Derivation property of the Lévy Laplacian

LUIGI ACCARDI
CENTRO VITO VOLterra
DIPARTIMENTO DI MATHEMATICA
UNIVERSITÀ DI ROMA TOR VERGATA
I-00133 ROMA, ITALY

NOBUAKI OBATA
DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCE
NAGOYA UNIVERSITY
NAGOYA 464-01, JAPAN

Introduction

In his book [11] P. Lévy introduced an infinite dimensional analogue of a finite dimensional Laplacian and developed an infinite dimensional potential theory, see also [12]. (For subsequent developments see e.g., [6], [7], [8], [9], [13], [15], and references cited therein.) The operator, presently called the Lévy Laplacian, is defined as the Cesàro mean of second order differential operators:

$$\Delta_L = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2},$$

where $x_1, x_2, \ldots$ constitute a coordinate system of the infinite dimensional vector space under consideration. Although the Lévy Laplacian inherits some typical properties of a finite dimensional Laplacian such as a natural relation with spherical means, it bears some pathological properties and has been discussed more or less in its own interests.

The situation is, however, changing with a recent series of works [1]-[3], [16]. The rediscovery of somehow unexpected relationship between the Lévy Laplacian and the Yang-Mills equation is opening a new approach to infinite dimensional stochastic analysis based on the Lévy Brownian motion and its quantization. (In fact, the relation was first found by Aref’eva and Volovich [4].)

The purpose of this paper is to clarify the derivation property of the Lévy Laplacian. It has been observed in a common discussion that the Lévy Laplacian behaves like a first order differential operator, i.e., a derivation. Moreover, this property is needed to characterize the Lévy Laplacian in terms of its group invariance [14]. However, as we shall show, this is typical when the Lévy Laplacian acts on functions on a Hilbert space. In this paper, employing some ideas in [10] where the Lévy Laplacian is defined as an operator acting on functions on a nuclear space, we study when the Lévy Laplacian is a derivation. As application we discuss the heat semigroup constructed in [2].
1 Lévy Laplacian on a nuclear space

Here we do not deal with a fully general nuclear space but a standard countably Hilbert nuclear space which is also known for the standard framework of white noise calculus.

Let $H$ be a real separable (infinite dimensional) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|_0 = |\cdot|$ and let $A$ be a positive selfadjoint operator in $H$ with Hilbert-Schmidt inverse. Then there exist a sequence of positive numbers

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

and a sequence of vectors $\{e_n\}_{n=1}^{\infty} \subset \text{Dom}(A)$ such that

$$Ae_n = \lambda_n e_n, \quad |e_n|_0 = 1, \quad \sum_{n=1}^{\infty} \lambda_n^{-2} = \| A^{-1} \|_{HS}^2 < \infty.$$ 

Note that $\{e_n\}_{n=1}^{\infty}$ forms a complete orthonormal system of $H$. For every $p \in \mathbb{R}$ we put

$$|\xi|_p^2 = \sum_{n=1}^{\infty} \langle \xi, e_n \rangle^2 \lambda_n^{2p} = |A^p \xi|_0^2, \quad \xi \in H.$$ 

For $p \geq 0$ the space $E_p$ of all $\xi \in H$ with $|\xi|_p < \infty$ becomes a Hilbert space with norm $|\cdot|_p$. Note that $H$ is no longer complete with respect to the norm $|\cdot|_{-p}$, $p \geq 0$. The completion $E_{-p}$ is then Hilbert space with norm $|\cdot|_{-p}$. We have thus constructed a chain of Hilbert spaces $\{E_p\}_{p \in \mathbb{R}}$ with natural inclusion relation. Since $A^{-1}$ is of Hilbert-Schmidt type, $E = \text{proj} \lim_{p \to \infty} E_p = \bigcap_{p \geq 0} E_p$ becomes a countably Hilbert nuclear space. Such a nuclear space constructed from an operator $A$ is called standard. For the strong dual space $E^*$ we have

$$E^* \cong \text{ind} \lim_{p \to \infty} E_{-p} \cong \bigcup_{p \geq 0} E_{-p}.$$ 

Thus we come to a Gelfand triple:

$$E \subset H \subset E^*.$$ 

Being compatible to the inner product of $H$, the canonical bilinear form on $E^* \times E$ is denoted by $\langle \cdot, \cdot \rangle$ again.

