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Derivation property of the Lévy Laplacian

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Introduction

In his book [11] P. Lévy introduced an infinite dimensional analogue of a finite dimensional Laplacian and developed an infinite dimensional potential theory, see also [12]. (For subsequent developments see e.g., [6], [7], [8], [9], [13], [15], and references cited therein.) The operator, presently called the Lévy Laplacian, is defined as the Cesàro mean of second order differential operators:

$$\Delta_L = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2},$$

where $x_1, x_2, \ldots$ constitute a coordinate system of the infinite dimensional vector space under consideration. Although the Lévy Laplacian inherits some typical properties of a finite dimensional Laplacian such as a natural relation with spherical means, it bears some pathological properties and has been discussed more or less in its own interests.

The situation is, however, changing with a recent series of works [1]-[3], [16]. The rediscovery of somehow unexpected relationship between the Lévy Laplacian and the Yang-Mills equation is opening a new approach to infinite dimensional stochastic analysis based on the Lévy Brownian motion and its quantization. (In fact, the relation was first found by Aref'eva and Volovich [4].)

The purpose of this paper is to clarify the derivation property of the Lévy Laplacian. It has been observed in a common discussion that the Lévy Laplacian behaves like a first order differential operator, i.e., a derivation. Moreover, this property is needed to characterize the Lévy Laplacian in terms of its group invariance [14]. However, as we shall show, this is typical when the Lévy Laplacian acts on functions on a Hilbert space. In this paper, employing some ideas in [10] where the Lévy Laplacian is defined as an operator acting on functions on a nuclear space, we study when the Lévy Laplacian is a derivation. As application we discuss the heat semigroup constructed in [2].
1 Lévy Laplacian on a nuclear space

Here we do not deal with a fully general nuclear space but a standard countably Hilbert nuclear space which is also known for the standard framework of white noise calculus.

Let $H$ be a real separable (infinite dimensional) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $| \cdot |_0 = | \cdot |$ and let $A$ be a positive selfadjoint operator in $H$ with Hilbert-Schmidt inverse. Then there exist a sequence of positive numbers

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

and a sequence of vectors $\{e_n\}_{n=1}^{\infty} \subset \text{Dom}(A)$ such that

$$Ae_n = \lambda_n e_n, \quad |e_n|_0 = 1, \quad \sum_{n=1}^{\infty} \lambda_n^{-2} = \|A^{-1}\|_{HS}^2 < \infty.$$ 

Note that $\{e_n\}_{n=1}^{\infty}$ forms a complete orthonormal system of $H$. For every $p \in \mathbb{R}$ we put

$$|\xi|^2_p = \sum_{n=1}^{\infty} \langle \xi, e_n \rangle^2 \lambda_n^{2p} = |A^p \xi|^2_0, \quad \xi \in H.$$ 

For $p \geq 0$ the space $E_p$ of all $\xi \in H$ with $|\xi|_p < \infty$ becomes a Hilbert space with norm $| \cdot |_p$. Note that $H$ is no longer complete with respect to the norm $| \cdot |_{-p}$, $p \geq 0$. The completion $E_{-p}$ is then Hilbert space with norm $| \cdot |_{-p}$. We have thus constructed a chain of Hilbert spaces $\{E_p\}_{p \in \mathbb{R}}$ with natural inclusion relation. Since $A^{-1}$ is of Hilbert-Schmidt type,

$$E = \text{proj lim}_{p \to \infty} E_p = \bigcap_{p \geq 0} E_p$$

becomes a countably Hilbert nuclear space. Such a nuclear space constructed from an operator $A$ is called standard. For the strong dual space $E^*$ we have

$$E^* \cong \text{ind lim}_{p \to \infty} E_{-p} \cong \bigcup_{p \geq 0} E_{-p}.$$

Thus we come to a Gelfand triple:

$$E \subset H \subset E^*.$$ 

Being compatible to the inner product of $H$, the canonical bilinear form on $E^* \times E$ is denoted by $\langle \cdot, \cdot \rangle$ again.

