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A Unified System of Quantum Stochastic Differential Equations

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1 Introduction

In this talk, I will present a unified system of quantum stochastic differential equations by means of a *canonical formalism* of non-equilibrium quantum systems, named Non-Equilibrium Thermo Field Dynamics (NETFD) [1]-[5]. This is a unified formalism which enables us to treat dissipative quantum systems (covering whole the aspects in non-equilibrium statistical mechanics, i.e. the *Boltzmann equation*, the *Fokker-Planck equation*, the *Langevin equation* and the *stochastic Liouville equation*[*]) by the method similar to the usual quantum mechanics and quantum field theory which accommodate the concept of the dual structure in the interpretation of nature, i.e. in terms of the *operator algebra* and the *representation space.*

The representation space of NETFD (named *thermal space*) is composed of the direct product of two Hilbert spaces, the one for *non-tilde* fields and the other for *tilde* fields. Any operator A in NETFD is accompanied by its partner (tilde) operator $\tilde{A}$. The *tilde conjugation* $\sim$ is defined by:

$$(A_1A_2)^\sim = \tilde{A}_1\tilde{A}_2, \tag{1}$$

$$(c_1A_1 + c_2A_2)^\sim = c_1^*\tilde{A}_1 + c_2^*\tilde{A}_2, \tag{2}$$

$$(\tilde{A})^\sim = A, \tag{3}$$

$$(A^\dagger)^\sim = \tilde{A}^\dagger, \tag{4}$$

where c_1 and c_2 are c-numbers. The tilde and non-tilde operators in the Schrödinger representation are mutually commutative:

$$[A,\ \tilde{B}] = 0. \tag{5}$$

The tilde and non-tilde operators are related with each other through the relation

$$\langle 1|A^\dagger = \langle 1|\tilde{A}, \tag{6}$$

[*] The stochastic Liouville equation in classical systems was introduced first by Kubo [6, 7].

where $\langle 1|$ is the thermal bra-vacuum (see (14) below).

It was revealed that *dissipation* is taken into account by a *rotation* in whole the two Hilbert spaces mixing the tilde and non-tilde operators. The terms constituted by the multiplication of tilde and non-tilde fields in the infinitesimal time-evolution generator take care of dissipative (i.e. irreversible) phenomena. This notion was discovered first when NETFD was constructed [1, 2].†

1.1 Technical Essentials

The unified framework, will be presented in the following, has been constructed essentially on a couple of well-known fundamental requirements. Namely, the general form of the time-evolution generator was derived upon the following characteristics associated with the Liouville equation

$$\frac{\partial}{\partial t}\rho(t) = -iL\rho(t). \tag{7}$$

D1. The hermiticy of the Liouville operator:

$$L^{\dagger} = L. \tag{8}$$

D2. The conservation of probability (tr $\rho = 1$):

$$\text{tr } LX = 0. \tag{9}$$

D3. The hermiticy of the density operator:

$$\rho^{\dagger}(t) = \rho(t). \tag{10}$$

Within the framework of NETFD, the above basics are termed as follows. The dynamical evolution of systems is described by the *Schrödinger equation* ($\hbar = 1$)

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}\ |0(t)\rangle. \tag{11}$$

We usually call the Schrödinger equation as Fokker-Planck equation, which is of the Schrödinger representation. Corresponding, respectively, to those of the Liouville equation listed above, we have the following characteristics of the Schrödinger equations.

B1. The *hat-Hamiltonian*, an infinitesimal time-evolution generator, $\hat{H}$ satisfies

$$\left(i\hat{H}\right)^{\sim} = i\hat{H}. \tag{12}$$

This characteristics is named *tildian.* The tildian hat-Hamiltonian is not necessarily an hermitian operator.

†This notion had not appeared in the formulation of the equilibrium thermo field dynamics (TFD) [8] which is an operator formalism of the Gibbs ensembles. There appears *no* dissipation in TFD. This is one of the essential difference between NETFD and TFD. Zubarev admired the method of NETFD, and he also started to use it for the investigation of non-equilibrium quantum phenomena [9].

B2. The hat-Hamiltonian has zero eigenvalue for the thermal bra-vacuum:

$$\langle 1|\hat{H} = 0. \tag{13}$$

This is the manifestation of the conservation of probability, i.e. $\langle 1|0(t)\rangle = 1$.

B3. The thermal vacuums $\langle 1|$ and $|0\rangle$ are *tilde invariant*:

$$\langle 1|^{\sim} = \langle 1|, \quad |0\rangle^{\sim} = |0\rangle, \tag{14}$$

and are normalized as $\langle 1|0\rangle = 1$.

It will be shown how the *semi-free* time-evolution generators both of the quantum Fokker-Planck equation and of the quantum stochastic Liouville equation for *stationary* quantum Wiener processes is built under these basic requirements. The semi-free generators are *bi-linear* and *globally gauge invariant*. With the former generator, we will demonstrate how to make a canonical theory for dissipative quantum systems. With the latter generator, we will present that a unified framework of quantum stochastic differential equations can be constructed. These generators are related with each other (see Fig. 1).

1.2 Overview of the Formulation

Leaving a detailed explanation to later sections, let us first give a perspective of NETFD. We put the corresponding equation numbers just for convenience.

In Fig. 1, we put the structure of the methods dealt in this paper. The relation between the Langevin equation and the stochastic Liouville equation is the same as the one between the Heisenberg equation and the Schrödinger equation in quantum mechanics and in quantum field theory. Since they are the stochastic differential equations, there are two types of stochastic multiplication, i.e. the Ito and the Stratonovich types. The Langevin equation (136) of the Stratonovich type has the same structure as the Heisenberg equation of motion for analytical quantities. Whereas, the Ito type (141) contains an extra term proportional to $dW(t)d\tilde{W}(t)$ due to the difference of stochastic differentiations. The random force operator $dW(t)$ is defined by (119). Although the stochastic Liouville equations both of the Ito and Stratonovich types, (112) and (113), have the same form, the latter is more convenient than the former to get the corresponding Fokker-Planck equation (125) by taking random average. It is due to the characteristics of the Ito multiplication.

The equation of motion for the dynamical variables taken both the random average and the vacuum expectation value can be obtained by two paths, i.e. the one from the Langevin equation directly by taking both the random average and the vacuum expectation, the other from the Fokker-Planck equation by taking the vacuum expectation of the operators corresponding to the dynamical variables.

With the help of the hat-Hamiltonian $\hat{H}$ for the Fokker-Planck equation (125), we can construct the Heisenberg equation for coarse grained operators. The existence of the Heisenberg equation of motion for coarse grained operators enabled us to construct the canonical formalism of the dissipative quantum fields.

It should be noted that the discovery of the stochastic Liouville equation is the key point for the construction of whole the unified quantum canonical formalism.