A function $F : E \to \mathbb{R}$ is called twice differentiable at $\xi \in E$ if there exist $F'(\xi) \in E^*$ and $F''(\xi) \in \mathcal{L}(E, E^*)$ such that

$$F(\xi + t \eta) = F(\xi) + \langle F'(\xi), \eta \rangle + \frac{1}{2} \langle F''(\xi)\eta, \eta \rangle + o(\eta), \quad \eta \in E,$$

where

$$\lim_{t \to 0} \frac{o(t \eta)}{t^2} = 0.$$ 

Let $C^2(E)$ be the space of everywhere twice differentiable functions $F : E \to \mathbb{R}$ such that both $\xi \mapsto F'(\xi) \in E^*$ and $\xi \mapsto F''(\xi) \in \mathcal{L}(E, E^*)$ are continuous. The topological isomorphisms:

$$(E \otimes E)^* \cong \mathcal{L}(E, E^*) \cong \mathcal{B}(E, E),$$
which follow from the kernel theorem, are often useful. Accordingly, we write

\[ (F''(\xi)\eta, \eta) = \langle F''(\xi), \eta \otimes \eta \rangle, \quad \eta \in E. \]

We set

\[ \mathcal{D} = \left\{ F \in C^2(E); \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle F''(\xi)e_n, e_n \rangle \text{ exists for all } \xi \in E \right\} \]

and

\[ \Delta_L F(\xi) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle F''(\xi)e_n, e_n \rangle, \quad \xi \in E, \quad F \in \mathcal{D}. \]

The operator \( \Delta_L \) is called the Lévy Laplacian on \( E \) (with respect to \( \{e_n\} \)). Note that the definition depends also on the arrangement of the complete orthonormal sequence \( \{e_n\} \).

A polynomial on \( E \) is by definition a finite linear combination of functions of the form:

\[ F(\xi) = \langle a, \xi^\otimes\nu \rangle, \quad a \in (E^\otimes\nu)^*, \quad \xi \in E. \]

The coefficient \( a \) is uniquely determined after symmetrization. Obviously, every polynomial belongs to \( C^2(E) \). In fact,

\begin{align*}
\langle F'(\xi), \eta \rangle &= \nu \langle a, \xi^{\otimes(\nu-1)} \otimes \eta \rangle = \nu \left( a \otimes_{\nu-1} \xi^{\otimes(\nu-1)}, \eta \right), \\
\langle F''(\xi), \eta \otimes \eta \rangle &= \nu(\nu - 1) \left( a, \xi^{\otimes(\nu-2)} \otimes \eta \otimes \eta \right) = \nu(\nu - 1) \left( a \otimes_{\nu-2} \xi^{\otimes(\nu-2)}, \eta \otimes \eta \right),
\end{align*}

where \( \otimes_{\nu} \) denotes the contraction of the tensor products. Hence,

\[ F'(\xi) = \nu a \otimes_{\nu-1} \xi^{\otimes(\nu-1)}, \quad F''(\xi) = \nu(\nu - 1)a \otimes_{\nu-2} \xi^{\otimes(\nu-2)}. \]

Not every polynomial belongs to \( \mathcal{D} \). In §5 we shall introduce particular classes of polynomials.

2 Derivation property

We begin with an immediate but important remark.

**Lemma 2.1** Let \( F_1, F_2 \in \mathcal{D} \). Then \( F_1F_2 \in \mathcal{D} \) if and only if the limit

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle F_1'(\xi), e_n \rangle \langle F_2'(\xi), e_n \rangle \]

exists for all \( \xi \in E \). Moreover,

\[ \Delta_L(F_1F_2) = (\Delta_L F_1)F_2 + F_1(\Delta_L F_2) \]

if and only if

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle F_1'(\xi), e_n \rangle \langle F_2'(\xi), e_n \rangle = 0, \quad \xi \in E. \]
PROOF. By definition for any $\xi, \eta \in E$,

$$\langle (F_1 F_2)'(\xi), \eta \rangle = \langle F_1'(\xi), \eta \rangle F_2(\xi) + F_1(\xi) \langle F_2'(\xi), \eta \rangle$$  \hspace{1cm} (1)$$

and

$$\langle (F_1 F_2)''(\xi), \eta \otimes \eta \rangle = \langle F_1''(\xi), \eta \otimes \eta \rangle F_2(\xi) + 2 \langle F_1'(\xi), \eta \rangle \langle F_2'(\xi), \eta \rangle + F_1(\xi) \langle F_2''(\xi), \eta \otimes \eta \rangle.$$  

