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The SAS Einstein Field Equations and a Central Extension of a Formal Loop Group

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0. Introduction

One of the most important things in research activities are among efforts in infinite dimensional analysis. From this point of view, in this note, we shall construct an infinite dimensional homogeneous space embedding in a formal loop group, which represents a kind of infinite dimensional symmetry.

We discussed in [HS2] that there is an elegant relation between the conformal factor in the stationary axisymmetric (SAS) Einstein-Maxwell field equations and a central extension of a formal loop group which is described by a group 2-cocycle on the formal loop group. This relation was first found by [BM] in the vacuum case. Also it is very important that the corresponding 2-cocyle on the Lie algebra of the formal loop group is the one which describes an affine Lie algebra [K], and that the space of formal solutions with conformal factors is a homogeneous space of a central extension of the Hauser group.

The crucial points of our discussions are as follows (see [HS2][HS1][S] for more details). The equations, which are derived from the stationary axisymmetric (SAS) Einstein-Maxwell field equations, become a σ-model. Then the theory of the σ-model is formulated in the category of formal power series by using Takasaki's formal loop group technique [T] and the linearization procedure investigated by Breitenlohner and Maison [BM]. The action of the centrally extended Hauser group $(\mathcal{G}^{(\infty)})^{\sim}$ or $(\mathcal{FH})^{\sim}$ on the potential space $(\mathcal{SP})^{\sim}$ with the conformal factor is defined with a decomposition of the formal loop group. It is expressed as the following commutative diagram for $g \in (\mathcal{FH})^{\sim}$:

$$\begin{tikzcd}
(\mathcal{FK})^{\sim}\backslash(\mathcal{FK})^{\sim}(\mathcal{SP})^{\sim} \arrow[r, "g"] \arrow[d] & (\mathcal{FK})^{\sim}\backslash(\mathcal{FK})^{\sim}(\mathcal{SP})^{\sim} \arrow[d] \\
(\mathcal{SP})^{\sim} \arrow[r] & (\mathcal{SP})^{\sim} .
\end{tikzcd}$$

In the present note, from the simplicity of our discussions, we restrict ourselves to the case in the Einstein vacuum field equations in the stationary axisymmetric space-time for couplings of the gravitational fields without any other field.

Now we derive the equations, which are our starting point, from the stationary axisymmetric Einstein field equations.

Let $ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$ be a metric on $\mathbb{R}^{1+3}$.

Then the Einstein field equations

$$R_{\mu\nu} = 0 \quad (\mu, \nu = 0, 1, 2, 3) \ ,$$

where $R_{\mu\nu}$ is the Ricci curvature.

We adopt the coordinates $(x^0, x^1, x^2, x^3) = (t, \phi, z, \rho)$ with t being time and (ϕ, z, ρ) the cylindrical coordinates of $\mathbb{R}^3$. Stationary axisymmetric space-times amount to the assumption that a metric is of the form

$$g = \begin{pmatrix} h_{00} & h_{01} & & \\ h_{10} & h_{11} & & \\ & & -\lambda & 0 \\ & & 0 & -\lambda \end{pmatrix}$$

$$\det h = -\rho^2,$$

where $\lambda > 0$, $h_{01} = h_{10}$ and $h = (h_{ij})$. The field λ is called the conformal factor.

Since we assume that the fields are stationary and axisymmetric, the functions h_{ij}'s and λ depend only on z and ρ. Furthermore, we fix the gauge as follows :

$$h|_{(z,\rho)=(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \tag{0.1}$$

Introducing the (complex) Ernst potential u constructed from h by the standard method (cf. [DO][E]), we obtain

Proposition 0.1. *The stationary axisymmetric Einstein-Maxwell field equations are equivalent to the following equation :*

$$f(d * du + \rho^{-1} d\rho \wedge *du) = du \wedge *du \tag{0.2}$$

$$\frac{\partial_z \lambda}{\lambda} = -\frac{\partial_z f}{2f} + \frac{\rho}{2f^2}(\partial_z f \partial_\rho f) - \frac{\rho}{2f^2}(\partial_\rho u - \partial_\rho f)(\partial_z u - \partial_z f) \tag{0.3}$$

$$\begin{aligned} \frac{\partial_\rho \lambda}{\lambda} = &- \frac{\partial_\rho f}{2f} + \frac{\rho}{4f^2}\{(\partial_\rho f)^2 - (\partial_z f)^2\} \\ &+ \frac{\rho}{4f^2}\{(\partial_z u - \partial_z f)^2 - (\partial_\rho u - \partial_\rho f)^2\}, \end{aligned} \tag{0.4}$$

where $f = \mathrm{Re}\, u$ *and* $*$ *is the Hodge operator given by* $*dz = d\rho, *d\rho = -dz$.