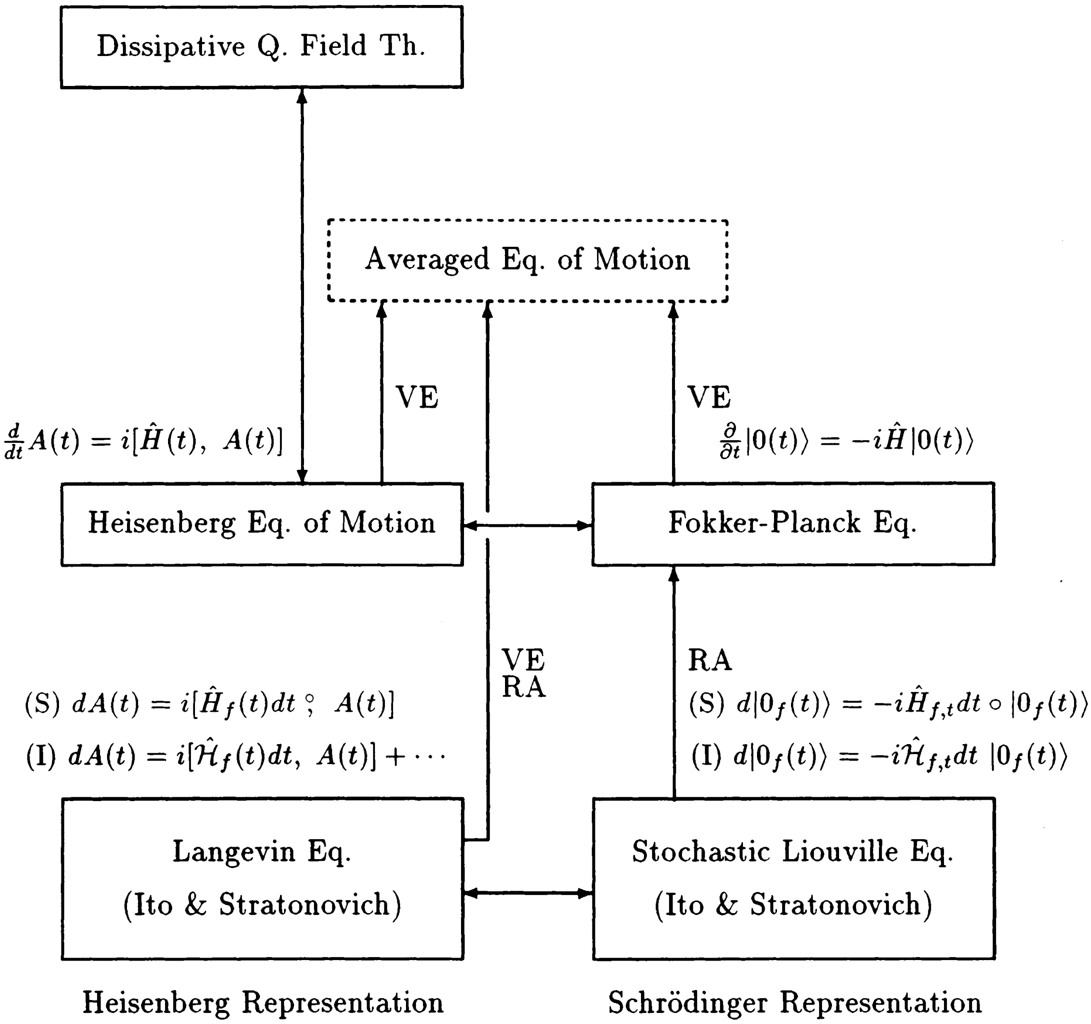

Figure 1: Structure of the Formalism. RA stands for the random average. VE stands for the vacuum expectation. (I) and (S) indicate Ito and Stratonovich types, respectively.

1.3 Questionnaire

We will see that the following basic questions associated with the quantum dissipative equations can be resolved (see [10] also for the criticisms against the quantum Langevin equations).

Q1. *Is it possible to construct a canonical formalism of quantum dissipative systems, where the time evolution is induced by the canonical transformation?*

A naive introduction of damping by putting imaginary part in the energy spectrum, i.e.

$$a(t) = a\mathrm{e}^{-(i\omega+\kappa)t}, \qquad a^\dagger(t) = a^\dagger \mathrm{e}^{(i\omega-\kappa)t}, \tag{15}$$

never preserves the equal-time commutation relation:

$$[a(t),\ a^\dagger(t)] = [a,\ a^\dagger]\ \mathrm{e}^{-2\kappa t}. \tag{16}$$

Q2. It is said that there should not be a white process in quantum systems, as it violates the KMS-condition [11, 12] for random force operators. For example, consider the Langevin equations

$$\frac{d}{dt}a(t) = -i\omega a(t) - \kappa a(t) + f(t), \tag{17}$$

$$\frac{d}{dt}a^\dagger(t) = i\omega a^\dagger(t) - \kappa a^\dagger(t) + f^\dagger(t), \tag{18}$$

where κ is a friction constant, and $a(t)$ and $a^\dagger(t)$ are, respectively, annihilation and creation *stochastic operators* in the Hilbert space which are supposed to satisfy the canonical commutation relation at $t = 0$:

$$[a(0),\ a^\dagger(0)] = 1. \tag{19}$$

The correlations of random force operators $f(t)$ and $f^\dagger(t)$ are specified by [13]-[15]

$$\langle f(t)\rangle = \langle f^\dagger(t)\rangle = 0, \tag{20}$$

$$\langle f^\dagger(t) f(s)\rangle = 2\kappa\bar{n}\delta(t-s), \tag{21}$$

$$\langle f(t) f^\dagger(s)\rangle = 2\kappa(\bar{n}+1)\delta(t-s), \tag{22}$$

with

$$\bar{n} = \frac{1}{\mathrm{e}^{\beta\omega}-1}. \tag{23}$$

Here, the symbol $\langle\cdots\rangle$ indicates a random average associated with the stochastic process specified by (20)–(22), and β is the inverse of the temperature T ($k_B = 1$) of environment. The equal-time canonical commutation relation preserves its form when one takes the random average over the stochastic process:

$$\langle [a(t),\ a^\dagger(t)]\rangle = 1. \tag{24}$$

The problem is if the correlations (21) and (22) satisfy the KMS-condition [11, 12]:

$$\int_{-\infty}^{\infty} dt\ \mathrm{e}^{-ik_0 t} \langle f^\dagger(t) f(0) \rangle = \mathrm{e}^{-\beta k_0} \int_{-\infty}^{\infty} dt\ \mathrm{e}^{-ik_0 t} \langle f(0) f^\dagger(t) \rangle. \tag{25}$$

The left hand side becomes

$$2\kappa\bar{n} = 2\kappa \frac{1}{\mathrm{e}^{\beta\omega} - 1}, \tag{26}$$

whereas the right hand side

$$2\kappa(\bar{n}+1) = 2\kappa \frac{\mathrm{e}^{\beta(\omega - k_0)}}{\mathrm{e}^{\beta\omega} - 1}. \tag{27}$$

We see that KMS-condition is not compatible with the correlations (21) and (22) unless $k_0 = \omega$ (a kind of on-mass-shell case).

However, there is the quantum master equations (or the quantum Fokker-Planck equations) derived by means of the damping theory from a microscopic point of view [16], which strongly suggests that there should be a Langevin equation induced by quantum white noise (see (28) and (41) below). *Are the quantum master equations widely used in many areas of study incorrect?*

Q3. The system described by a Langevin equation should be constituted by at least two interacting sub-systems. One for a relevant sub-system and the other for the irrelevant one associated with random force. As will be shown in the following, the stochastic Liouville equation is of the Schrödinger representation, whereas the Langevin equation is of the Heisenberg representation. The specification of the stochastic process, like (20)–(22), should be performed in the Schrödinger representation where no dynamical effect of the interaction between the two sub-systems comes out. *Can we put the same correlation of the random forces to the Langevin equation as the one in the Schrödinger representation?*

Q4. *How can we construct the representation space for stochastic processes?*

2 Quantum Statistical Mechanics

Before proceeding to an explanation of NETFD, let us have a look at the method which had been performed before NETFD was constructed.

2.1 Quantum Fokker-Planck Equation

The quantum master equation (or the quantum Fokker-Planck equation) for a damped harmonic oscillator is given by [16]

$$\frac{\partial}{\partial_t} \rho_S(t) = -i \left(H_S^\times + i\Pi \right) \rho_S(t), \tag{28}$$

with the symbol $H_S^\times X = [H_S,\ X]$, where H_S is the Hamiltonian of the system we are interested in:

$$H_S = \omega a^\dagger a, \qquad \omega = \epsilon - \mu, \tag{29}$$

with ϵ and μ being the one-particle energy and the chemical potential, respectively, and where Π is the damping operator:

$$\Pi X = \kappa \left\{ [aX,\ a^\dagger] + [a,\ Xa^\dagger] \right\} + 2\kappa\bar{n}[a,\ [X,\ a^\dagger]], \tag{30}$$

with

$$\kappa = \Re\mathrm{e}\ g^2 \int_0^\infty dt \sum_{\mathbf{k}} \left\langle [R_{\mathbf{k}}(t),\ R_{\mathbf{k}}^\dagger(0)] \right\rangle_R e^{i\omega t}, \tag{31}$$

$$\bar{n} = \frac{1}{e^{\beta\omega} - 1}. \tag{32}$$

We introduced the average, $\langle \cdots \rangle_R = \mathrm{tr}_R \cdots \rho_R$, where the density operator for a reservoir is given by $\rho_R = Z_R^{-1} e^{-\beta H_R}$, with $Z_R = \mathrm{tr}_R\ e^{-\beta H_R}$. The coupling constant g represents the strength of the interaction between the damped harmonic oscillator and the reservoir whose temperature is $T = \beta^{-1}$. The Boltzmann constant has been put to equal to 1.

