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THE AXIOMS OF PROBABILITY

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Introduction

A simple set of physically meaningful axioms, unifying classical and quantum probability, is proposed. A classification theorem exhibits all the models for these axioms. These include the usual classical and quantum models, but new interesting models arise.

Bayes' definition of conditional probability plays in probability theory the same role played by the parallel axiom in Euclidean geometry. As the geometric invariants (e.g., curvature) distinguish Euclidean and non-Euclidean models, the statistical invariants distinguish Kolmogorovian and non-Kolmogorovian models. The statistical invariants show the non-Kolmogorovianity of the quantum models. They distinguish real and complex numbers. They show that, even in the simplest situation, there are new unexplored possibilities.

For several years the physicists have applied some Kolmogorovian rules to some non-Kolmogorovian (quantum) data: this has lead to the famous "paradoxes of quantum theory". This story is reviewed.

The interpretative problems of quantum mechanics have led to a new analysis of the axioms of classical probability theory. The main result of this analysis is that some basic probabilistic notions, considered for centuries intrinsic to the very notion of probability, are in fact model dependent statements, like Euclid's parallel postulate. The implicit axiom, which played for probability theory the role of the parallel postulate in geometry, is Bayes' elementary definition of conditional probability. We can now prove experimentally, using data arising in quantum mechanical experiments, that this postulate is not adequate to describe several simple quantum phenomena. A mathematical analysis of Bayes' formula defining conditional probability shows that it is equivalent to several properties which are by far not self-evident: several examples of violation of these axioms in the quantum probabilistic model are considered and their experimental evidence is discussed.

The program of classifying probabilistic models in terms of "statistical invariants" is outlined and the results obtained up to now in this direction are surveyed. This probabilistic analysis suggests a natural solution to the interpretative problems of quantum theory as well as a new approach to the problem of its mathematical foundations.

1 The quantum probabilistic approach to the foundations of quantum theory

The main conclusions of the quantum probabilistic approach to the foundations of quantum theory, developed in the last fifteen years, can be summarized (necessarily in a *very* schematic way) as follows:

(1) All the paradoxes of quantum theory (reality, locality, separability, $\cdots$) have their roots in the statement that if a system is in a superposition state with respect to a given observable A, then this observable cannot actually assume any of its values and only the act of measurement *collapses* the physical state of the system so that one and only one of the values A is assumed.

(2) There exists, in the literature, only one *proof* (up to minor variations) of the *theoretical and experimental necessity* of accepting the statement of item (1). This proof is based on the implicit assumption of the applicability of an important elementary formula of classical probability theory. Quantum probability proves that the application of this formula in the context of the above mentioned proofs is mathematically unwarranted. In other words, quantum probability shows that the apparent contradictions met in quantum physics arise when one mixes the rules of the new quantum probability calculus with those of the old classical one.

(3) As a corollary of the result of item (2) it follows that *there is no need to assert that the collapse of the wave packet corresponds to a real physical phenomenon.* In particular, the so called paradoxes of quantum theory, which are all constructed by means of variations on the theme of the collapse of the wave packet, are cut at their roots.

(4) A simple set of physically meaningful axioms, unifying classical and quantum probability, is proposed. A classification theorem exhibits all the models for these axioms. These include the usual classical and quantum models, but new interesting models arise.

(5) A consistent physical interpretation of the new probability calculus can be developed entirely within the conceptual framework of classical physics. Some trends, open for the philosophical and epistemological meditations by these results are outlined.

2 Algebraic probability theory

Some fundamental results of classical probability theory have a universal validity, being based on purely combinatorial properties. This statement is illustrated with the quantum probabilistic generalization of three basic probabilistic results: the law of large numbers; the central limit; theorem and De Finetti's theorem.

In paricular, the quasi-free states of quantum field theory are shown to arise from quantum central limit theorems just as the usual Gaussian measures arise from classical central limit theorems. Also the Heisenberg commutation relations are shown to be a quantum central limit effect. Finally a quantum invariance principle is proved from which one deduces the quantum Brownian motions, introduced in the sixties in laser theory.

The general quantum central limit theorems, applied to the quantum Bernoulli process, lead to the quantum harmonic oscillator.

A theory of quantum Markov chains is developed. Several results of the classical theory are extended to a quantum environment. Several examples are produced and they are used to construct some nontrivial (i.e., with a nontrivial interaction) models of signal transmission in a quantum environment.

An axiomatic theory of quantum noise is proposed. From the axioms one deduces a quantum stochastic differential equation which generalizes the usual Schrödinger equation in interaction representation. A consequence of this equation is the quantum generalization of the Einstein fluctuation-dissipation relation.

From this generalized Schrödinger equation the general form of the quantum Langevin equation is deduced. When the abstract theory is particularized to the class of quantum noises usually considered in laser theory, the quantum Langevin equations are shown to be in 1-1 correspondence to certain Lie algebras of observables. The position-momentum Lie algebra produces the equation of the dumped harmonic oscillator proposed by Senitzky and Lax in 1968. The angular momentum Lie algebra produces a nonlinear generalization of the equation proposed by Block in 1946 to describe paramagnetic resonance. Numerical experiments show that this nonlinearity leads to a dynamical phase transition which is strongly reminiscent of the phenomenon of parametric laser amplification.