Then the assertion is immediate. \hspace{1cm} qed

In particular, note that $\mathcal{D}$ is not an algebra, i.e., not closed under pointwise multiplication. Now we put

$$\mathcal{D}_0 = \left\{ F \in \mathcal{D}; \limsup_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} |\langle F'_{1}(\xi), e_j \rangle|^2 = 0 \right\}.$$

Theorem 2.2 The space $\mathcal{D}_0$ is closed under pointwise multiplication, i.e., is an algebra, on which the Lévy Laplacian acts as derivation.

PROOF. Suppose that $F_1, F_2 \in \mathcal{D}_0$. We first prove that $F_1 F_2 \in \mathcal{D}$. Observe that

$$\frac{1}{N} \left| \sum_{n=1}^{N} \langle F_1'(\xi), e_n \rangle \langle F_2'(\xi), e_n \rangle \right| \leq \frac{1}{N} \left( \sum_{n=1}^{N} |\langle F_1'(\xi), e_n \rangle|^2 \right)^{1/2} \left( \sum_{n=1}^{N} |\langle F_2'(\xi), e_n \rangle|^2 \right)^{1/2} = \left( \frac{1}{N} \sum_{n=1}^{N} |\langle F_1'(\xi), e_n \rangle|^2 \right)^{1/2} \left( \frac{1}{N} \sum_{n=1}^{N} |\langle F_2'(\xi), e_n \rangle|^2 \right)^{1/2} \to 0 \quad \text{as} \quad N \to \infty.$$

It then follows from Lemma 2.1 that $F_1 F_2 \in \mathcal{D}$. We next show that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\langle (F_1 F_2)'(\xi), e_n \rangle|^2 = 0.$$

In fact, since

$$\langle (F_1 F_2)'(\xi), e_n \rangle = \langle F_1'(\xi), e_n \rangle F_2(\xi) + F_1(\xi) \langle F_2'(\xi), e_n \rangle,$$

by Minkowskii’s inequality we obtain

$$\left( \sum_{n=1}^{N} |\langle (F_1 F_2)'(\xi), e_n \rangle|^2 \right)^{1/2} \leq \left( \sum_{n=1}^{N} |\langle F_1'(\xi), e_n \rangle F_2(\xi)|^2 \right)^{1/2} + \left( \sum_{n=1}^{N} |F_1(\xi) \langle F_2'(\xi), e_n \rangle|^2 \right)^{1/2}.$$
and therefore
\[
\left( \frac{1}{N} \sum_{n=1}^{N} | \langle (F_1 F_2)'(\xi), e_n \rangle |^2 \right)^{1/2} \\
\leq \left( \frac{1}{N} \sum_{n=1}^{N} | \langle F_1'(\xi), e_n \rangle |^2 \right)^{1/2} |F_2(\xi)| + \left( \frac{1}{N} \sum_{n=1}^{N} | \langle F_2'(\xi), e_n \rangle |^2 \right)^{1/2} |F_1(\xi)|
\]
\[\rightarrow 0 \quad \text{as} \quad N \rightarrow \infty,\]
as desired. We have thus proved that $F_1F_2 \in \mathcal{D}_0$. Finally it follows immediately from Lemma 2.1 that $\Delta_L(F_1F_2) = \Delta_L F_1 \cdot F_2 + F_1 \cdot \Delta_L F_2$, namely that the Lévy Laplacian acts on $\mathcal{D}_0$ as derivation.

Here is an immediate consequence.

Corollary 2.3 For $p \geq 0$ we put
\[
\mathcal{A}_p = \{ F \in \mathcal{D}; F'(\xi) \in E_p, \xi \in E \}.
\]
Then $\mathcal{A}_p$ is a subalgebra of $\mathcal{D}_0$. In particular, $\Delta_L$ acts on $\mathcal{A}_p$ as derivation.