A function $F : E \to \mathbb{R}$ is called twice differentiable at $\xi \in E$ if there exist $F'(\xi) \in E^*$ and $F''(\xi) \in \mathcal{L}(E, E^*)$ such that

$$F(\xi + \eta) = F(\xi) + \langle F'(\xi), \eta \rangle + \frac{1}{2} \langle F''(\xi)\eta, \eta \rangle + o(\eta), \quad \eta \in E,$$

where

$$\lim_{t \to 0} \frac{o(t\eta)}{t^2} = 0.$$ 

Let $C^2(E)$ be the space of everywhere twice differentiable functions $F : E \to \mathbb{R}$ such that both $\xi \mapsto F'(\xi) \in E^*$ and $\xi \mapsto F''(\xi) \in \mathcal{L}(E, E^*)$ are continuous. The topological isomorphisms:

$$(E \otimes E)^* \cong \mathcal{L}(E, E^*) \cong \mathcal{B}(E, E),$$
which follow from the kernel theorem, are often useful. Accordingly, we write

\[ (F''(\xi)\eta, \eta) = \langle F''(\xi), \eta \otimes \eta \rangle, \quad \eta \in E. \]

We set

\[ \mathcal{D} = \left\{ F \in C^2(E); \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle F''(\xi)e_n, e_n \rangle \text{ exists for all } \xi \in E \right\} \]

and

\[ \Delta_L F(\xi) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle F''(\xi)e_n, e_n \rangle, \quad \xi \in E, \quad F \in \mathcal{D}. \]

The operator \( \Delta_L \) is called the Lévy Laplacian on \( E \) (with respect to \( \{e_n\} \)). Note that the definition depends also on the arrangement of the complete orthonormal sequence \( \{e_n\} \).

A polynomial on \( E \) is by definition a finite linear combination of functions of the form:

\[ F(\xi) = \langle a, \xi^\otimes \nu \rangle, \quad a \in (E^\otimes \nu)^*, \quad \xi \in E. \]

The coefficient \( a \) is uniquely determined after symmetrization. Obviously, every polynomial belongs to \( C^2(E) \). In fact,

\[ \langle F'(\xi), \eta \rangle = \nu \langle a, \xi^\otimes(\nu-1) \otimes \eta \rangle = \nu \langle a \otimes_{\nu-1} \xi^\otimes(\nu-1), \eta \rangle, \]

\[ \langle F''(\xi), \eta \otimes \eta \rangle = \nu(\nu - 1) \langle a, \xi^\otimes(\nu-2) \otimes \eta \otimes \eta \rangle = \nu(\nu - 1) \langle a \otimes_{\nu-2} \xi^\otimes(\nu-2), \eta \otimes \eta \rangle, \]

where \( \otimes_\nu \) denotes the contraction of the tensor products. Hence,

\[ F'(\xi) = \nu a \otimes_{\nu-1} \xi^\otimes(\nu-1), \quad F''(\xi) = \nu(\nu - 1) a \otimes_{\nu-2} \xi^\otimes(\nu-2). \]

Not every polynomial belongs to \( \mathcal{D} \). In §5 we shall introduce particular classes of polynomials.

### 2 Derivation property

We begin with an immediate but important remark.

**Lemma 2.1** Let \( F_1, F_2 \in \mathcal{D} \). Then \( F_1F_2 \in \mathcal{D} \) if and only if the limit

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle F'_1(\xi), e_n \rangle \langle F'_2(\xi), e_n \rangle \]

exists for all \( \xi \in E \). Moreover,

\[ \Delta_L(F_1F_2) = (\Delta_L F_1)F_2 + F_1(\Delta_L F_2) \]

if and only if

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle F'_1(\xi), e_n \rangle \langle F'_2(\xi), e_n \rangle = 0, \quad \xi \in E. \]
**Proof.** By definition for any \( \xi, \eta \in E \),
\[
\langle (F_1 F_2)'(\xi), \eta \rangle = \langle F_1'(\xi), \eta \rangle F_2(\xi) + F_1(\xi) \langle F_2'(\xi), \eta \rangle
\]  
(1)
and
\[
\langle (F_1 F_2)''(\xi), \eta \otimes \eta \rangle = \langle F_1''(\xi), \eta \otimes \eta \rangle F_2(\xi) + 2 \langle F_1'(\xi), \eta \rangle \langle F_2'(\xi), \eta \rangle + F_1(\xi) \langle F_2''(\xi), \eta \otimes \eta \rangle.
\]