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3 Quantum fluctuations

In the past years a new kind of perturbation theory was developed in a long series of papers dealing with an increasingly complex sequence of quantum models. The theory is well suited for the study of *scaling limits* of various kinds of quantum models. At present the most studied types of scaling limits are the *weak coupling* (or van Hove) limit and the *low density* limit. In the following, in order to simplify the discussion, we shall only discuss the weak coupling limit case, and the perturbation parameter shall be called the *coupling constant.*

The starting point of the new perturbation theory is, like for the usual one, the iterated series solution of Schrödinger's equation in interaction representation, which is supposed to be convergent in some weak topology. The basic features which distinguish the new approach from the usual one are the following:

(1) One considers limits of matrix elements of the solution of the Schrödinger equation with respect to states which depend themselves on the coupling constant. The choice of these vectors is determined by first order (usual) perturbation theory and by theoretical considerations inspired to the quantum central limit theorems.

(2) In each term of the iterated series expansion of these matrix elements, one distinguishes two pieces, one of which is negligible in the limit, the other not.

(3) One shows that the series obtained after the limit can be resummed and the result satisfies a quantum stochastic differential equation.

Usually the theory, which at the moment is nonrelativistic, is applied to describing a quantum system (called *the small system*) interacting with a second quantized system (called the *reservoir*). In the limit the reservoir system becomes a quantum stochastic process which strongly depends on the form of the interaction between the small system and the reservoir. Such a process is usually called a *quantum noise* and the quantum stochastic differential equation is a generalization of the usual Schrödinger equation. From this point of view, a quantum noise is a scaling approximation to a quantum field. Typically, i.e. for several models, one finds a quantum Brownian motion in the weak coupling limit; a quantum Poisson process in the low density limit.

The interactions considered up to now in the weak coupling limit case are known approximations of the standard quantum electrodynamical interaction: dipole approximation, rotating wave approximation, $\cdots$. The full QED interaction (without dipole approximation) was first considered one year ago and has led to the introduction of some dramatically new features into the picture:

(1) The quantum noise arising from the full QED interaction is of a completely new type. In particular, the vacuum distribution of the noise field operator is not the usual Gaussian, but a convex combination of *Wigner semicircle laws.*

(2) The quantum noise does not live on a Hilbert space, but on a *Hilbert module* over the momentum algebra of the small system.

(3) From the limiting procedure a new kind of Fock space arises, called the *interacting Fock space* because the scalar product in each of the n-particles subspaces is not the usual one, coming from the tensor product of Hilbert spaces, but a new one, for which the n particles are no longer independent.

These new features have yet to be fully understood, but the non independent, i.e. strongly interacting, nature of the quanta in each of the n-particle spaces in the interacting Fock space, suggests the speculation that this type of space might be a good candidate to describe the state space of the quarks.

4 The Quantum Brownian motion as approximations of quantum fields

The free quantum electromagnetic field and the Quantum Brownian motions (QBM) are both examples of quantum Gaussian fields. The QBM were introduced in the sixties, in the physical literature on laser theory, as approximations of the free quantum electromagnetic field. The problem of giving a precise meaning to this statement was recently solved with quantum probabilistic techniques. The basic result was that in the weak coupling limit (WCL) of a system interacting with a Gaussian quantum field, the usual Hamiltonian equations are approximated by a quantum stochastic differential equation driven by a QBM.

The situation in the low density limit (LDL) is considerably more difficult, due to the presence in the interaction of a finite intensity term, not tending to zero with the density. In the probabilistic analogy, the WCL interaction arises from a sum of uniformly infinitesimal quantum fields – a situation strongly reminding the classical central limit theorems; while the finite intensity term in the LDL corresponds to rare individual events (low density), a situation which reminds the classical Poisson limit theorems. On the basis of this analogy Frigerio and Maassen conjectured that this finite intensity term should give rise, in some limiting sense to be specified, to a quantum Poisson process (QPP). However, while in the physical literature there are several arguments which justify at least at a nonrigorous level the origins of the QBM, nothing similar can be found for the QPP (a posteriori we now understand why: some basic features of the QBM already appear at the level of second order perturbation theory – precisely these effects were discovered by the laser theorists – while any effect related to the QPP receives contributions from the whole perturbative series). The solution of the problem has gone through 3 basic steps:

(1) One isolates the finite intensity term and shows that in the low density limit this gives rise to a quantum stochastic equation driven by a pure jump quantum noise.

(2) One identifies the coefficients of the quantum noise in step (1) to the matrix elements of the scattering operator between the system and the one particle space of the field on the zero energy shell.

(3) One splits the full interaction into a weak coupling and a low density part (plus higher order terms) and shows that in the limit, the two parts interact in such a way to produce exactly the compensation term, between the quantum diffusion (due to the WCL) and the quantum jump part (due to the LDL) required by the quantum Ito formula to guarantee unitarity of the evolution.

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