We see that the one-particle distribution function, defined by

$$n(t) = \mathrm{tr}\ a^\dagger a \rho_S(t), \tag{33}$$

satisfies the Boltzmann equation

$$\frac{d}{dt} n(t) = -2\kappa \left[n(t) - \bar{n} \right], \tag{34}$$

with $\bar{n}$ being defined by (32).

The above master equation (28) can be obtained by projecting out the reservoir within the long-time limit (or the van Hove limit) by means of the damping theory [16], starting with the Liouville equation:

$$\frac{\partial}{\partial t} \rho(t) = -iH^\times \rho(t), \tag{35}$$

with the model given by the Hamiltonian

$$H = H_S + H_R + H_I. \tag{36}$$

Here, H_I is the Hamiltonian describing the interaction between the system and the reservoir:

$$H_I = g \sum_{\mathbf{k}} \left(a R_{\mathbf{k}}^\dagger + \mathrm{h.c.} \right), \tag{37}$$

with $R_{\mathbf{k}}^\dagger$ and $R_{\mathbf{k}}$ being the operators of the reservoir, and H_R is the Hamiltonian of the reservoir the explicit form of which needs not be specified to get the quantum Fokker-Planck equation (28). The coarse-grained density operator $\rho_S(t)$ in (28) is defined by

$$\rho_S(t) = \mathrm{tr}_R\ \rho(t). \tag{38}$$

2.2 Coherent State Representation

As it is a hard task to treat the quantum master equation (28), one usually maps it to a c-number partial differential equation.

Introducing the boson coherent state representation of the anti-normal ordering [17]-[19] through

$$\rho_S(t) = \int \frac{d^2z}{\pi} f_S(z,t)|z\rangle\langle z|, \tag{39}$$

with the boson coherent state $|z\rangle$, defined by

$$a|z\rangle = z|z\rangle, \tag{40}$$

we can map the master equation (28) into a partial differential equation for the c-number function $f_S(z,t)$ as [16]

$$\frac{\partial}{\partial t} f_S(z,t) = \left[-i\omega \left(\frac{\partial}{\partial z^*} z^* - \text{c.c.} \right) + \kappa \left(\frac{\partial}{\partial z^*} z^* + \text{c.c.} \right) + 2\kappa\bar{n} \frac{\partial}{\partial z^*} \frac{\partial}{\partial z} \right] f_S(z,t). \tag{41}$$

This is nothing but a Fokker-Planck equation. This form suggests that there should be a quantum white process, as mentioned before.

With the transformation

$$F(z,t) = e^{it\omega\left(\frac{\partial}{\partial z^*} z^* - \frac{\partial}{\partial z} z\right)} f_S(z,t), \tag{42}$$

the Fokker-Planck equation (41) is transformed into

$$\frac{\partial}{\partial t} F(\xi,t) = 2\kappa \left(\frac{\partial}{\partial \xi} \xi + \bar{n} \frac{\partial}{\partial \xi} \xi \frac{\partial}{\partial \xi} \right) F(\xi,t), \tag{43}$$

where $\xi = |z|^2$. Putting

$$F(\xi,t) = R(\xi/\bar{n}) e^{-2\kappa\lambda t}, \tag{44}$$

in (43) and changing the variable as $\zeta = \xi/\bar{n}$, we have an eigen-value equation for the right-hand side eigen-functions

$$\zeta R'' + (1+\zeta)R' + R = -\lambda R. \tag{45}$$

The differential equation (an eigen-value equation for the left-hand side eigen-functions) adjoint of (45) turns out to be

$$\zeta L'' + (1-\zeta)L' = -\lambda L. \tag{46}$$

Note that a further transformation

$$R(\zeta) = e^{-\zeta} f(\zeta), \tag{47}$$

in (45) gives us

$$\zeta f'' + (1-\zeta)f' = -\lambda f. \tag{48}$$

Now, we remember that the Laguerre polynomials defined by

$$\sum_{\ell=0}^{\infty} L_\ell(\zeta)x^\ell = \frac{1}{1-x}\mathrm{e}^{-\zeta\frac{x}{1-x}}, \tag{49}$$

or

$$L_\ell(\zeta) = \frac{1}{\ell!}\,\mathrm{e}^{\zeta}\,\frac{d^\ell}{d\zeta^\ell}\left(\mathrm{e}^{-\zeta}\zeta^\ell\right) = \sum_{k+0}^{\ell}(-)^k\begin{pmatrix} \ell \\ \ell-k \end{pmatrix}\frac{\zeta^k}{k!}, \tag{50}$$

satisfy the differential equation

$$\zeta L_\ell'' + (1-\zeta)L_\ell' + \ell L_\ell = 0. \tag{51}$$

We notice in comparison of (51) with (48) and (46) that the eigen-value λ should be

$$\lambda = \ell, \qquad (\ell = 0, 1, 2, \cdots), \tag{52}$$

and that the right and left eigen-functions belonging to the same eigen-value, say ℓ, are given respectively by

$$R_\ell(\zeta) = L_\ell(\zeta)\mathrm{e}^{-\zeta}, \quad \text{and} \quad L_\ell(\zeta) = L_\ell(\zeta). \tag{53}$$

These eigen-functions form an ortho-normal complete set satisfying

$$\int_0^\infty d\zeta L_\ell(\zeta)R_{\ell'}(\zeta) = \delta_{\ell,\ell'}, \tag{54}$$

$$\sum_{\ell=0}^{\infty} R_\ell(\zeta)L_\ell(\zeta') = \delta(\zeta-\zeta'). \tag{55}$$

Note that the right-hand side eigen-functions $R_\ell(\zeta)$ are of $L_2(\mathbf{R}_+)$, whereas the left-hand side eigen-functions $L_\ell(\zeta)$ are not. We may say that $R_\ell(\zeta)$ and $L_\ell(\zeta)$ belong, respectively, to the *space of test functions* and its *conjugate space* in the Gel'fand triplet (or the rigged Hilbert space).

Expanding the desired function $F(\xi, t)$ as

$$F(\zeta\bar{n}, t) = \sum_{\ell=0}^{\infty} a_\ell R_\ell(\zeta)\mathrm{e}^{-2\kappa\ell t}, \tag{56}$$

and obtaining the expressions of the coefficients a_ℓ with the help of the initial condition

$$F(\xi, 0) = f_S(\xi, 0) = \frac{1}{n}\mathrm{e}^{-\xi/n}, \tag{57}$$

we can solve (43) in the form

$$F(\xi, t) = \frac{1}{n(t)}\mathrm{e}^{-\xi/n(t)}, \tag{58}$$

with

$$n(t) = \bar{n} + (n-\bar{n})\,\mathrm{e}^{-2\kappa t}, \tag{59}$$

which satisfies the Boltzmann equation (34) with the initial condition $n(0) = n$. In deriving the expression (58), we used the generating function (49) of the Laguerre polynomials.

Substituting (58) into (42), and putting the obtained $f_S(z, t)$ into (39), we have

$$\rho_S(t) = \frac{1}{n(t)}\int \frac{d^2z}{\pi}\mathrm{e}^{-|z|^2/n(t)}|z\rangle\langle z|. \tag{60}$$

3 Non-Equilibrium Thermo Field Dynamics

Now, we will demonstrate how the formulation of quantum statistical mechanics for open systems, presented in the previous section, was raised to a canonical operator formalism.

With the general basics (12)–(14) and the information of final state, we can derive the Schrödinger equation within NETFD for a damped harmonic oscillator as [20]

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle, \tag{61}$$

with the hat-Hamiltonian

$$\begin{aligned}\hat{H} &= \omega\left(a^\dagger a - \tilde{a}^\dagger \tilde{a}\right) - i\kappa\left[(1+2\bar{n})\left(a^\dagger a + \tilde{a}^\dagger \tilde{a}\right) - 2(1+\bar{n})\,a\tilde{a} - 2\bar{n}a^\dagger \tilde{a}^\dagger\right] - i2\kappa\bar{n} \\ &= (\omega - i\kappa)\,\bar{a}^\mu a^\mu - i2\kappa\bar{a}^\mu \bar{n}^{\mu\nu} a^\nu + \omega + i\kappa,\end{aligned} \tag{62}$$

where we introduced the thermal doublet notation $a^{\mu=1} = a$, $a^{\mu=2} = \tilde{a}^\dagger$, $\bar{a}^{\mu=1} = a^\dagger$, $\bar{a}^{\mu=2} = -\tilde{a}$, and the matrix

$$\bar{n}^{\mu\nu} = \begin{pmatrix} \bar{n} & -\bar{n} \\ 1+\bar{n} & -(1+\bar{n}) \end{pmatrix}. \tag{63}$$

The initial ket-thermal vacuum, $|0\rangle = |0(0)\rangle$, is specified by

$$a|0\rangle = f\tilde{a}^\dagger|0\rangle, \tag{64}$$

where $f = n/(1+n)$ with $n = n(0)$. It was noticed first by Crawford [21] that the introduction of two kinds of operators for each operator enables us to handle the Liouville equation as the Schrödinger equation.