Then the assertion is immediate. \( \quad \text{qed} \)

In particular, note that \( \mathcal{D} \) is not an algebra, i.e., not closed under pointwise multiplication. Now we put
\[
\mathcal{D}_0 = \left\{ F \in \mathcal{D}; \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\langle (F_1 F_2)'(\xi), e_n \rangle|^2 = 0 \right\}.
\]

**Theorem 2.2** The space \( \mathcal{D}_0 \) is closed under pointwise multiplication, i.e., is an algebra, on which the Lévy Laplacian acts as derivation.

**Proof.** Suppose that \( F_1, F_2 \in \mathcal{D}_0 \). We first prove that \( F_1 F_2 \in \mathcal{D} \). Observe that
\[
\frac{1}{N} \left| \sum_{n=1}^{N} \langle F_1'(\xi), e_n \rangle \langle F_2'(\xi), e_n \rangle \right|
\leq \frac{1}{N} \left( \sum_{n=1}^{N} |\langle F_1'(\xi), e_n \rangle|^2 \right)^{1/2} \left( \sum_{n=1}^{N} |\langle F_2'(\xi), e_n \rangle|^2 \right)^{1/2}
= \left( \frac{1}{N} \sum_{n=1}^{N} |\langle F_1'(\xi), e_n \rangle|^2 \right)^{1/2} \left( \frac{1}{N} \sum_{n=1}^{N} |\langle F_2'(\xi), e_n \rangle|^2 \right)^{1/2}
\longrightarrow 0 \quad \text{as} \quad N \to \infty.
\]

It then follows from Lemma 2.1 that \( F_1 F_2 \in \mathcal{D} \). We next show that
\[
\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\langle (F_1 F_2)'(\xi), e_n \rangle|^2 = 0.
\]

In fact, since
\[
\langle (F_1 F_2)'(\xi), e_n \rangle = \langle F_1'(\xi), e_n \rangle F_2(\xi) + F_1(\xi) \langle F_2'(\xi), e_n \rangle,
\]
by Minkowskii's inequality we obtain
\[
\left( \sum_{n=1}^{N} |\langle (F_1 F_2)'(\xi), e_n \rangle|^2 \right)^{1/2}
\leq \left( \sum_{n=1}^{N} |\langle F_1'(\xi), e_n \rangle F_2(\xi)|^2 \right)^{1/2} + \left( \sum_{n=1}^{N} |F_1(\xi) \langle F_2'(\xi), e_n \rangle|^2 \right)^{1/2}
\]
and therefore
\[
\left(\frac{1}{N} \sum_{n=1}^{N} |\langle (F_{1}F_{2})'(\xi), e_n \rangle|^2\right)^{1/2} \leq \left(\frac{1}{N} \sum_{n=1}^{N} |\langle F_{1}'(\xi), e_n \rangle|^2\right)^{1/2} |F_{2}(\xi)| + \left(\frac{1}{N} \sum_{n=1}^{N} |\langle F_{2}'(\xi), e_n \rangle|^2\right)^{1/2} |F_{1}(\xi)|
\rightarrow 0 \quad \text{as} \quad N \rightarrow \infty,
\]
as desired. We have thus proved that $F_{1}F_{2} \in \mathcal{D}_{0}$. Finally it follows immediately from Lemma 2.1 that $\Delta_{L}(F_{1}F_{2}) = \Delta_{L}F_{1} \cdot F_{2} + F_{1} \cdot \Delta_{L}F_{2}$, namely that the Lévy Laplacian acts on $\mathcal{D}_{0}$ as derivation.

Here is an immediate consequence.

**Corollary 2.3** For $p \geq 0$ we put
\[
\mathcal{A}_{p} = \{ F \in \mathcal{D}; F'(\xi) \in E_{p}, \xi \in E \}.
\]
Then $\mathcal{A}_{p}$ is a subalgebra of $\mathcal{D}_{0}$. In particular, $\Delta_{L}$ acts on $\mathcal{A}_{p}$ as derivation.