The first equation is called the Ernst equation.

Corresponding to the gauge fixing (0.1), we shall consider the solutions under the condition

$$u|_{(z,\rho)=(0,0)} = 1. \tag{0.5}$$

It is essential to introduce the function $\tau = f^{1/2}\lambda$ and we shall consider τ, in stead of λ, throughout the paper.

1. Formal Loop Groups and Central Extension

In this section we shall discuss and describe those concepts and results from the theory of finite dimensional Lie group, formal loop groups and central extension which are needed for our mathematical formulation and to fix our notations in this paper.

Let θ be Cartan involution of $G = SL(2,\mathbb{R})$ defined by $g \mapsto {}^t g^{-1}$ and K be the subgroup of G such that each element of K is fixed by θ.
Then K is a maximal subgroup of G.

We fix subgroups A and N of G as follows :

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} ; a > 0 \right\}, \qquad N = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} ; x \in \mathbb{R} \right\}.$$

Then we have $G = KAN$ (Iwasawa decomposition).

Let R be a ring of formal power series in z and ρ over $\mathbb{R}$ i.e. $R = \mathbb{R}[[z,\rho]]$, and G_R be a subgroup of $GL(2,R)$ defined by $\{g \in \mathfrak{gl}(2,R)\,;\det g = 1\}$. Then, corresponding to $G = KAN$, G_R decomposes as $G_R = K_R A_R N_R$, where K_R, A_R and N_R denote subgroups of G_R consisting of matrices with values in K, A and N respectively, each of whose components is an element of R.

Put $F_0 = R = \mathbb{R}[[z,\rho]]$ and $F_n = \rho^{|n|}R$ for a nonzero integer n. We introduce a topology in R by declaring that $\{F_n\}_{n\geq 0}$ forms a fundamental neighborhoods system of 0. Note that $F_m F_n \subset F_{m+n}$ for $m, n \geq 0$.

We define a formal loop group $\mathcal{F}\mathcal{G}_0$, following [T], by

$$\mathcal{F}\mathcal{G}_0 = \left\{ g = \sum_{n\in\mathbb{Z}} g_n t^n \in \mathcal{F}\mathfrak{gl};\ \det g = 1\ g_0(0,0) = 1 \right\}, \tag{1.1}$$

and its subgroups by

$$\mathcal{F}\mathcal{K} = \left\{ k = \sum_{n\in\mathbb{Z}} k_n t^n \in \mathcal{F}\mathcal{G}_0\,;\ \theta^{(\infty)} k = k \right\} \tag{1.2}$$

$$\mathcal{FP} = \left\{ p = \sum_{n \in \mathbb{Z}} p_n t^n \in \mathcal{FG}_0 \,;\, p_0 \in A_R N_R,\, p_n = 0 \text{ if } n < 0 \right\}, \tag{1.3}$$

where $\mathcal{F}\mathfrak{gl}$ is defined below, t is a spectral parameter and a Cartan involution $\theta^{(\infty)}$ of $\mathcal{FG}_0$ is defined by $\theta^{(\infty)}(g) = \theta(g(-1/t))$ for $g \in \mathcal{FG}_0$.