The operators a, $\tilde{a}^\dagger$, etc. satisfy the canonical commutation relation:

$$[a,\ a^\dagger] = 1, \qquad [\tilde{a},\ \tilde{a}^\dagger] = 1. \tag{65}$$

The tilde and non-tilde operators are mutually commutative.

Note that the Fokker-Planck equation (61) is derived first [1, 2] by mapping the master equation (28) with the help of the principle of correspondence [22, 1, 2]:

$$\rho_S(t) \longleftrightarrow |0(t)\rangle, \tag{66}$$

$$A_1\rho_S(t)A_2 \longleftrightarrow A_1\tilde{A}_2^{\dagger}|0(t)\rangle. \tag{67}$$

Let us introduce annihilation and creation operators, $\gamma^{\mu=1} = \gamma_t$, $\gamma^{\mu=2} = \tilde{\gamma}^{\natural}$ and $\bar{\gamma}^{\mu=1} = \gamma^{\natural}$, $\bar{\gamma}^{\mu=2} = -\tilde{\gamma}_t$, by

$$\gamma_t^\mu = B(t)^{\mu\nu} a^\nu, \qquad \bar{\gamma}_t^\mu = \bar{a}^\nu B^{-1}(t)^{\nu\mu}, \tag{68}$$

with the *time-dependent Bogoliubov transformation*:

$$B(t)^{\mu\nu} = \begin{pmatrix} 1+n(t) & -n(t) \\ -1 & 1 \end{pmatrix}. \tag{69}$$

They annihilate the thermal vacuums:

$$\gamma_t|0(t)\rangle = 0, \qquad \langle 1|\tilde{\gamma}^{\natural} = 0. \tag{70}$$

Since the hat-Hamiltonian (62) is expressed in terms of the annihilation and creation operators as

$$\hat{H} = \omega\left(\gamma^{\natural}\gamma_t - \tilde{\gamma}^{\natural}\tilde{\gamma}_t\right) - i\kappa\left(\gamma^{\natural}\gamma_t + \tilde{\gamma}^{\natural}\tilde{\gamma}_t + 2\left[n(t) - \bar{n}\right]\gamma^{\natural}\tilde{\gamma}^{\natural}\right), \tag{71}$$

it is easy to see that the solution of the Fokker-Planck equation (61) becomes

$$|0(t)\rangle = \exp\left[\left[n(t) - n(0)\right]\gamma^{\natural}\tilde{\gamma}^{\natural}\right]|0\rangle. \tag{72}$$

This contains the same information as the reduced density operator $\rho_S(t)$ given by (60). In deriving (72), we used the Boltzmann equation (34).

The attractive expression (72), which was obtained first in [23], led us to the notion of a mechanism named the *spontaneous creation of dissipation* [24, 25]. We can obtain the result (72) only by algebraic manipulations. This technical convenience of the operator algebra in NETFD, which is very much similar to that of the usual quantum mechanics, enables us to treat open systems in far-from-equilibrium state simpler and more transparent (see [4] for the references about applications).

The operators in the interaction representation are defined by

$$a(t) = \hat{S}^{-1}(t)a\hat{S}(t), \qquad \tilde{a}^{\dagger\dagger}(t) = \hat{S}^{-1}(t)\tilde{a}^{\dagger}\hat{S}(t), \tag{73}$$

where

$$\frac{d}{dt}\hat{S}(t) = -i\hat{H}\hat{S}(t), \qquad (i\hat{H})^{\sim} = i\hat{H}, \tag{74}$$

with $\hat{S}(0) = 1$. The semi-free operators satisfy

$$\langle 1|a^{\dagger\dagger}(t) = \langle 1|\tilde{a}(t), \qquad a(t)|0\rangle = \frac{n(t)}{1 + n(t)}\tilde{a}^{\dagger\dagger}(t)|0\rangle. \tag{75}$$

Since the semi-free hat-Hamiltonian $\hat{H}$ is not necessarily hermite, we introduced the symbol $\dagger\dagger$ in order to distinguish it from the hermite conjugation $\dagger$. However, we will use $\dagger$ instead of $\dagger\dagger$, for simplicity, unless it is confusing.

The annihilation and creation operators, $\gamma(t)^{\mu=1} = \gamma(t)$, $\gamma(t)^{\mu=2} = \tilde{\gamma}^{\natural}(t)$ and $\bar{\gamma}(t)^{\mu=1} = \gamma^{\natural}(t)$, $\bar{\gamma}(t)^{\mu=2} = -\tilde{\gamma}(t)$, are introduced by

$$\gamma(t)^{\mu} = \hat{S}^{-1}(t)\gamma^{\mu}\hat{S}(t) = B(t)^{\mu\nu}a(t)^{\nu}, \tag{76}$$

$$\bar{\gamma}(t)^{\mu} = \hat{S}^{-1}(t)\bar{\gamma}^{\mu}\hat{S}(t) = \bar{a}(t)^{\nu}B^{-1}(t)^{\nu\mu}. \tag{77}$$

They have the properties

$$\gamma(t)|0\rangle = 0, \qquad \langle 1|\tilde{\gamma}^{\natural}(t) = 0. \tag{78}$$

The two-point function $G(t,t')^{\mu\nu}$ can be derived only by algebraic manipulations as

$$G(t,t')^{\mu\nu} = -i\langle 1|T\left[a(t)^{\mu}\bar{a}(t')^{\nu}\right]|0\rangle = \left[B^{-1}(t)\mathcal{G}(t,t')B(t')\right]^{\mu\nu}, \tag{79}$$

where

$$\mathcal{G}(t,t')^{\mu\nu} = -i\langle 1|T\left[\gamma(t)^{\mu}\bar{\gamma}(t')^{\nu}\right]|0\rangle = \begin{pmatrix} G^R(t,t') & 0 \\ 0 & G^A(t,t') \end{pmatrix}, \tag{80}$$

with

$$G^R(t,t') = -i\theta(t-t')\mathrm{e}^{(-i\omega-\kappa)(t-t')}, \tag{81}$$

$$G^A(t,t') = i\theta(t'-t)\mathrm{e}^{(-i\omega+\kappa)(t-t')}. \tag{82}$$

The representation space (the *thermal space*) of NETFD is the vector space spanned by the set of bra and ket state vectors which are generated, respectively, by cyclic operations of the annihilation operators $\gamma(t)$ and $\tilde{\gamma}(t)$ on $\langle 1|$, and of the creation operators $\gamma^{\ddagger}(t)$ and $\tilde{\gamma}^{\ddagger}(t)$ on $|0\rangle$.

The normal product is defined by means of the annihilation and the creation operators, i.e. $\gamma^{\ddagger}(t)$, $\tilde{\gamma}^{\ddagger}(t)$ stand to the left of $\gamma(t)$, $\tilde{\gamma}(t)$. The process, rewriting physical operators in terms of the annihilation and creation operators, leads to a Wick-type formula, which in turn leads to Feynman-type diagrams for multi-point functions in the renormalized interaction representation. The internal line in the Feynman-type diagrams is the unperturbed two-point function (79).