PROOF. We first prove that $\mathcal{A}_p \subset \mathcal{D}_0$. Suppose $F \in \mathcal{A}_p$. Then, since $0 < \lambda_1 \leq \lambda_2 \leq \cdots$,
\[
\frac{1}{N} \sum_{n=1}^{N} | \langle F'(\xi), e_n \rangle |^2 = \frac{1}{N} \sum_{n=1}^{N} | \langle F'(\xi), e_n \rangle |^2 \lambda_n^{2p} \lambda_n^{-2p} \\
\leq \frac{1}{N} \sum_{n=1}^{N} | \langle F'(\xi), e_n \rangle |^2 \lambda_n^{2p} \lambda_1^{-2p} \\
\leq \frac{\lambda_1^{-2p}}{N} \left\| F'(\xi) \right\|_p^2 \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty.
\]
Therefore $F \in \mathcal{D}_0$. It is then straightforward to verify that $\mathcal{A}_p$ is a subalgebra of $\mathcal{D}_0$. qed

In particular, $\mathcal{A}_0$ is an algebra of functions on $E$ on which the Lévy Laplacian acts as derivation. This is the reason why the Lévy Laplacian acting on functions on a Hilbert space is a derivation (note that $E_0 = H$), see e.g., [10], [13], [14], [15].

The derivation property is also observed in a slightly different manner.

Proposition 2.4 Let $F_1, F_2 \in \mathcal{D}$ and fix $\xi \in E$. If there exists $p \geq 0$ such that
\[
|F_1'(\xi)|_p < \infty, \quad |F_2'(\xi)|_{-p} < \infty,
\]
then
\[
\Delta_L(F_1F_2)(\xi) = \Delta_L F_1(\xi) \cdot F_2(\xi) + F_1(\xi) \cdot \Delta_L F_2(\xi).
\]

PROOF. We see that
\[
\left| \frac{1}{N} \sum_{n=1}^{N} \langle F_1'(\xi), e_n \rangle \langle F_2'(\xi), e_n \rangle \right| \\
\leq \frac{1}{N} \left( \sum_{n=1}^{N} | \langle F_1'(\xi), e_n \rangle |^2 \lambda_n^{2p} \lambda_n^{-2p} \right)^{1/2} \left( \sum_{n=1}^{N} | \langle F_2'(\xi), e_n \rangle |^2 \lambda_n^{-2p} \right)^{1/2} \\
\leq \frac{1}{N} \left\| F_1'(\xi) \right\|_p \left\| F_2'(\xi) \right\|_{-p} \rightarrow 0, \quad \text{as} \quad N \rightarrow \infty.
\]
Then we need only to apply Lemma 2.1. qed
3 Lévy Laplacian on positive definite functions

There is an interesting class of functions on $E$ which are related to finite measures on $E^*$. Let $\mathcal{B}$ be the $\sigma$-field on $E^*$ generated by linear functions:

$$x \mapsto \langle x, \xi \rangle, \quad x \in E^*,$$

where $\xi$ runs over $E$. It is easily seen that $\mathcal{B}$ coincides with the topological $\sigma$-field induced from the strong dual topology of $E^*$.

Let $\mathcal{B}(E)$ be the space of all signed measures on $(E^*, \mathcal{B})$ with finite variation. Every element in $M(E^*)$ is written as $\mu_1 - \mu_2$, $\mu_1, \mu_2 \in M_+(E^*)$. If $\mu \in M(E^*)$, then its Fourier transform $\hat{\mu}$ is a function on $E$ defined by

$$\hat{\mu}(\xi) = \int_{E^*} e^{i\langle x, \xi \rangle} \mu(dx), \quad \xi \in E. \quad (2)$$

We here recall a fundamental result.

**Theorem 3.1 (Bochner-Minlos)** There is a one-to-one correspondence between $M_+(E^*)$ and the space $\mathcal{B}_+(E)$ of all continuous positive definite functions on $E$ through the Fourier transform (2).