**Proof.** We first prove that $\mathcal{A}_{p} \subset \mathcal{D}_{0}$. Suppose $F \in \mathcal{A}_{p}$. Then, since $0 < \lambda_{1} \leq \lambda_{2} \leq \cdots$,
\[
\frac{1}{N} \sum_{n=1}^{N} |\langle F'(\xi), e_n \rangle|^2 = \frac{1}{N} \sum_{n=1}^{N} |\langle F'(\xi), e_n \rangle|^2 \lambda_{n}^{2p} \lambda_{n}^{-2p} \leq \frac{1}{N} \sum_{n=1}^{N} |\langle F'(\xi), e_n \rangle|^2 \lambda_{n}^{2p} \lambda_{1}^{-2p} \leq \frac{\lambda_{1}^{-2p}}{N} |F'(\xi)|_{p}^{2} \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty.
\]
Therefore $F \in \mathcal{D}_{0}$. It is then straightforward to verify that $\mathcal{A}_{p}$ is a subalgebra of $\mathcal{D}_{0}$.

In particular, $\mathcal{A}_{0}$ is an algebra of functions on $E$ on which the Lévy Laplacian acts as derivation. This is the reason why the Lévy Laplacian acting on functions on a Hilbert space is a derivation (note that $E_{0} = H$), see e.g., [10], [13], [14], [15].

The derivation property is also observed in a slightly different manner.

**Proposition 2.4** Let $F_{1}, F_{2} \in \mathcal{D}$ and fix $\xi \in E$. If there exists $p \geq 0$ such that
\[
|F_{1}'(\xi)|_{p} < \infty, \quad |F_{2}'(\xi)|_{-p} < \infty,
\]
then
\[
\Delta_{L}(F_{1}F_{2})(\xi) = \Delta_{L}F_{1}(\xi) \cdot F_{2}(\xi) + F_{1}(\xi) \cdot \Delta_{L}F_{2}(\xi).
\]

**Proof.** We see that
\[
\left|\frac{1}{N} \sum_{n=1}^{N} \langle F_{1}'(\xi), e_n \rangle \langle F_{2}'(\xi), e_n \rangle \right| \leq \frac{1}{N} \left( \sum_{n=1}^{N} |\langle F_{1}'(\xi), e_n \rangle|^{2} \lambda_{n}^{2p} \right)^{1/2} \left( \sum_{n=1}^{N} |\langle F_{2}'(\xi), e_n \rangle| \lambda_{n}^{-2p} \right)^{1/2} \leq \frac{1}{N} |F_{1}'(\xi)|_{p} |F_{2}'(\xi)|_{-p} \rightarrow 0, \quad \text{as} \quad N \rightarrow \infty.
\]
Then we need only to apply Lemma 2.1.

qed
3 Lévy Laplacian on positive definite functions

There is an interesting class of functions on $E$ which are related to finite measures on $E^*$. Let $\mathcal{B}$ be the $\sigma$-field on $E^*$ generated by linear functions:

$$x \mapsto \langle x, \xi \rangle, \quad x \in E^*,$$

where $\xi$ runs over $E$. It is easily seen that $\mathcal{B}$ coincides with the topological $\sigma$-field induced from the strong dual topology of $E^*$.

Let $M_+(E^*)$ be the space of finite measures on $E^*$ and let $M(E^*)$ be the space of all signed measures on $(E^*, \mathcal{B})$ with finite variation. Every element in $M(E^*)$ is written as $\mu_1 - \mu_2$, $\mu_1, \mu_2 \in M_+(E^*)$. If $\mu \in M(E^*)$, then its Fourier transform $\hat{\mu}$ is a function on $E$ defined by

$$\hat{\mu}(\xi) = \int_{E^*} e^{i\langle x, \xi \rangle} \mu(dx), \quad \xi \in E. \quad (2)$$

We here recall a fundamental result.

**Theorem 3.1 (Bochner-Minlos)** There is a one-to-one correspondence between $M_+(E^*)$ and the space $\mathcal{B}_+(E)$ of all continuous positive definite functions on $E$ through the Fourier transform (2).