The Heisenberg equation of motion for a coarse grained operator $A(t) = \hat{S}^{-1}(t)A\hat{S}(t)$ is given by

$$\frac{d}{dt}A(t) = i[\hat{H}(t),\ A(t)], \tag{83}$$

with

$$\hat{H}(t) = \hat{S}^{-1}(t)\hat{H}\hat{S}(t). \tag{84}$$

We would like to emphasize here that the existence of the Heisenberg equation of motion (83) for coarse grained operators is one of the notable features of NETFD. This enabled us to construct a *canonical formalism of the dissipative quantum field theory*, where the coarse grained operator $a(t)$ etc. in the Heisenberg representation satisfies the equal-time canonical commutation relation

$$[a(t),\ a^{\dagger}(t)] = 1, \qquad [\tilde{a}(t),\ \tilde{a}^{\dagger}(t)] = 1. \tag{85}$$

For the present model, we have

$$\frac{d}{dt}a(t) = -i\omega a(t) - \kappa\left[(1+2\bar{n})\,a(t) - 2\bar{n}\tilde{a}^{\dagger}(t)\right], \tag{86}$$

$$\frac{d}{dt}a^{\dagger}(t) = i\omega a^{\dagger}(t) + \kappa\left[(1+2\bar{n})\,a^{\dagger}(t) - 2\,(1+\bar{n})\,\tilde{a}(t)\right]. \tag{87}$$

We see that the equation of motion for $a^{\dagger}(t)$ is not the hermite conjugate of the one for $a(t)$.

The hat-Hamiltonian (62) can be also written in the form

$$\hat{H} = \omega\left(d^{\dagger}d - \tilde{d}^{\dagger}\tilde{d}\right) - i\kappa\left(d^{\dagger}d + \tilde{d}^{\dagger}\tilde{d}\right), \tag{88}$$

where $d^{\mu=1} = d$, $d^{\mu=2} = \tilde{d}^\dagger$ and $\bar{d}^{\mu=1} = d^\dagger$, $\bar{d}^{\mu=2} = -\tilde{d}$ are defined by

$$d^\mu = Q^{-1\mu\nu} a^\nu, \qquad \bar{d}^\mu = \bar{a}^\nu Q^{\nu\mu}, \tag{89}$$

with

$$Q^{\mu\nu} = \begin{pmatrix} 1 & \bar{n} \\ 1 & 1+\bar{n} \end{pmatrix}. \tag{90}$$

The initial thermal state condition (64) is expressed in terms of d and $\tilde{d}^\dagger$ as

$$d|0\rangle = (n - \bar{n})\, \tilde{d}^\dagger|0\rangle. \tag{91}$$

It is easy to see from the diagonalized form (88) of $\hat{H}$ that

$$d(t) = \hat{S}^{-1}(t)\, d\, \hat{S}(t) = d\, e^{-(i\omega+\kappa)t}, \tag{92}$$

$$\tilde{d}^{\dagger\!\dagger}(t) = \hat{S}^{-1}(t)\, \tilde{d}^\dagger\, \hat{S}(t) = \tilde{d}^\dagger\, e^{-(i\omega-\kappa)t}. \tag{93}$$

On the other hand, it is easy to see from the normal product form (71) of $\hat{H}$ that it satisfies $\langle 1|\hat{H} = 0$, since the annihilation and creation operators satisfy (70). The difference between the operators which diagonalize $\hat{H}$ and the ones which make $\hat{H}$ in the form of normal product is one of the features of NETFD, and shows the point that the formalism is quite different from usual quantum mechanics and quantum field theory. This is a manifestation of the fact that the hat-Hamiltonian is a time-evolution generator for irreversible processes.

Let us check here the irreversibility of the system. The entropy of the system is given by

$$S(t) = -\left\{ n(t)\ln n(t) - [1+n(t)]\ln[1+n(t)] \right\}, \tag{94}$$

whereas the heat change of the system is given by

$$d'Q = \omega dn. \tag{95}$$

Thermodynamics tells us that

$$dS = dS_e + dS_i, \qquad dS_e = d'Q/T, \tag{96}$$

$$dS_i \geq 0. \tag{97}$$

The latter inequality (97) is the second law of thermodynamics. Putting (94) and (95) in (96), for dS and dS_e, respectively, we have a relation for the entropy production rate [27]

$$\frac{dS_i}{dt} = \frac{dS}{dt} - \frac{dS_e}{dt} = 2\kappa\,[n(t) - \bar{n}] \ln \frac{n(t)[1+\bar{n}]}{\bar{n}[1+n(t)]} \geq 0. \tag{98}$$

It is easy to see that the expression on the right-hand side of the second equality satisfies the last inequality which is consistent with (97). The equality realizes either for the thermal equilibrium state, $n(t) = \bar{n}$, or for the quasi-stationary process, $\kappa \to 0$.

It may be worthy to note here that the entropy of the system can be defined by means of the *norm* [20]:

$$\|\ |0(t)\rangle\ \|^2 = \langle 1|0^\dagger(t)|0(t)\rangle. \tag{99}$$

It may be also worthwhile to note that the relation of the operator algebra for a harmonic oscillator within quantum mechanics to the Hermite polynomials is very much similar to the relation of the operator algebra for a damped harmonic oscillator within NETFD to the Laguerre polynomials.

4 Quantum Stochastic Liouville Equations

4.1 Basics for Stochastic Semi-Free Hamiltonian

The general form of the stochastic semi-free hat-Hamiltonian $\hat{\mathcal{H}}_{f,t}$, appeared in the stochastic Liouville equation of the Ito type, and the correlation of the random force operators can be derived under the following basic requirements [28]:

A1. The stochastic semi-free operators are defined by

$$a(t) = \hat{S}_f^{-1}(t) a \hat{S}_f(t), \qquad \tilde{a}^{\dagger\!\dagger}(t) = \hat{S}_f^{-1}(t) \tilde{a}^\dagger \hat{S}_f(t), \tag{100}$$

where

$$d\hat{S}_f(t) = -i\hat{\mathcal{H}}_{f,t} dt \; \hat{S}_f(t), \tag{101}$$

with $\hat{S}_f(0) = 1$. Here, it is assumed that, at $t = 0$, the relevant system starts to contact with the irrelevant system representing the stochastic process described by the random force operators $dF(t)$, etc. defined in **A3** below.[‡] The stochastic operators a, $a^\dagger, \tilde{a}$ and $\tilde{a}^\dagger$ in the Schrödinger representation satisfy the canonical commutation relation:

$$[a,\ a^\dagger] = 1, \qquad [\tilde{a},\ \tilde{a}^\dagger] = 1. \tag{102}$$

The stochastic semi-free operators (100) keep the equal-time canonical commutation relation:

$$[a(t),\ a^{\dagger\!\dagger}(t)] = 1, \qquad [\tilde{a}(t),\ \tilde{a}^{\dagger\!\dagger}(t)] = 1. \tag{103}$$

The tildian nature for $\hat{\mathcal{H}}_{f,t}dt$ is consistent with the definition (100) of the semi-free operators. Since the tildian hat-Hamiltonian $\hat{\mathcal{H}}_{f,t}dt$ is not necessarily hermite, we introduced the symbol $\dagger\!\dagger$ in order to distinguish it from the hermite conjugation $\dagger$. However, we will use $\dagger$ instead of $\dagger\!\dagger$, for simplicity, unless it is confusing. We use here the same notation $a(t)$ etc. for the stochastic semi-free operators as those for the coarse grained semi-free operators. We expect that there will be no confusion between them.

A2. The stochastic semi-free operators satisfy:

$$\langle 1 | a^{\dagger\!\dagger}(t) = \langle 1 | \tilde{a}(t). \tag{104}$$

A3. The random force operators $dF(t)$ etc. are of the stationary quantum Wiener process whose first and second cumulants are given by real c-numbers:

$$\left\langle dF(t) \right\rangle = \left\langle dF^\dagger(t) \right\rangle = 0, \tag{105}$$

$$\langle dF(t) dF(t) \rangle = \left\langle dF^\dagger(t) dF^\dagger(t) \right\rangle = 0, \tag{106}$$

$$\left\langle dF^\dagger(t) dF(s) \right\rangle = 2\kappa \bar{n} \delta(t-s) dt ds, \tag{107}$$

$$\left\langle dF(t) dF^\dagger(s) \right\rangle = 2\kappa \left(\bar{n}+1\right) \delta(t-s) dt ds, \tag{108}$$

[‡]Within the formalism, the random force operators $dF(t)$ and $dF^\dagger(t)$ are assumed to commute with any relevant system operator A in the Schrödinger representation: $[A,\ dF(t)] = [A,\ dF^\dagger(t)] = 0$.

where $\langle\cdots\rangle = \langle|\cdots|\rangle$ represents the random average referring to the random force operators $dF(t)$.