Let $\mathcal{B}(E)$ be the space of the Fourier transform of $\mu \in M(E^*)$. Note that $M(E^*)$ is an algebra with convolution product:

$$\int_{E^*} \phi(x) \mu * \nu(dx) = \int_{E^* \times E^*} \phi(x+y) \mu(dx) \nu(dy).$$

Through the Fourier transform $\mathcal{B}(E)$ becomes an algebra with pointwise multiplication. Thus, $\mathcal{B}(E)$ becomes a closed subalgebra of $L^\infty(E)$ and therefore it is an abelian $C^*$-algebra for itself.

The support of $\mu$ is related to the continuity of $\hat{\mu}$.

**Theorem 3.2** If a positive definite function $C : E \to \mathbb{C}$ admits a continuous extension to $E_p$, $p \geq 0$, the corresponding measure $\mu$ is concentrated on $E_{-(p+q)}$ for any $q \geq 0$ such that the canonical injection $E_{p+q} \to E_p$ is of Hilbert-Schmidt type.

**Lemma 3.3** Let $F$ be the Fourier transform of $\mu \in M_+(E^*)$. If

$$\int_{E^*} |x|_p \mu(dx) < \infty \quad (3)$$

for some $p \in \mathbb{R}$, then $F''(\xi) \in E_p$ for any $\xi \in E$.

**Proof.** Since

$$|i \langle x, \eta \rangle e^{i\langle x, \xi \rangle}| \leq |x|_p |\eta|_{-p},$$

it follows from Lebesgue's convergence theorem that

$$\langle F''(\xi), \eta \rangle = \int_{E^*} i \langle x, \eta \rangle e^{i\langle x, \xi \rangle} \mu(dx), \quad \eta \in E.$$
Moreover,
\[ |\langle F'(\xi), \eta\rangle| \leq \int_{E^*} |x|_p \eta|_{-p} \mu(dx) = |\eta|_{-p} \int_{E^*} |x|_p \mu(dx), \]
which implies that \( F'(\xi) \in E_p \).

\textbf{Remark.} It follows from (3) that \( \mu(E_p) = 1 \). In fact, there exists a null set \( N \) such that \( |x|_p < \infty \) for any \( x \in E^* - N \). Hence \( E^* - N \subset E_p \) and therefore \( 1 = \mu(E^* - N) \leq \mu(E_p) \).

Note also that \( p \) in (3) can be replaced with an arbitrary smaller one.

\textbf{Example.} Let \( \mu_{\alpha} \) be the Gaussian measure with variance \( \alpha^2 \). Then
\[ F(\xi) = \hat{\mu}(\xi) = \exp\left(-\frac{\alpha^2}{2} |\xi|_0^2\right), \quad \xi \in E. \]

By a direct calculation we obtain
\[ F'(\xi) = -\alpha^2 e^{-\alpha^2 |\xi|_0^2/2} \xi = -\alpha^2 F(\xi) \xi, \]
and therefore \( F'(\xi) \in E = \bigcap_{p \geq 0} E_p \). Consequently, \( F = \hat{\mu}_{\alpha} \in \mathcal{A}_{p} \) for any \( p \geq 0 \).

\section{Cauchy problem and semigroup}

We recapitulate some results obtained in [2]. For the fixed complete orthonormal basis \( \{e_n\}_{n=1}^{\infty} \) of \( H \), which are in fact contained in \( E \), let \( S \) denote the shift with respect to the basis \( \{e_n\} \), i.e., the unique linear continuous (in fact isometric) map from \( H \) to \( H \) such that
\[ Se_n = e_{n+1}, \quad n = 1, 2, \ldots. \]

We note the following

\textbf{Lemma 4.1} \( S \in \mathcal{L}(E, E) \) if and only if
\[ \sup_{n \geq 0} \frac{\lambda_{n+1}}{\lambda_n^{1+r}} < \infty \]
for some \( r \geq 0 \).