Let $\mathcal{B}(E)$ be the space of the Fourier transform of $\mu \in M(E^*)$. Note that $M(E^*)$ is an algebra with convolution product:

$$\int_{E^*} \phi(x)\mu * \nu(dx) = \int_{E^* \times E^*} \phi(x + y)\mu(dx)\nu(dy).$$

Through the Fourier transform $\mathcal{B}(E)$ becomes an algebra with pointwise multiplication. Thus, $\mathcal{B}(E)$ becomes a closed subalgebra of $L^\infty(E)$ and therefore it is an abelian $C^*$-algebra for itself.

The support of $\mu$ is related to the continuity of $\hat{\mu}$.

**Theorem 3.2** If a positive definite function $C : E \rightarrow \mathbb{C}$ admits a continuous extension to $E_p$, $p \geq 0$, the corresponding measure $\mu$ is concentrated on $E_{-(p+q)}$ for any $q \geq 0$ such that the canonical injection $E_{p+q} \rightarrow E_p$ is of Hilbert-Schmidt type.

**Lemma 3.3** Let $F$ be the Fourier transform of $\mu \in M_+(E^*)$. If

$$\int_{E^*} |x|_p \mu(dx) < \infty \quad (3)$$

for some $p \in \mathbb{R}$, then $F'(\xi) \in E_p$ for any $\xi \in E$.

**Proof.** Since

$$\left| i \langle x, \eta \rangle e^{i\langle x, \xi \rangle} \right| \leq |x|_p |\eta|_{-p},$$

it follows from Lebesgue's convergence theorem that

$$\langle F'(\xi), \eta \rangle = \int_{E^*} i \langle x, \eta \rangle e^{i\langle x, \xi \rangle} \mu(dx), \quad \eta \in E.$$
Moreover,
\[ |(F'(\xi), \eta)| \leq \int_{E^*} |x|_p |\eta|_{-p} \mu(dx) = |\eta|_{-p} \int_{E^*} |x|_p \mu(dx), \]
which implies that \( F'(\xi) \in E_p \).
\[ \text{qed} \]

**Remark.** It follows from (3) that \( \mu(E_p) = 1 \). In fact, there exists a null set \( N \) such that \( |x|_p < \infty \) for any \( x \in E^* - N \). Hence \( E^* - N \subset E_p \) and therefore \( 1 = \mu(E^* - N) \leq \mu(E_p) \). Note also that \( p \) in (3) can be replaced with an arbitrary smaller one.

**Example.** Let \( \mu_\alpha \) be the Gaussian measure with variance \( \alpha^2 \). Then
\[ F(\xi) = \hat{\mu}(\xi) = \exp\left(-\frac{\alpha^2}{2} |\xi|_0^2\right), \quad \xi \in E. \]
By a direct calculation we obtain
\[ F'(\xi) = -\alpha^2 e^{-\alpha^2 |\xi|_0^2/2} \xi = -\alpha^2 F(\xi) \xi, \]
and therefore \( F'(\xi) \in E = \cap_{p \geq 0} E_p \). Consequently, \( F = \hat{\mu}_\alpha \in A_p \) for any \( p \geq 0 \).

## 4 Cauchy problem and semigroup

We recapitulate some results obtained in [2]. For the fixed complete orthonormal basis \( \{e_n\}_{n=1}^\infty \) of \( H \), which are in fact contained in \( E \), let \( S \) denote the shift with respect to the basis \( \{e_n\} \), i.e., the unique linear continuous (in fact isometric) map from \( H \) to \( H \) such that
\[ Se_n = e_{n+1}, \quad n = 1, 2, \ldots. \]
We note the following

**Lemma 4.1** \( S \in \mathcal{L}(E, E) \) if and only if
\[ \sup_{n \geq 0} \frac{\lambda_{n+1}}{\lambda_n^{1+r}} < \infty \]
for some \( r \geq 0 \).