A4. The random force operators satisfy:

$$\langle|dF^\dagger(t) = \langle|d\tilde{F}(t). \tag{109}$$

A5. The stochastic semi-free operators and the random force operators satisfy the *orthogonality*

$$\left\langle a(t)d\mathcal{F}^\dagger(t)\right\rangle = 0, \quad \text{etc.}, \tag{110}$$

where the random force operator $d\mathcal{F}^\dagger(t)$ in the *Heisenberg* representation[§] is defined by

$$d\mathcal{F}^\dagger(t) = \hat{S}_f^{-1}(t)dF^\dagger(t)\hat{S}_f(t). \tag{111}$$

4.2 Expressions of Quantum Stochastic Liouville Equations

The quantum stochastic Liouville equation of the Ito type and the Stratonovich type are given, respectively, by

$$d|0_f(t)\rangle = -i\hat{\mathcal{H}}_{f,t}dt\ |0_f(t)\rangle, \tag{112}$$

and

$$d|0_f(t)\rangle = -i\hat{H}_{f,t}dt \circ |0_f(t)\rangle, \tag{113}$$

where the stochastic semi-free hat-Hamiltonians $\hat{\mathcal{H}}_{f,t}$ and $\hat{H}_{f,t}$ are given, respectively, by [29, 30, 4, 5]

$$\begin{aligned}
\hat{\mathcal{H}}_{f,t}dt &= \hat{H}_S dt - i\kappa\left[(a^\dagger - \tilde{a})(\mu a + \nu\tilde{a}^\dagger) + \text{t.c.}\right]dt \\
&\quad + i2\kappa(\bar{n} + \nu)(a^\dagger - \tilde{a})(\tilde{a}^\dagger - a) \\
&\quad + i\left[(a^\dagger - \tilde{a})dW(t) + \text{t.c.}\right], \qquad (114) \\
\hat{H}_{f,t}dt &= \hat{\mathcal{H}}_{f,t}dt + i(a^\dagger - \tilde{a})(\tilde{a}^\dagger - a)dW(t)d\tilde{W}(t) \qquad (115) \\
&= \hat{H}_S dt + \left[(a^\dagger - \tilde{a})\left\{id(\mu a + \nu\tilde{a}^\dagger) + [\hat{H}_S,\ \mu a + \nu\tilde{a}^\dagger]dt\right\} - \text{t.c.}\right], \qquad (116)
\end{aligned}$$

with[¶] $\hat{H}_S = H_S - \tilde{H}_S$, $H_S = \omega a^\dagger a$. The flow operators da and $d\tilde{a}^\dagger$ are specified by[‖]

$$\begin{aligned}
da &= i[\hat{H}_S,\ a]dt - \kappa[(\mu - \nu)a + 2\nu\tilde{a}^\dagger]dt + dW(t), \qquad (117) \\
d\tilde{a}^\dagger &= i[\hat{H}_S,\ \tilde{a}^\dagger]dt - \kappa[2\mu a - (\mu - \nu)\tilde{a}^\dagger]dt + dW(t). \qquad (118)
\end{aligned}$$

The symbol $\circ$ represents the Stratonovich multiplication.

[§]It can be the interaction representation when one includes non-linear terms in the hat-Hamiltonian, and performs a perturbational calculation. As we are dealing with only the bi-linear case in this talk, we call the representation as the Heisenberg one.

[¶]The following formulation is valid for the cases where H_S has non-linear terms.

[‖]The flow equations (117) and (118) read $d(\mu a + \nu\tilde{a}^\dagger) = i[\hat{H}_S,\ \mu a + \nu\tilde{a}^\dagger]dt - \kappa(\mu a + \nu\tilde{a}^\dagger)dt + dW(t)$, $d(a - \tilde{a}^\dagger) = i[\hat{H}_S,\ a - \tilde{a}^\dagger]dt + \kappa(a - \tilde{a}^\dagger)dt$.

The correlations of the random force operator

$$dW(t) = \mu dF(t) + \nu d\tilde{F}^{\dagger}(t), \quad \mu + \nu = 1, \tag{119}$$

are given by

$$\left\langle dW(t) \right\rangle = \left\langle d\tilde{W}(t) \right\rangle = 0, \tag{120}$$

$$\left\langle dW(t)dW(s) \right\rangle = \left\langle d\tilde{W}(t)d\tilde{W}(s) \right\rangle = 0, \tag{121}$$

$$\begin{aligned}\left\langle dW(t)d\tilde{W}(s) \right\rangle &= \left\langle d\tilde{W}(s)dW(t) \right\rangle \\ &= \mu \left\langle dF^{\dagger}(s)dF(t) \right\rangle + \nu \left\langle dF(t)dF^{\dagger}(s) \right\rangle \qquad (122) \\ &= 2\kappa \left(\bar{n} + \nu\right) \delta(t-s)dtds. \qquad (123)\end{aligned}$$

The general form (114) of the semi-free hat-Hamiltonian, which is bi-linear in a, $a^{\dagger}$, $dF(t)$, $dF^{\dagger}(t)$ and their tilde conjugates, and is invariant under the phase transformation $a \to ae^{i\theta}$, and $dF(t) \to dF(t)\, e^{i\theta}$, was derived with the help of the basic requirements given in the previous sub-section [4, 5]. The stochastic process here is a *stationary* Gaussian white one.

The *one-particle distribution function*

$$n(t) = \langle\!\langle 1|a^{\dagger\!\dagger}(t)a(t)|0_f\rangle\!\rangle, \tag{124}$$

satisfies the Boltzmann equation (34). The symbol $\langle\!\langle\cdots\rangle\!\rangle = \left\langle \langle 1|\cdots|0_f\rangle \right\rangle$ indicates to take both the vacuum expectation and the average with respect to the random process.

4.3 Relation to Quantum Fokker-Planck Equation

Taking the random average of the Ito stochastic Liouville equation (112) with (114), we obtain the corresponding Fokker-Planck equation (61) [1, 2, 4, 5]:

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle, \tag{125}$$

with $\hat{H}dt = \left\langle \hat{\mathcal{H}}_{f,t}dt \right\rangle$ being given by (62), and $|0(t)\rangle = \left\langle |0_f(t)\rangle \right\rangle$. Here, we used the characteristics of the Ito multiplication: $\left\langle dW(t)\hat{S}_f(t) \right\rangle = 0$, etc..

The Fokker-Planck equation (125) can be derived also by taking random average of the Stratonovich stochastic Liouville equation (113) with (116).

Note that the hat-Hamiltonian (114) of the Ito type can be expressed as

$$\begin{aligned}\hat{\mathcal{H}}_{f,t}dt &= \hat{H}_{f,t}dt + i(a^{\dagger} - \tilde{a})(\tilde{a}^{\dagger} - a)dW(t)d\tilde{W}(t) \qquad (126) \\ &= \hat{H}dt + i\left\{(a^{\dagger} - \tilde{a})dW(t) + \text{t.c.}\right\}. \qquad (127)\end{aligned}$$

4.4 An Interpretation in the Line of Accardi

We have known that the stochastic time-evolution generator $\hat{S}_f(t)$ satisfies (101):

$$d\hat{S}_f(t) = -i\hat{\mathcal{H}}_{f,t}dt\ \hat{S}_f(t), \tag{128}$$

with (127). Multiplying $\hat{S}_f^{-1}(t)$ on (128) from the right, we have

$$\hat{S}_f(t+dt,t) - 1 = \hat{H}dt + i\left\{(a^\dagger - \tilde{a})dW(t) + \text{t.c.}\right\}, \tag{129}$$

where we introduced

$$\hat{S}_f(t,s) = \hat{S}_f(t)\hat{S}_f^{-1}(s). \tag{130}$$

With the definition of the forward derivative D_+ [31], we see that

$$\begin{aligned} D_+\hat{S}_f(t) &= \lim_{\epsilon\to 0^+} E_{t]}\left(\frac{\hat{S}_f(t+\epsilon,0) - \hat{S}_f(t,0)}{\epsilon}\right) \\ &= \lim_{\epsilon\to 0^+} E_{t]}\left(\frac{\hat{S}_f(t+\epsilon,t) - 1}{\epsilon}\right)\hat{S}_f(t) \\ &= \hat{H}\hat{S}_f(t), \end{aligned} \tag{131}$$

where we used the property

$$E_{t]}\left(dW(t)\right) = 0. \tag{132}$$

The *conditional expectation* $E_{t]}$ is defined, for example, by

$$E_{t]}(X) = e_{t]}Xe_{t]} \otimes 1_{[t}, \tag{133}$$

where

$$e_{t]} = 1_{t]} = \int_{-\infty}^{t} d\tau|\tau\rangle\langle\tau|, \qquad 1_{[t} = \int_{t}^{\infty} d\tau|\tau\rangle\langle\tau|. \tag{134}$$

We are assuming that a thermal space $\mathcal{H}\otimes\mathcal{H}$ can be represented by an orthonormal complete basis $\{|\tau\rangle,\ -\infty < \tau < \infty\}$ satisfying

$$\int_{-\infty}^{\infty} d\tau|\tau\rangle\langle\tau| = 1, \qquad \langle\tau|\tau'\rangle = \delta(\tau-\tau'). \tag{135}$$

The forward time derivative of $\hat{S}_f(t)$ is derived by the infinitesimal time-evolution generator $\hat{H}$ which appeared in the Fokker-Planck equation (125).