Then, using the Birkhoff decomposition ((3.17), [T]), we can decompose uniquely an element $g \in \mathcal{FG}$ as

$$g = kp \quad (k \in \mathcal{FK},\ p \in \mathcal{FP}). \tag{1.4}$$

The formal loop algebra $\mathcal{F}\mathfrak{gl}$, which is defined by

$$\mathcal{F}\mathfrak{gl} = \left\{ X = \sum_{n \in \mathbb{Z}} X_n t^n \,;\, X_n \in \mathfrak{gl}(2, F_n) \right\}, \tag{1.5}$$

becomes a Lie algebra with Lie bracket $[X, Y] = XY - YX$. The map

$$\exp : \mathcal{F}\mathfrak{gl} \longrightarrow \mathcal{F}GL \quad (\exp X = e^X = \sum_{n \geq 0} \frac{X^n}{n!}) \tag{1.6}$$

is called the *formal exponential map*.

For X, Y in $\mathcal{F}\mathfrak{gl}$, let $c_n(X, Y)$ $(n = 1, 2, \cdots)$ be the elements in $\mathcal{F}\mathfrak{gl}$ which are determined by

$$\exp vX \exp vY = \exp \sum_{n \geq 0} c_n(X, Y) v^n,$$

where v is an indeterminate and c_n's are uniquely determined by a recursion formulas (see [V]). We set $C(X, Y) = \sum_{n \geq 1} c_n(X, Y)$. Then $C(X, Y)$ is a well-defined element of $\mathcal{F}\mathfrak{gl}$ for X, Y such that $X_0, Y_0 \in \mathfrak{gl}(2, \mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal of R. Furthermore, there exists a $\mathcal{F}\mathfrak{gl}$-valued function $L(\cdot, \cdot)$ such that

$$C(X, Y) = X + Y + [X, L(X, Y)] + [Y, L(-Y, -X)].$$

Now we introduce a group 2-cocycle on $\mathcal{FG}_0$, following [BM], by use of the above defined L and a R-valued 2-cocycle ω on $\mathcal{F}\mathfrak{gl}$ given by

$$\omega(X, Y) = \mathrm{Res}_t \,\mathrm{Re}\, \mathrm{tr} X \partial_t Y$$

for $X, Y \in \mathcal{F}\mathfrak{gl}$, where Res_t indicates a formal residue with respect to t.

Note that any element $g \in \mathcal{FG}_0$ can be uniquely written as $g = e^X$ for $X \in \mathcal{F}\mathfrak{gl}$ with $X_0 \in \mathfrak{gl}(2, \mathfrak{m})$.

Definition. Let Ξ be a R-valued function on $\mathcal{FG}_0 \times \mathcal{FG}_0$ defined by

$$\Xi(e^X, e^Y) = \omega(X, L(X,Y)) + \omega(Y, L(-Y,-X)).$$

Then Ξ defines a 2-*cocycle* on $\mathcal{FG}_0$, i.e. satisfies the cocycle condition :

$$\Xi(e^X, e^Y) + \Xi(e^X e^Y, e^Z) = \Xi(e^Y, e^Z) + \Xi(e^X, e^Y e^Z) \tag{1.7}$$

for $X, Y, Z \in \mathcal{F}\mathfrak{gl}$.

Thus we define a central extension of $\mathcal{FG}_0$ in terms of the cocycle Ξ.

Definition. Let $(\mathcal{FG}_0)^\sim$ be the set given by

$$(\mathcal{FG}_0)^\sim = \{(g, e^\mu)\,;\; g \in \mathcal{FG}_0, \mu \in R\}$$

with a product of any two elements of $(\mathcal{FG}_0)^\sim$ by

$$(g_1, e^{\mu_1}) \cdot (g_2, e^{\mu_2}) = (g_1 g_2, e^{\mu_1+\mu_2+\Xi(g_1,g_2)}) \tag{1.8}$$

for $(g_1, e^{\mu_1}), (g_2, e^{\mu_2}) \in (\mathcal{FG}_0)^\sim$. Since Ξ satisfies the cocycle condition (1.7), $(\mathcal{FG}_0)^\sim$ forms a group with group multiplication given by (1.8). Namely, $(\mathcal{FG}_0)^\sim$ is a *central extension* of $\mathcal{FG}_0$.