**Proof.** Suppose first that \( S \in \mathcal{L}(E, E) \). Take an arbitrary \( p > 0 \). Then there exist \( q \geq 0 \) and \( C \geq 0 \) such that
\[ |S\xi|_p \leq C |\xi|_{p+q}, \quad \xi \in E. \]
In particular, putting \( \xi = e_n \) we have
\[ |e_{n+1}|_p = |Se_n|_p \leq C |e_n|_{p+q}. \]
Hence
\[ \lambda_{n+1}^p \leq C \lambda_n^{p+q}, \quad n = 1, 2, \ldots, \]
and
\[ \sup_{n \geq 1} \frac{\lambda_{n+1}}{\lambda_n^{1+q/p}} \leq C^{1/p} < \infty, \]
as desired. Conversely, we assume that there exists \( r \geq 0 \) with
\[
M = \sup_{n \geq 0} \frac{\lambda_{n+1}}{\lambda_{n}^{1+r}} < \infty.
\]
Consider an element \( \xi \in E \) which admits an expansion:
\[
\xi = \sum_{n=1}^{\infty} c_n e_n,
\]
where \( c_n = 0 \) except finitely many \( n \). Then by definition,
\[
S\xi = \sum_{n=1}^{\infty} c_n S e_n = \sum_{n=1}^{\infty} c_n e_{n+1}.
\]
For any \( p \geq 0 \) we have
\[
|S\xi|_p^2 = \sum_{n=1}^{\infty} |c_n|^2 |e_{n+1}|_p^2 = \sum_{n=1}^{\infty} |c_n|^2 \lambda_{n+1}^{2p} \leq M^2 \sum_{n=1}^{\infty} |c_n|^2 \lambda_n^{2p(1+r)} = M^2 |\xi|_{p(1+r)}^2.
\]
This implies that \( S \) is a continuous operator on \( E \).

From now on we assume that \( S \in \mathcal{L}(E, E) \). Then the adjoint \( S^* \in \mathcal{L}(E^*, E^*) \) becomes a measurable map from \( E^* \) into \( E^* \). Let \( M_S(E^*) \subset M(E^*) \) be the space of measures on \( E^* \) which are invariant under \( S^* \). We put
\[
M_S^2(E^*) = \{ \mu \in M_S(E^*) : \int_{E^*} |\langle x, \eta \rangle|^2 \mu(dx) < \infty \text{ for all } \eta \in E \}.
\]
Let \( \mathcal{H} \) be the subspace of all \( x \in E^* \) such that the limit
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\langle x, e_n \rangle|^2 < \infty
\]
exists. Then,
\[
\|x\| = \lim_{N \to \infty} \left( \frac{1}{N} \sum_{n=1}^{N} |\langle x, e_n \rangle|^2 \right)^{1/2}, \quad x \in \mathcal{H}.
\]
becomes a seminorm of \( \mathcal{H} \).

**Lemma 4.2** Let \( \mu \in M_S^2 \). Then \( x \in \mathcal{H} \) for \( \mu \)-a.e. \( x \in E^* \). In other words, the limit
\[
\|x\|^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\langle x, e_n \rangle|^2 < \infty
\]
exists for \( \mu \)-a.e. \( x \in E^* \). Moreover, the limit converges in \( L^1(E^*, \mu) \).

**Proof.** For simplicity we put
\[
F(x) = |\langle x, e_1 \rangle|^2.
\]
Then, clearly $F \in L^1(E^*, \mu)$. Since $S^*$ is a $\mu$-preserving measurable map from $E^*$ into itself, it follows from the ergodic theorem (e.g., [5, Chap.VIII]) that

$$ F^*(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} F(S^{*(n-1)}x) $$

converges $\mu$-a.e. $x \in E^*$ as well as in the $L^1$-sense. In that case $F^* \in L^1(E^*, \mu)$. On the other hand, since

$$ \sum_{n=1}^{N} F(S^{*(n-1)}x) = \sum_{n=1}^{N} \langle S^{*(n-1)}x, e_1 \rangle^2 = \sum_{n=1}^{N} \langle x, S^{*(n-1)}e_1 \rangle^2 = \sum_{n=1}^{N} \langle x, e_n \rangle^2, $$

we see that $F^*(x) = \|x\|^2$. The assertion then follows immediately. qed

In a similar manner,

**Lemma 4.3** Let $\mu, \nu \in M^2_\Sigma(E^*)$. Then the limit

$$ \langle \langle x, y \rangle \rangle = \frac{1}{N} \sum_{n=1}^{N} \langle x, e_n \rangle \langle y, e_n \rangle $$

exists for $\mu \times \nu$-a.e. $(x, y) \in E^* \times E^*$.