\textbf{Proof.} Suppose first that \( S \in \mathcal{L}(E, E) \). Take an arbitrary \( p > 0 \). Then there exist \( q \geq 0 \) and \( C \geq 0 \) such that
\[ |S\xi|_p \leq C |\xi|_{p+q}, \quad \xi \in E. \]
In particular, putting \( \xi = e_n \) we have
\[ |e_{n+1}|_p = |Se_n|_p \leq C |e_n|_{p+q}. \]
Hence
\[ \lambda_{n+1}^p \leq C \lambda_n^{p+q}, \quad n = 1, 2, \ldots, \]
and
\[ \sup_{n \geq 1} \frac{\lambda_{n+1}}{\lambda_n^{1+q/p}} \leq C^{1/p} < \infty, \]
as desired. Conversely, we assume that there exists \( r \geq 0 \) with

\[
M = \sup_{n \geq 0} \frac{\lambda_{n+1}}{\lambda_{n}^{1+r}} < \infty.
\]

Consider an element \( \xi \in E \) which admits an expansion:

\[
\xi = \sum_{n=1}^{\infty} c_{n}e_{n},
\]

where \( c_{n} = 0 \) except finitely many \( n \). Then by definition,

\[
S\xi = \sum_{n=1}^{\infty} c_{n}Se_{n} = \sum_{n=1}^{\infty} c_{n}e_{n+1}.
\]

For any \( p \geq 0 \) we have

\[
|S\xi|_{p}^{2} = \sum_{n=1}^{\infty} |c_{n}|^{2}|e_{n+1}|_{p}^{2} = \sum_{n=1}^{\infty} |c_{n}|^{2}\lambda_{n+1}^{2p} \leq M^{2} \sum_{n=1}^{\infty} |c_{n}|^{2}\lambda_{n}^{2p(1+r)} = M^{2} |\xi|_{p(1+r)}^{2}.
\]

This implies that \( S \) is a continuous operator on \( E \).

qed

From now on we assume that \( S \in \mathcal{L}(E, E) \). Then the adjoint \( S^{*} \in \mathcal{L}(E^{*}, E^{*}) \) becomes a measurable map from \( E^{*} \) into \( E^{*} \). Let \( M_{S}(E^{*}) \subset M(E^{*}) \) be the space of measures on \( E^{*} \) which are invariant under \( S^{*} \). We put

\[
M_{S}^{2}(E^{*}) = \{ \mu \in \Lambda I_{S}(E^{*}) ; \int_{E^{*}} |\langle x, \eta \rangle|^{2}\mu(dx) < \infty \text{ for all } \eta \in E \}.
\]

Let \( \mathcal{H} \) be the subspace of all \( x \in E^{*} \) such that the limit

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\langle x, e_{n} \rangle|^{2} < \infty
\]

exists. Then,

\[
\| x \| = \lim_{N \to \infty} \left( \frac{1}{N} \sum_{n=1}^{N} |\langle x, e_{n} \rangle|^{2} \right)^{1/2}, \quad x \in \mathcal{H}.
\]

becomes a seminorm of \( \mathcal{H} \).

**Lemma 4.2** Let \( \mu \in M_{S}^{2} \). Then \( x \in \mathcal{H} \) for \( \mu \)-a.e. \( x \in E^{*} \). In other words, the limit

\[
\| x \|^{2} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\langle x, e_{n} \rangle|^{2} < \infty
\]

exists for \( \mu \)-a.e. \( x \in E^{*} \). Moreover, the limit converges in \( L^{1}(E^{*}, \mu) \).

**Proof.** For simplicity we put

\[
F(x) = |\langle x, e_{1} \rangle|^{2}.
\]
Then, clearly $F \in L^1(E^*, \mu)$. Since $S^*$ is a $\mu$-preserving measurable map from $E^*$ into itself, it follows from the ergodic theorem (e.g., [5, Chap.VIII]) that

$$F^*(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} F(S^{*(n-1)}x)$$

converges $\mu$-a.e. $x \in E^*$ as well as in the $L^1$-sense. In that case $F^* \in L^1(E^*, \mu)$. On the other hand, since

$$\sum_{n=1}^{N} F(S^{*(n-1)}x) = \sum_{n=1}^{N} \langle S^{*(n-1)}x, e_1 \rangle^2 = \sum_{n=1}^{N} \langle x, S^{(n-1)}e_1 \rangle^2 = \sum_{n=1}^{N} \langle x, e_n \rangle^2,$$

we see that $F^*(x) = \|x\|^2$. The assertion then follows immediately.