Let $\tilde{\theta}^{(\infty)}$ be an involution of $(\mathcal{FG}_0)^\sim$ given by $\tilde{\theta}^{(\infty)}(g, e^\mu) = (\theta^{(\infty)}(g), e^{-\mu})$. If we denote by $(\mathcal{FK})^\sim$ the subgroup of $(\mathcal{FG}_0)^\sim$ consisting of elements which are fixed by $\tilde{\theta}^{(\infty)}$, then we have

$$(\mathcal{FK})^\sim = \{(k, 1) \in (\mathcal{FG}_0)^\sim\,;\; k \in \mathcal{FK}\}\,.$$

Let $(\mathcal{FP})^\sim$ be a subgroup of $(\mathcal{FG}_0)^\sim$ given by

$$(\mathcal{FP})^\sim = \{(p, e^\mu) \in (\mathcal{FG}_0)^\sim\,;\; p \in \mathcal{FP}, \mu \in R\}\,.$$

It follows immediately from the decomposition (1.4) of $\mathcal{FG}$ that $(\mathcal{FG}_0)^\sim$ has a unique decomposition :

$$(\mathcal{FG}_0)^\sim = (\mathcal{FK})^\sim \cdot (\mathcal{FP})^\sim. \tag{1.9}$$

2. Linearization and Potential Spaces

In this section we shall discuss a linearization of the Ernst equation and define potential spaces realized in formal loop groups.

First we parametrize an element of $A_R N_R$ as follows :

$$P = \begin{pmatrix} \sqrt{f} & 0 \\ \dfrac{\psi}{\sqrt{f}} & \dfrac{1}{\sqrt{f}} \end{pmatrix}, \tag{2.1}$$

where f is the same ones as in (0.2), and $\psi = \mathrm{Im}\, u$.

The following fact is well known.

Proposition 2.1. *Under the parametrization of (2.1), we put $M = \theta(P^{-1})P$. Then the Ernst equation (0.2) is equivalent to the following equation:*

$$d(\rho * dMM^{-1}) = 0. \tag{2.2}$$

Moreover the function τ is a solution of (0.3) and (0.4) if and only if it is a solution of the following equations :

$$\tau^{-1}\partial_z\tau = \frac{\rho}{4}\mathrm{tr}(\partial_z MM^{-1}\partial_\rho MM^{-1}) \tag{2.3}$$

$$\tau^{-1}\partial_\rho\tau = \frac{\rho}{8}\mathrm{tr}((\partial_\rho MM^{-1})^2 - (\partial_z MM^{-1})^2). \tag{2.4}$$

The integrability of τ follows easily from (2.3) and (2.4).

It is also known that the equation (2.2) can be rewritten as the integrability condition of a 1-form with values in $\mathfrak{g}$ each of whose component is an element of $\mathbb{R}(z,\rho) \otimes_{\mathbb{R}} \mathbb{R}[[t]]$, where $\mathbb{R}(z,\rho)$ is the quotient field of $R = \mathbb{R}[[z,\rho]]$ and t an indeterminate called "spectral parameter". Namely, let $\mathcal{A}$ and $\mathcal{I}$ be 1-forms defined by

$$\mathcal{A} = \frac{1}{2}(dPP^{-1} + \theta_R(dPP^{-1})), \quad \mathcal{I} = \frac{1}{2}(dPP^{-1} - \theta_R(dPP^{-1}))$$

for any $P \in A_R N_R$, and put

$$\Omega_P = \mathcal{A} + \left(\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}*\right)\mathcal{I},$$

where $*$ is the Hodge operator given by $*dz = d\rho$, $*d\rho = -dz$. We extend the canonical exterior derivative d on $\mathbb{C}(z,\rho)$ to that on $\mathbb{R}(z,\rho)\otimes_{\mathbb{R}}\mathbb{R}[[t]]$ by defining

$$dt = \frac{t}{(1+t^2)\rho}\left((1-t^2)d\rho + 2tdz\right). \tag{2.5}$$

Note then that $d^2 t = 0$. Now we have

Proposition 2.2. *Ω_P satisfies the integrability condition, i.e.,*

$$d\Omega_P - \Omega_P \wedge \Omega_P = 0 \tag{2.6}$$

if and only if P is a solution of (2.2).

