sup. Sér. 4* **19** (1986), 57–106.

[28] M. Pimsner and S. Popa, Iterating the basic construction, *Trans. Amer. Math. Soc.* **310** (1988), 127–133.

[29] S. Popa, Classification of subfactors: the reduction to commuting squares, *Invent. Math.* **101** (1990), 19–43.

[30] S. Popa, Sur la classification des sous-facteurs d'indice fini du facteur hyperfini, *C. R. Acad. Sci. Paris Sér. I* **311** (1990), 95–100.

[31] S. Popa, Classification of amenable subfactors of type II, preprint.

[32] H. S. Yin, Entropy of certain noncommutative shifts, *Rockey Mountain J. Math.* **20** (1990), 651–656.