**Proposition 4.4** If $\mu \in M^2_\Sigma(E^*)$, then $F = \hat{\mu} \in D$ and

$$ \Delta_L F(x, t) = -\int_{E^*} \|x\|^2 e^{i\langle x, \xi \rangle} \mu(dx). $$

**Proof.** It is easily verified from definition that

$$ \langle F''(\xi), e_n \otimes e_n \rangle = -\int_{E^*} \langle x, e_n \rangle^2 e^{i\langle x, \xi \rangle} \mu(dx). $$

Then we need only to apply Lemma 4.2. qed

Consider the Cauchy problem for the Laplace equation:

$$ \frac{\partial}{\partial t} F(x, t) = \Delta_L F(x, t), \quad F(x, 0) = F_0(x), $$

(4)

where $F_0$ is a certain function on $E$. For some particular initial condition the Cauchy problem is solved satisfactorily in Accardi-Roselli-Smolyanov [2].

**Theorem 4.5** Let $\mu \in M^2_\Sigma(E^*)$ and put $F_0 = \hat{\mu}$. Then the solution of the Cauchy problem (4) is given as

$$ F(x, t) = \overline{\mu_t}(\xi), \quad \mu_t(dx) = e^{-\|x\|^2} \mu(dx), \quad t \geq 0. $$

**Proof.** By Lemma 4.2 $\mu_t$ is well defined and belongs to $M_+ (E^*)$. Moreover, obviously $\mu_t$ is $S^*$-invariant and

$$ \int_{E^*} \langle x, \eta \rangle^2 \mu_t(dx) \leq \int_{E^*} \langle x, \eta \rangle^2 \mu(dx) < \infty, $$

and
namely, $\mu_t \in M^2_S(E^*)$. It then follows from Proposition 4.4 that $\tilde{\mu}_t \in \mathcal{D}$ and

$$\Delta_L F(\xi, t) = -\int_{E^*} \|x\|^2 e^{-t\|x\|^2} e^{i(x, \xi)} \mu(dx).$$

On the other hand, since $\|x\|^2$ belongs to $L^1(E^*, \mu)$ by Lemma 4.2, we see by Lebesgue’s theorem that

$$\frac{\partial F}{\partial t} = -\int_{E^*} \|x\|^2 e^{-t\|x\|^2} e^{i(x, \xi)} \mu(dx).$$

Therefore $F(\xi, t) = \tilde{\mu}_t(\xi)$ is a solution of the Cauchy problem under consideration. qed

We put

$$\tilde{P}^t(\mu)(dx) = e^{-t\|x\|^2} \mu(dx), \quad \mu \in M^2_S(E^*), \quad t \geq 0.$$ 

Then $\tilde{P}^t$ constitutes a one-parameter semigroup of transformations on $M^2_S(E^*)$.

Let $B^2_S(E)$ be the image space of $M^2_S(E^*)$ under the Fourier transform. The induced one-parameter semigroup of transformations on $B^2_S(E)$ is denoted by $P^t$. This is called the heat semigroup of the Lévy Laplacian $\Delta_L$.

We note the following

**Proposition 4.6** The subspace $M^2_S(E^*)$ is closed under convolution. Therefore $B^2_S(E)$ is closed pointwise multiplication.

However, the Lévy Laplacian is not a derivation on $B^2_S(E)$ and $\tilde{P}^t$ is not multiplicative; namely,

$$\tilde{P}^t(\mu * \nu) \neq \tilde{P}^t \mu * \tilde{P}^t \nu$$

does not holds in general. In fact, $\tilde{\mu}$ belongs to $\mathcal{D}$ but not to $\mathcal{D}_0$ on which the Lévy Laplacian acts as derivation, see Theorem 2.2.