In the view of Proposition 2.2, we can define $\mathcal{SP}$ with the gauge fixing condition (0.6), which is called *the space of potentials.*

Definition. Let $\mathcal{FP}$ be as in (1.3). We define $\mathcal{SP}$ to be a subset of $\mathcal{FP}$ consisting of elements $p = \sum_{n\geq 0} p_n t^n$ which satisfy the following conditions :

$$dp = \Omega_{p_0} \cdot p \quad \text{and} \quad p_0|_{(z,\rho)=(0,0)} = 1. \tag{2.7}$$

Note that the compatibility condition of (2.7) is just (2.2). Hence p_0 is a solution of the Ernst equation (2.2) for $p = \sum_{n\geq 0} p_n t^n \in \mathcal{SP}$.

Finally, we define $(\mathcal{SP})^{\sim}$, which is called *the space of potentials with conformal factor.*

Definition. $(\mathcal{SP})^{\sim}$ being the subset of $(\mathcal{FP})^{\sim}$ is defined by

$$(\mathcal{SP})^{\sim} = \Big\{ (p, e^{\mu}) \in (\mathcal{FP})^{\sim} ;\, p = \sum_{n\geq 0} p_n t^n \in \mathcal{SP},$$
$$\tau = e^{-\mu} \text{ satisfies (1.8) and (1.9) with } P = p_0 \Big\}. \tag{2.8}$$

3. Hauser Group and its Central Extension

In this section we define the Hauser group and its central extension, which is the trivial extension.

Define an infinite dimensional group $\mathcal{G}^{(\infty)}$, which we call Hauser group, by

$$\mathcal{G}^{(\infty)} = \left\{ g = \sum_{n\geq 0} g_n s^n \in \mathfrak{gl}(2, \mathbb{R}[[s]]) \,;\, \det g = 1, g_0 = 1 \right\},$$

where s is an indeterminate and $\mathbb{R}[[s]]$ is a ring of formal power series in s over $\mathbb{R}$.

Let j be a homomorphism of $\mathcal{G}^{(\infty)}$ into $\mathcal{FG}_0$ given by

$$j : g = \sum_{n\geq 0} g_n s^n \longmapsto j(g) = \sum_{n\geq 0} g_n \left(\rho(\frac{1}{t} - t) + 2z \right)^n .$$

Then it is easy to see that j is injective and that the image of $\mathcal{G}^{(\infty)}$ by j is in $\mathcal{FG}_0$. We denote by $\mathcal{FH}$ the image of $\mathcal{G}^{(\infty)}$ by j. The following equations characterize the elements of $\mathcal{FH}$ in $\mathcal{FG}_0$.

Lemma 3.1. *An element $g \in \mathcal{FG}_0$ belongs to $\mathcal{FH}$ if and only if g satisfies the following equations :*

$$\partial_t g = -\, \rho \left(\partial_z + \frac{1}{t} \partial_\rho \right) g \tag{3.1}$$

$$\partial_t g = -\, \frac{\rho}{2} \left(1 + \frac{1}{t^2} \right) \partial_z g. \tag{3.2}$$

This characterization will play an important role in the proof of our main theorem.

Now we define another infinite dimensional group $(\mathcal{G}^{(\infty)})^{\sim}$, which is the (trivial) central extension of the Hauser group, by

$$(\mathcal{G}^{(\infty)})^{\sim} = \mathcal{G}^{(\infty)} \times \{e^{\gamma} ; \gamma \in \mathbb{R}\}$$

with a direct product of groups. Furthermore, put

$$(\mathcal{FH})^{\sim} = \{(g, e^{\gamma}) \in (\mathcal{FG}_0)^{\sim} ; g \in \mathcal{FH}, \gamma \in \mathbb{R}\},$$

which is the image of $(\mathcal{G}^{(\infty)})^{\sim}$ by the homomorphism $j \times i$. It follows from Lemma 3.2 in [HS2] that $\mathcal{FH}$ can be regarded as a subgroup of $(\mathcal{FH})^{\sim}$ by

$$\mathcal{FH} \longrightarrow (\mathcal{FH})^{\sim}, \quad g \longmapsto (g, 1).$$

4. Infinite Dimensional Homogeneous spaces

So far we have discussed and defined the potential spaces and the Hauser group with relation to the central extension of formal loop groups. In this section we prove that the potential spaces have the structure of an infinite dimensional homogeneous space.