5 Quantum Langevin Equations

5.1 Expressions of Quantum Langevin Equations

For the dynamical quantity $A(t) = \hat{S}_f^{-1}(t)A\hat{S}_f(t)$, the quantum Langevin equation of the Stratonovich type is given as the Heisenberg equation of motion [30]:

$$dA(t) = i[\hat{H}_f(t)dt \,\overset{\circ}{,}\, A(t)] \tag{136}$$

$$\begin{aligned} &= i[\hat{H}_S(t),\ A(t)]dt + \kappa\left\{[\left(a^\dagger(t) - \tilde{a}(t)\right)\left(\mu a(t) + \nu\tilde{a}^\dagger(t)\right),\ A(t)]\right. \\ &\quad \left. + [\left(\tilde{a}^\dagger(t) - a(t)\right)\left(\mu\tilde{a}(t) + \nu a^\dagger(t)\right),\ A(t)]\right\}dt \\ &\quad - \left\{[a^\dagger(t) - \tilde{a}(t),\ A(t)]\circ dW(t) + [\tilde{a}^\dagger(t) - a(t),\ A(t)]\circ d\tilde{W}(t)\right\}, \end{aligned} \tag{137}$$

where

$$\hat{H}_f(t) = \hat{S}_f^{-1}(t)\hat{H}_{f,t}\hat{S}_f(t), \qquad \hat{H}_S(t) = \hat{S}_f^{-1}(t)\hat{H}_S\hat{S}_f(t), \tag{138}$$

$$[X(t) \,\overset{\circ}{,}\, Y(t)] = X(t) \circ Y(t) - Y(t) \circ X(t), \tag{139}$$

for arbitrary operators $X(t)$ and $Y(t)$, and use has been made of the fact that

$$\hat{S}_f^{-1}(t)dW(t)\hat{S}_f(t) = dW(t). \tag{140}$$

The last property can be proven by the fact that the random force operator $dW(t)$ is commutative with $\hat{S}_f(t)$ due to the property (123), since there appear only the random force operators dW and $d\tilde{W}$ in $\hat{S}_f(t)$.

The corresponding quantum Langevin equation of the Ito type has the form

$$\begin{aligned} dA(t) &= i[\hat{\mathcal{H}}_f(t)dt,\ A(t)] + \Big\{\big(a^\dagger(t) - \tilde{a}(t)\big)\,[\tilde{a}^\dagger(t) - a(t),\ A(t)] \\ &\quad + \big(\tilde{a}^\dagger(t) - a(t)\big)\,[a^\dagger(t) - \tilde{a}(t),\ A(t)]\Big\}\,dW(t)d\tilde{W}(t) \qquad (141) \\ &= i[\hat{H}_S(t),\ A(t)]dt + \kappa\Big\{\big[\big(a^\dagger(t) - \tilde{a}(t)\big)\big(\mu a(t) + \nu\tilde{a}^\dagger(t)\big),\ A(t)\big] \\ &\quad + \big[\big(\tilde{a}^\dagger(t) - a(t)\big)\big(\mu\tilde{a}(t) + \nu a^\dagger(t)\big),\ A(t)\big]\Big\}\,dt \\ &\quad + 2\kappa\,(\bar{n} + \nu)\,[\tilde{a}^\dagger(t) - a(t),\ [a^\dagger(t) - \tilde{a}(t),\ A(t)]]dt \\ &\quad - \Big\{[a^\dagger(t) - \tilde{a}(t),\ A(t)]dW(t) + [\tilde{a}^\dagger(t) - a(t),\ A(t)]d\tilde{W}(t)\Big\}, \qquad (142) \end{aligned}$$

where $\hat{\mathcal{H}}_f(t)dt = \hat{S}_f^{-1}(t)\hat{\mathcal{H}}_{f,t}dt\hat{S}_f(t)$.

Note that, using (137), we can readily verify that

$$d[A(t)B(t)] = dA(t) \circ B(t) + A(t) \circ dB(t), \tag{143}$$

for arbitrary relevant system operators A and B [32]. This fact proves that the quantum stochastic differential equation (137) is of the Stratonovich type. On the other hand, with the help of (142), we can find the calculus rule of the Ito type

$$d[A(t)B(t)] = dA(t) \cdot B(t) + A(t) \cdot dB(t) + dA(t)dB(t), \tag{144}$$

for arbitrary relevant stochastic operators A and B [33]. This proves that the quantum stochastic differential equation (142) is in fact of the Ito type. Furthermore, since (142) is the time-evolution equation for any relevant stochastic operator $A(t)$, it is *Ito's formula* for quantum systems.

Putting a and $\tilde{a}^\dagger$ for A, we see that (137) and (142) reduce to

$$da(t) = i[\hat{H}_S(t),\ a(t)]dt - \kappa[(\mu - \nu)a(t) + 2\nu\tilde{a}^\dagger(t)]dt + dW(t), \tag{145}$$

$$d\tilde{a}^\dagger(t) = i[\hat{H}_S(t),\ \tilde{a}^\dagger(t)]dt - \kappa[2\mu a(t) - (\mu - \nu)\tilde{a}^\dagger(t)]dt + dW(t), \tag{146}$$

whose formal structures are the same as the flows (117) and (118), respectively.

In the Langevin equation approach, the dynamical behavior of systems is specified when one characterizes the correlations of random forces. The quantum Langevin equation is the equation in the Heisenberg representation, therefore the characterization of random force operators should be performed in this representation. This cannot be done in terms of $d\mathcal{F}(t)$ etc., since the information of the stochastic process is masked by the dynamics generated by $\hat{H}_f(t)$ in these operators. Whereas, the specification of the correlation between $dW(t)$, etc. directly characterizes the stochastic process thanks to the relations in (140). This solved the problem **Q3**. Note that as can be seen by (123) only the commutative set $dW(t)$ and $d\tilde{W}(t)$ takes part in the quantum Langevin equations (137) and (142). Then, **Q2** turns out to be no problem.

5.2 Averaged Equation of Motion

Applying the random force bra-vacuum $\langle|$ and the bra-vacuum $\langle 1|$ to (142) from the left, we can derive the stochastic equation of motion of Ito type for the bra-vector state $\langle\!\langle 1|A(t)$ in the form

$$
\begin{aligned}
d\langle\!\langle 1|A(t) = i\langle\!\langle 1|[H_S(t),\ A(t)]dt - \kappa\left\{\langle\!\langle 1|[A(t),\ a^\dagger(t)]a(t) + \langle\!\langle 1|a^\dagger(t)[a(t),\ A(t)]\right\} dt \\
+2\kappa\bar{n}\langle\!\langle 1|[a(t),\ [A(t),\ a^\dagger(t)]]dt \\
+\langle\!\langle 1|[A(t),\ a^\dagger(t)]dF(t) + \langle\!\langle 1|[a(t),\ A(t)]dF^\dagger(t),
\end{aligned} \tag{147}
$$

where we used the property $\langle|dW(t) = \langle|dF(t)$ and $\langle|d\tilde{W}(t) = \langle|dF^\dagger(t)$. This equation should be interpreted as the differential equation for the vector state in the conjugate space of the Gel'fand triplet. The equation of motion for the bra-vector state may be intimately related with the Langevin equation given by Gardiner and Collett [34].