In a similar manner,

**Lemma 4.3** Let $\mu, \nu \in M^2_S(E^*)$. Then the limit

$$\langle\langle x, y \rangle\rangle = \frac{1}{N} \sum_{n=1}^{N} \langle x, e_n \rangle \langle y, e_n \rangle$$

exists for $\mu \times \nu$-a.e. $(x, y) \in E^* \times E^*$.

**Proposition 4.4** If $\mu \in M^2_S(E^*)$, then $F = \hat{\mu} \in \mathcal{D}$ and

$$\Delta L F(\xi) = - \int_{E^*} \|x\|^2 e^{ix, \xi} \mu(dx).$$

**PROOF.** It is easily verified from definition that

$$\langle F''(\xi), e_n \otimes e_n \rangle = - \int_{E^*} \langle x, e_n \rangle^2 e^{ix, \xi} \mu(dx).$$

Then we need only to apply Lemma 4.2.

Consider the Cauchy problem for the Laplace equation:

$$\frac{\partial}{\partial t} F(\xi, t) = \Delta L F(\xi, t), \quad F(\xi, 0) = F_0(\xi),$$

(4)

where $F_0$ is a certain function on $E$. For some particular initial condition the Cauchy problem is solved satisfactorily in Accardi-Roselli-Smolyanov [2].

**Theorem 4.5** Let $\mu \in M^2_S(E^*)$ and put $F_0 = \hat{\mu}$. Then the solution of the Cauchy problem (4) is given as

$$F(\xi, t) = \overline{\mu_t}(\xi), \quad \mu_t(dx) = e^{-\|x\|^2} \mu(dx), \quad t \geq 0.$$

**PROOF.** By Lemma 4.2 $\mu_t$ is well defined and belongs to $M_+(E^*)$. Moreover, obviously $\mu_t$ is $S^*$-invariant and

$$\int_{E^*} \langle x, \eta \rangle^2 \mu_t(dx) \leq \int_{E^*} \langle x, \eta \rangle^2 \mu(dx) < \infty,$$
namely, $\mu_t \in M^2_\mathcal{S}(E^*)$. It then follows from Proposition 4.4 that $\hat{\mu}_t \in \mathcal{D}$ and

$$\Delta_L F(\xi, t) = -\int_{E^*} \|x\|^2 e^{-t\|x\|^2} e^{i(x, \xi)} \mu(dx).$$

On the other hand, since $\|x\|^2$ belongs to $L^1(E^*, \mu)$ by Lemma 4.2, we see by Lebesgue's theorem that

$$\frac{\partial F}{\partial t} = -\int_{E^*} \|x\|^2 e^{-t\|x\|^2} e^{i(x, \xi)} \mu(dx).$$

Therefore $F(\xi, t) = \hat{\mu}_t(\xi)$ is a solution of the Cauchy problem under consideration. qed

We put

$$(\hat{P}t\mu)(dx) = e^{-t\|x\|^2} \mu(dx), \quad \mu \in M^2_\mathcal{S}(E^*), \quad t \geq 0.$$  

Then $\hat{P}t$ constitutes a one-parameter semigroup of transformations on $M^2_\mathcal{S}(E^*)$.

Let $\mathcal{B}^2_\mathcal{S}(E)$ be the image space of $M^2_\mathcal{S}(E^*)$ under the Fourier transform. The induced one-parameter semigroup of transformations on $\mathcal{B}^2_\mathcal{S}(E)$ is denoted by $\hat{P}t$. This is called the heat semigroup of the Lévy Laplacian $\Delta_L$.

We note the following

**Proposition 4.6** The subspace $M^2_\mathcal{S}(E^*)$ is closed under convolution. Therefore $\mathcal{B}^2_\mathcal{S}(E)$ is closed pointwise multiplication.

However, the Lévy Laplacian is not a derivation on $\mathcal{B}^2_\mathcal{S}(E)$ and $\hat{P}t$ is not multiplicative; namely,

$$\hat{P}t(\mu * \nu) = \hat{P}t\mu * \hat{P}t\nu$$

does not holds in general. In fact, $\hat{\mu}$ belongs to $\mathcal{D}$ but not to $\mathcal{D}_0$ on which the Lévy Laplacian acts as derivation, see Theorem 2.2.