First we define an action of the Hauser group $\mathcal{G}^{(\infty)}$ or $\mathcal{FH}$ on the potential space $\mathcal{SP}$.

Theorem 4.1. *Let p be an element of $\mathcal{FP}$. Then p belongs to $\mathcal{SP}$ if and only if $\theta^{(\infty)}(p^{-1})p \in \mathcal{FH}$.*

Let $p \in \mathcal{SP}$ and $g \in \mathcal{G}^{(\infty)}$. By (1.4) there exist $k \in \mathcal{FK}$ and $p_g \in \mathcal{FP}$ such that

$$p \cdot j(g) = k^{-1} \cdot p_g. \tag{4.1}$$

Then, it follows immediately from Theorem 4.1 that p_g is in $\mathcal{SP}$. Thus we can define an action of the Hauser group $\mathcal{G}^{(\infty)}$ on $\mathcal{SP}$ to the right by

$$\mathcal{SP} \times \mathcal{G}^{(\infty)} \longrightarrow \mathcal{SP} \quad (p, g) \longmapsto p_g, \tag{4.2}$$

where p_g is given by (4.1).

From the fact that an element $g = \sum_{n \in \mathbb{Z}} g_n t^n \in \mathcal{FH}$ such that $\theta^{(\infty)}(g^{-1}) = g$ and g_0 is positive definite decomposes as $g = h^* h$ for some $h \in \mathcal{G}^{(\infty)}$, we have

Corollary 4.2. *The action of $\mathcal{G}^{(\infty)}$ on $\mathcal{SP}$ given by (4.2) is transitive.*

Remark. As we mentioned in [S], our group $\mathcal{G}^{(\infty)}$ is too small to obtain all solutions of the Ernst equation (2.2) through the action (4.2).

Now we proceed to the discussions of an action of the centrally extended Hauser group $(\mathcal{G}^{(\infty)})^{\sim}$ or $(\mathcal{FH})^{\sim}$ on the potential space $(\mathcal{SP})^{\sim}$.
For any $p \in \mathcal{SP}$, we can find an element $g \in \mathcal{FH}$ which sends the identity element $1 \in \mathcal{SP}$ to p by Corollary 4.2. Then we have $p = kg$ for some $k \in \mathcal{FK}$.

Proposition 4.3. *For $p = \sum_{n\geq 0} p_n t^n \in \mathcal{SP}$, let $g \in \mathcal{FH}$ and $k \in \mathcal{FK}$ be such that $p = kg$. Let τ be a solution of (2.3) and (2.4) corresponding to $P = p_0$. Then we have the following relations:*

$$\tau^{-1}\partial_z\tau = \partial_z\Xi(kg, g^{-1}) \tag{4.3}$$

$$\tau^{-1}\partial_\rho\tau = \partial_\rho\Xi(kg, g^{-1}). \tag{4.4}$$

Proposition 4.4. *For $p \in \mathcal{SP}$, let $k \in \mathcal{FK}$ and $g \in \mathcal{FH}$ be as above, i.e. $p = kg$. Then we have*

$$\Xi(\theta^{(\infty)}(p^{-1}), p) = 2\,\Xi(kg, g^{-1}). \tag{4.5}$$

Therefore, any element of $(\mathcal{SP})^{\sim}$ can be written as $(p, e^{-\frac{1}{2}\Xi(\theta^{(\infty)}(p^{-1}),p)+\gamma})$ for $p \in \mathcal{SP}, \gamma \in \mathbb{R}$.

Define an action of $(\mathcal{FH})^{\sim}$ on the space of potentials with conformal factor $(\mathcal{SP})^{\sim}$ to the right through the decomposition (1.9) :

$$(\mathcal{SP})^{\sim} \times (\mathcal{FH})^{\sim} \longrightarrow (\mathcal{SP})^{\sim}, \quad ((p, e^{\mu}), (g, e^{\gamma})) \longmapsto (p_g, e^{\alpha}). \tag{4.6}$$