Putting the random force ket-vacuum $|\rangle$ and the ket-vacuum $|0\rangle$ to (147) from the right, we obtain the equation of motion for the expectation value of an arbitrary operator $A(t)$ of the relevant system as

$$
\begin{aligned}
\frac{d}{dt}\langle\!\langle A(t)\rangle\!\rangle = i\,\langle\!\langle [H_S(t),\ A(t)]\rangle\!\rangle + \kappa\left(\langle\!\langle a^\dagger(t)[A(t),\ a(t)]\rangle\!\rangle + \langle\!\langle [a^\dagger(t),\ A(t)]a(t)\rangle\!\rangle\right) \\
+2\kappa\bar{n}\langle\!\langle [a^\dagger(t),\ [A(t),\ a(t)]]\rangle\!\rangle,
\end{aligned} \tag{148}
$$

This is the exact equation of motion for systems with linear-dissipative coupling to reservoir, which can be also derived by means of Fokker-Planck equation (125). Here, we used the property $\langle a(t)dW(t)\rangle = 0$, etc., which are the characteristics of the Ito multiplication [33]. Note that (148) was derived for general $\hat{H}_S$ including non-linear interaction terms.

6 An Interpretation of the Mori Formula

Let us consider a column vector

$$
A = \begin{pmatrix} A_1 \\ A_2 \\ \cdots \\ A_n \end{pmatrix}, \tag{149}
$$

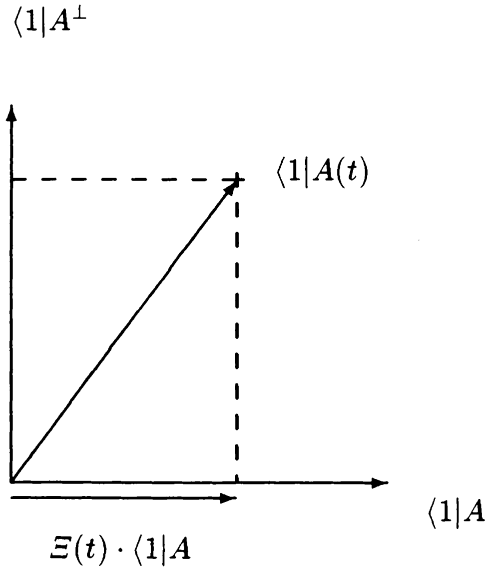

Figure 2: Schematic diagram of the thermal space decomposing the bra-vector state $\langle 1|A(t)$ into two sub-spaces $\langle 1|A$ and $\langle 1|A^{\perp}$.

of a set of operators $\{A_i\ (i=1,2,\cdots,n)\}$ corresponding to gross variables. It satisfies the Heisenberg equation within NETFD:

$$\frac{d}{dt}A(t) = i[\hat{H},\ A(t)], \tag{150}$$

where the hat-Hamiltonian annihilates the bra-vacuum:

$$\langle 1|\hat{H} = 0. \tag{151}$$

The Heisenberg equation is formally solved to give

$$A(t) = \mathrm{e}^{i\hat{H}t}A\mathrm{e}^{-i\hat{H}t}. \tag{152}$$

We will decompose the thermal space, where the bra-vector $\langle 1|A(t)$ is embedded, into two sub-spaces (see Fig. 2). One is the space spanned by $\{\langle 1|A_i\ (i=1,2,\cdots,n)\}$ and the other is the space perpendicular to it. We will denotes the latter space by $\langle 1|A^{\perp}$. We assume that the ket-vacuum $|0\rangle$ satisfies

$$\hat{H}|0\rangle = 0. \tag{153}$$

The equation of motion for the ket-vector $\langle 1|A(t)$ can be rewritten in the form [5]

$$\frac{d}{dt}\langle 1|A(t) = i\Omega\langle 1|A(t) - \int_0^{\infty} ds\Gamma(t-s)\langle 1|A(s) + \langle 1|R(t), \tag{154}$$

where

$$i\Omega = \langle 1|\dot{A}A^{\dagger}|0\rangle \cdot \langle 1|AA^{\dagger}|0\rangle^{-1}, \tag{155}$$

$$\Gamma(t) = \langle 1|R(t)R^{\dagger}(0)|0\rangle \cdot \langle 1|AA^{\dagger}|0\rangle^{-1}, \tag{156}$$

$$R(t) = \mathrm{e}^{i\hat{H}t(1-P)}F\mathrm{e}^{-i\hat{H}t(1-P)}, \tag{157}$$

$$F = \dot{A} - i\Omega A, \quad \dot{A} = i[\hat{H},\ A]. \tag{158}$$

Note that $R(t)$ satisfies

$$\langle 1|R(t)A^{\dagger}|0\rangle = 0, \tag{159}$$

which shows that the vector $\langle 1|R(t)$ belongs to the sub-space $\langle 1|A^{\perp}$.

The equation (154) may be intimately related to the Mori formula [35], and may give its reasonable interpretation. The *Langevin equation* (154) of the ket-vector states $\langle 1|A(t)$ may have a deeper meaning in the sense of the Gel'fand triplet where the bra-vector states belong to the space conjugate to the ket-vector space which is spanned only by a set of observable states generated on the ket-vacuum $|0\rangle$. The conjugate space is wider than the observable ket-vector space, therefore the sub-space $\langle 1|A^{\perp}$ can have a rich variety. This variety may take care of the jump in the re-interpretation of the Mori formula from an ordinary differential equation to a stochastic differential equation. Its detailed investigation will be published elsewhere.

With the orthogonality (159) between the vectors $\langle 1|R(t)$ and $A^{\dagger}|0\rangle$, we have the equation of motion for the correlation matrix

$$\Xi(t) = \langle 1|A(t)A^{\dagger}|0\rangle \cdot \langle 1|AA^{\dagger}|0\rangle^{-1}, \tag{160}$$

in the form

$$\frac{d}{dt}\Xi(t) = i\Omega\Xi(t) + \int_0^{\infty} ds\Gamma(t-s)\Xi(s). \tag{161}$$

We see that $i\Omega$ is related with $\Xi(t)$ by the relation

$$i\Omega = \frac{d}{dt}\Xi(t)\Big|_{t=0}. \tag{162}$$

Note that since

$$\langle 1|A(t)|0\rangle = \langle 1|A|0\rangle, \tag{163}$$

(see (151) and (153)), we can make $\Xi(t)$ a second cumulant matrix by putting $\langle 1|A|0\rangle = 0$.

7 Summary

With the unified formalism presented in this paper, the problems associated with the quantum stochastic differential equations were cleared up. The question **Q1** was unraveled as can be seen by (85). The questions **Q2** and **Q3** were solved because of the commutativity (123) (see the comment at the end of sub-section 5.1). The question **Q4** has been resolved by mathematicians [31], [36]-[38]. The notion of the representation space of the quantum stochastic differential equations may be deepened by the interpretation of the Mori formula in connection with the rigged Hilbert space.

An axiomatic reconstruction of the present formulation of quantum stochastic differential equations is possible [39] with the help of its mathematical formulation proposed by Accardi [31] as was briefly mentioned in this paper.

A generalization of the present formalism to the *non-stationary* quantum Wiener process is possible [4], including the cases of fermion and of spin systems. An application to spin systems provides us with an attractive result especially in connection with the manipulation performed by Shibata and Hashitsume [40]. The formalism can be also extended to *non-white* stochastic processes. They will be reported elsewhere.

The phase-space method within NETFD was constructed [4, 5, 41, 42] where the consistency of the unified formalism of NETFD was proven by means of the mapped c-number space. The effect of the non-linear interaction within relevant systems on the correlation of random forces is investigated in [43].

Boltzmann tried to explain the irreversibility of nature based on the microscopic and reversible Newton's mechanics. It was revealed that he had introduced a stochastic manipulation, what is called the *molecular chaos*, without knowing it in the course of the derivation of the Boltzmann equation (see [27] for a brief review of the irreversibility in statistical mechanics). Besides the technical transparency of our new method, we expect that its dual structure, as a quantum theory of dissipative fields, may provide us with a breakthrough to realize Boltzmann's original dream. The duality was not recognized in Boltzmann's days. We expect that the present unified framework of NETFD may open a new field of dissipative quantum field theory which will provide us with a deeper insight of nature.

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