## 5 Normal polynomials

In this section we introduce particular classes of polynomials under an additional structure of $E$, namely, multiplication. We assume that $E$ is equipped with a multiplication which makes $E$ a commutative algebra. Furthermore we assume that the multiplication is continuous (since $E$ is a Fréchet space, there is no difference between joint and separate continuity) and that

$$\langle \xi\eta, \zeta \rangle = \langle \xi, \eta\zeta \rangle, \quad \xi, \eta, \zeta \in E.$$  

This situation often occurs when $E$ is a function space (the multiplication above is the usual pointwise multiplication of functions). By duality multiplication of $f \in E^*$ and $\xi \in E$, denoted by $f\xi = \xi f$, is defined as a unique element in $E^*$ such that

$$\langle f\xi, \eta \rangle = \langle f, \xi\eta \rangle, \quad \eta \in E.$$  

Obviously, the multiplicication $E^* \times E \to E^*$ is an extension of $E \times E \to E$.

Consider a quadratic function $\xi \mapsto \langle f, \xi^2 \rangle$, where $f \in E^*$ is fixed. Since $\langle \xi, \eta \rangle \mapsto \langle f, \xi\eta \rangle$ is a continuous bilinear form on $E \times E$, there exists $g \in (E \otimes E)^*$ such that $\langle f, \xi\eta \rangle = \langle g, \xi \otimes \eta \rangle$. Thus, $\langle f, \xi^2 \rangle = \langle g, \xi^{\otimes 2} \rangle$ and there occurs no new quadratic function in this manner. On the contrary, using the new product in $E$ we may introduce a subclass of polynomials. Namely,
if \( f \in E^* \) is "regular," the corresponding quadratic functions constitute a certain class of quadratic functions. This is immediately generalized to polynomials of any degree. Thus, a normal polynomial on \( E \) is a finite linear combination of functions of the form:

\[
(f, \xi^{\nu_1} \otimes \cdots \otimes \xi^{\nu_n}), \quad \nu_1, \ldots, \nu_n = 0, 1, 2, \ldots,
\]

where \( f \in (E^\otimes n)^* \) is a regular element. Here the tensor product and the multiplication of \( E \) should be carefully distinguished.

We now go into a typical situation. Consider a one dimensional torus \( T = \mathbb{R}/\mathbb{Z} \). Put \( H = L^2(T) \) and consider \( d/dt \). Then \( E = C^\infty(T) \) and \( \{e_n\} \) consists of trigonometric functions. In that case \( \{e_n\} \) possesses additional properties: first \( \{e_n\} \) is uniformly bounded:

\[
\sup_n \sup_{t \in T} |e_n(t)| < \infty;
\]

Second, it is equally dense, i.e.,

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \int_{0}^{1} f(t)e_n(t)^2 dt = \int_{0}^{1} f(t) dt, \quad f \in L^\infty(T).
\]

Moreover, the pointwise multiplication gives a continuous bilinear map from \( E \times E \) into \( E \). We say that \( f \in (E^\otimes n)^* \) is regular if \( f \in L^1(T^n) \). This is the usual definition of a regular distribution. Then we have the space of normal polynomials. In other words, a normal polynomial on \( E \) is by definition a linear combination of functions of the form:

\[
F(\xi) = \int_{T^n} k(t_1, \cdots, t_n)\xi(t_1)^{\nu_1} \cdots \xi(t_n)^{\nu_n} dt_1 \cdots dt_n, \quad \xi \in E,
\]

where \( k \) is an integrable function on \( T^n \). If \( \nu_i = 1 \) for all \( i \), the polynomial is called regular after Lévy's original definition.

**Lemma 5.1** Consider a normal polynomial of the form:

\[
F(\xi) = (f, \xi^\nu), \quad f \in E^*.
\]

Then \( F \in D_0 \) if and only if

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |(f\xi^{\nu-1}, e_n)|^2 = 0
\]

for any \( \xi \in E \).

The proof is immediate. Then we come to the following

**Proposition 5.2** Every normal polynomial belongs to \( D_0 \).

The above result generalizes the known fact that the Lévy Laplacian is a derivation on normal polynomials, see [10, Proposition 3.2].
References