## 5 Normal polynomials

In this section we introduce particular classes of polynomials under an additional structure of $E$, namely, multiplication. We assume that $E$ is equipped with a multiplication which makes $E$ a commutative algebra. Furthermore we assume that the multiplication is continuous (since $E$ is a Fréchet space, there is no difference between joint and separate continuity) and that

$$\langle \xi \eta, \zeta \rangle = \langle \xi, \eta \zeta \rangle, \quad \xi, \eta, \zeta \in E.$$ 

This situation often occurs when $E$ is a function space (the multiplication above is the usual pointwise multiplication of functions). By duality multiplication of $f \in E^*$ and $\xi \in E$, denoted by $f \xi = \xi f$, is defined as a unique element in $E^*$ such that

$$\langle f \xi, \eta \rangle = \langle f, \xi \eta \rangle, \quad \eta \in E.$$ 

Obviously, the multiplicatication $E^* \times E \rightarrow E^*$ is an extension of $E \times E \rightarrow E$.

Consider a quadratic function $\xi \mapsto \langle f, \xi^2 \rangle$, where $f \in E^*$ is fixed. Since $\langle \xi, \eta \rangle \mapsto \langle f, \xi \eta \rangle$ is a continuous bilinear form on $E \times E$, there exists $g \in (E \otimes E)^*$ such that $\langle f, \xi \eta \rangle = \langle g, \xi \otimes \eta \rangle$. Thus, $\langle f, \xi^2 \rangle = \langle g, \xi \otimes \eta \rangle$ and there occurs no new quadratic function in this manner. On the contrary, using the new product in $E$ we may introduce a subclass of polynomials. Namely,
if $f \in E^*$ is "regular," the corresponding quadratic functions constitute a certain class of quadratic functions. This is immediately generalized to polynomials of any degree. Thus, a normal polynomial on $E$ is a finite linear combination of functions of the form:

$$\langle f, \xi^{\nu_1} \otimes \cdots \otimes \xi^{\nu_n} \rangle, \quad \nu_1, \ldots, \nu_n = 0, 1, 2, \ldots,$$

where $f \in (E^{\otimes n})^*$ is a regular element. Here the tensor product and the multiplication of $E$ should be carefully distinguished.

We now go into a typical situation. Consider a one dimensional torus $T = \mathbb{R}/\mathbb{Z}$. Put $H = L^2(T)$ and consider $d/dt$. Then $E = C^\infty(T)$ and $\{e_n\}$ consists of trigonometric functions. In that case $\{e_n\}$ possesses additional properties: first $\{e_n\}$ is uniformly bounded:

$$\sup_{n} \sup_{t \in T} |e_n(t)| < \infty;$$

Second, it is equally dense, i.e.,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \int_{0}^{1} f(t) e_n(t)^2 dt = \int_{0}^{1} f(t) dt, \quad f \in L^\infty(T).$$

Moreover, the pointwise multiplication gives a continuous bilinear map from $E \times E$ into $E$. We say that $f \in (E^{\otimes n})^*$ is regular if $f \in L^1(T^n)$. This is the usual definition of a regular distribution. Then we have the space of normal polynomials. In other words, a normal polynomial on $E$ is by definition a linear combination of functions of the form:

$$F(\xi) = \int_{T^n} k(t_1, \cdots, t_n) \xi(t_1)^{\nu_1} \cdots \xi(t_n)^{\nu_n} dt_1 \cdots dt_n, \quad \xi \in E,$$

where $k$ is an integrable function on $T^n$. If $\nu_i = 1$ for all $i$, the polynomial is called regular after Lévy’s original definition.

**Lemma 5.1** Consider a normal polynomial of the form:

$$F(\xi) = \langle f, \xi^\nu \rangle, \quad f \in E^*.$$

Then $F \in D_0$ if and only if

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\langle f \xi^{\nu-1}, e_n \rangle|^2 = 0$$

for any $\xi \in E$.

The proof is immediate. Then we come to the following

**Proposition 5.2** Every normal polynomial belongs to $D_0$.

The above result generalizes the known fact that the Lévy Laplacian is a derivation on normal polynomials, see [10, Proposition 3.2].
References