Namely, we can find a unique element $(k, 1) \in (\mathcal{FK})^{\sim}$ and $(p_g, e^{\alpha}) \in (\mathcal{FP})^{\sim}$ such that

$$(p, e^{\mu})(g, e^{\gamma}) = (k, 1)^{-1}(p_g, e^{\alpha}),$$

where k and p_g are the elements given in (4.1). Since we have

$$\tilde{\theta}^{(\infty)}((p, e^{\mu})(g, e^{\gamma}))^{-1} \cdot (p, e^{\mu})(g, e^{\gamma}) = (g^* p^* p g, e^{2(\mu+\gamma)+\Xi(p^*,p)})$$

and

$$\tilde{\theta}^{(\infty)}(p_g, e^{\alpha})^{-1} \cdot (p_g, e^{\alpha}) = (p_g^* p_g, e^{2\alpha+\Xi(p_g^*,p_g)}),$$

we obtain

$$\begin{aligned}\alpha =& \mu + \gamma + \frac{1}{2}(\Xi(p^*, p) - \Xi(p_g^*, p_g)) \\ =& \gamma' - \frac{1}{2}\Xi(p_g^*, p_g)\end{aligned}$$

for some $\gamma' \in \mathbb{R}$, where we used Proposition 4.4. Thus (p_g, e^{α}) belongs to $(\mathcal{SP})^{\sim}$, i.e. the action (4.6) of $(\mathcal{FH})^{\sim}$ is well-defined.

Now we state our main theorem :

Theorem 4.5. *The group $(\mathcal{FH})^{\sim}$ acts transitively on the space of potentials with conformal factor $(\mathcal{SP})^{\sim}$ by (4.6).*

References

[BM] P. Breitenlohner and D. Maison, *On the Geroch group*, Ann. Inst. Henri Poincaré **46** (1987), 215–246.

[BMG] P.Breitenlohner, D.Maison and G.Gibbons, *4-Dimensional black holes from Kaluza-Klein theories*, Commun. Math. Phys. **120** (1988), 295–333.

[DO] H. Doi and K. Okamoto, *A survey of the generalized Geroch conjecture*, Adv. Stud. in Pure Math. **14** (1988), 379–393.

[E] F.J. Ernst, *New formulation of the axially symmetric gravitational field problem II*, Phys. Rev. **168** (1968), 1415–1417.

[G] R. Geroch, *A method for generating new solutions of Einstein's equations II*, J. Math. Phys. **13** (1972), 394–404.

[GX] M.Gürses and B.C.Xanthopoulos, *Axially symmetric, static self-dual SU(3) gauge fields and stationary Einstein-Maxwell metrics*, Phys. Rev. D **26** (1982), 1912–1915.

[HS1] T. Hashimoto and R. Sawae, *A linearization of* $S(U(1)\times U(2))\backslash SU(1,2)$ σ*-model*, Hiroshima Math. J. **22** (1992), 445–459.

[HS2] T. Hashimoto and R. Sawae, *The Conformal Factor in the SAS Einstein-Maxwell Field Equations and a Central Extension of a Formal Loop Group*, Commun. Math. Phys. (to appear).

[HE] I. Hauser and F.J. Ernst, *Proof of a generalized Geroch conjecture*, in "Galaxies axisymmetric systems and relativity edited by M.A.H. MacCallum," Cambridge Univ. Press 115, Cambridge, 1985.

[K] V.G. Kac, "Infinite dimensional Lie algebras," Birkhäuser, Boston, 1983.

[Ki] W. Kinnersley, *Symmetries of the stationary Einstein-Maxwell field equations I*, J. Math. Phys. **18** (1977), 1529–1537.

[Kn] A.W. Knapp, "Representation theory of semisimple groups — An overview based on examples," Princeton Univ. Press, Princeton, 1986.

[N] K. Nagatomo, *The Ernst equation as a motion on a universal Grassmann manifold*, Commun. Math. Phys. **122** (1989), 423–453.

[S] R. Sawae, *The conformal factor and a central extension of a formal loop group with values in* $PSL(2,\mathbb{R})$, Hiroshima Math. J. (to appear).

[T] K. Takasaki, *A new approach to the self-dual Yang-Mills equations II*, Saitama Math. J. **3** (1985), 11–40.

[V] V.S. Varadarajan, "Lie groups, Lie algebras, and their representations," Springer-Verlag, 1984.