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On some results of the cohomology of extra special p-groups

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Extra special p-groups are central extensions of $\mathbb{Z}/p$ by elementary abelian p-groups. These groups occupy a distinctive in the cohomology and representation theories of finite groups. Quillen decided mod 2 cohomology of the extra special 2-groups [Q]. However the corresponding calculation for odd $p$ is still unknown. Tezuka-Yagita studied the varieties defined from its mod $p$ cohomology [T-Y]. Extending these results, Benson-Carlson decided the mod $p$ cohomology modulo Jacobson radical [B-C]. The radical parts seem very difficult. For the group of the order $p^3$, Lewis decided the integral cohomology and Leary wrote down the mod $p$ cohomology completely [Lw], [L2]. Minh computed the mod 3 cohomology of the group with the order $3^8$ and of the exponent $3^9$ [M].

One of main results of this paper, is to give the additive structure of the mod $p$ cohomology of the group with the order $p^n$ and its exponent $p$ (Theorem 8.25). The another results are existence of groups and their modules which period is exactly $2p^n$ for each $n$.

§1. Extra special p-group.

An extra special p-group $G$ is a group such that its center is $\mathbb{Z}/p$ and there is the central extension

\[ 1 \rightarrow \mathbb{Z}/p \rightarrow G \xrightarrow{\pi} V \rightarrow 1 \]

where $V = \mathbb{Z}/Z/p$.

Such group is isomorphic to the $n$-th central product $E \ldots E = E_n$ or $E_{n-1} \cdot M$ where $E$ (resp. $M$) is the non abelian group of the order $p^n$ and exponent $p$ (resp. $p^2$). Hence we can explicitly write
(1.2) $E_n = \langle a_1, \ldots, a_{2n}, c | [a_{2i-1}, a_{2i}] = c, c \in \text{Center} \rangle$

$[a_i, a_j] = 1$ for $i < j, (i, j) = (2k-1, 2k)$

$a_k^p = c^p = 1 >$

The group $E_{n-1}M$ is written similarly except for $a_{2n}^p = c$.

Let us write by $x_i \in H^1(V) = \text{Hom}(V, Z/p)$ the dual of $\pi_i(a_i)$ and write $y_i = \varphi x_i$

Then the cohomology of $V$ is

$H^*(V) \cong S_{2n} \otimes \Lambda_{2n}$ with $S_{2n} = Z/p[y_1, \ldots, y_{2n}]$ and $\Lambda_{2n} = \Lambda(x_1, \ldots, x_{2n})$.

Proposition 1.3. The extension (1.1) represents element in $H^*(V)$

$f = \sum x_{2i-1} x_{2i}$ (resp. $\sum x_{2i-1} x_{2i} y_{2n}$) for $G = E_n$ (resp. $E_{n-1}M$).

We consider spectral sequence induced from (1.1)

(1.4) $E_\infty^{p,q} = H^*(V; H^*(Z/p))$

$\cong S_{2n} \otimes \Lambda_{2n} \otimes Z/p[u] \otimes \Lambda(z) \rightarrow H^q(G)$

with $\varphi z = u$. From Proposition 1.3, we know

(1.5) $d_z z = f$.

By transgression theorem,

(1.6) $d_z u = \varphi d_z z = \sum y_{2i-1} x_{2i} - y_{2i} x_{2i+1}$

(1.7) $d_z p^{s+1} u^{s+1} = p^{s+1} \varphi d_z u = \sum y_{2i-1} p^{s+1} x_{2i} - y_{2i} x_{2i+1}$

By Kudo's transgression theorem

(1.8) $d_z p^{s+1} \otimes (u^{s+1} \otimes d_z p^{s+1} + u^{s+1}) = p^{s+1} \varphi d_z p^{s+1} + u^{s+1} = \sum y_{2i-1} p^{s+1} y_{2i} y_{2i+1} x_{2i+1}$

Let us write (1.6) = $z_n(l)$, (1.7) = $z_n(s+1)$, (1.8) = $w_n(s+1)$. Hence $E_\infty^{p,q}$ is a quotient of

(1.9) $E = S_{2n} \otimes \otimes Z/n/(f, z_n(1), \ldots, z_n(n), w_n(1), \ldots, w_n(n))$

We also know that $u^{*H}$ is a permanent cycle because which represents $p^{s+1}$-th Chern class of induced representation from a maximal elementary abelian $p$-group. Write by $u'$ a corresponding element in $H^*(G)$. Then $H^*(G)$ is a $E_{\partial Z/p[u']}$-module. From Benson-Carlson [B-C].

$H^*(G)/J = E_{Z/p[u']}/J$ for the Jacobson radical $J$.

Note the regularity of the sequence $w_1(1), \ldots, w_n(n)$ in $S_{2n}$ is shown in Tezuka-Yagita [T-Y].
§2. $\tilde{G}$; the central product of $G$ and $S^1$.

The spectral sequence (1.4) is very complicated even for $E$ (see [K-S-T-Y] (5.11)). Hence we consider another arguments which are used by Kropholler, Leary, Huebshmann and Moselle. Embed $c \cong Z/p \zeta S^1$ and cosider the central product

(2.1) \[ G = G \times_{c} S^1. \]

Note that $E_{n} \cong E_{n-1} \mathbb{W}$, indeed, take $a_{2n}c^{-1/p}$ as $a_{2n}$, if $a_{2n}p = c$. Then we have the exact sequence

(2.2) \[ 1 \rightarrow S^1 \rightarrow G \rightarrow V \rightarrow 1 \]

and induced spectral sequence

(2.3) \[ E_{2}^{*,*} \cong H^*(V; H^*(BS^1)) \]

\[ \cong S_{2n} \otimes \Lambda_{2n} \otimes Z/p[u] \rightarrow H^*(\tilde{G}) \]

Then differentials (1.6)-(1.8) also hold but $d_2 = 0$ by the dimensional reason.

Given $H^*(\tilde{G})$, to see $H^*(G)$ we use the following fibration induced from (2.1)

(2.4) \[ S^1 \cong \widetilde{G}/G \rightarrow BG \rightarrow BG. \]

The induced spectral sequence is

(2.5) \[ E_{2}^{*,*} \cong H^*(\tilde{G}; H^*(S^1)) \]

\[ \cong H^*(\tilde{G}) \otimes \Lambda(z) \rightarrow H^*(G) \]

and $d_2z \neq f$ (1.5). Therefore

Proposition 2.6. \[ H^*(G) \cong (\text{Ker}(f) | H^*(G))/\{z\} \otimes H^*(G)/(f). \]

§3. $2p$-terms for $E_n$ for $n \lt p$.

In sections 3-5, we consider spectral sequence (2.3) for $E_n$ for $p \lt n$. Given graded algebra $A$ and $z \in A^{n+n}$, we define homology $H(A, z)$ with the differential $d_z$ $(a) = za$. The first non-zero differential in (2.3) is $d_2u = z_n(1)$. Hence

\[ E_{2}^{*,*} \cong \begin{cases} S_{2n} \otimes A_{2n}/(z_n(1)) & j=0 \mod p \\ H(S_{2n} \otimes A_{2n}, z_n(1)) & 1 \leq j \leq 2 \\ \text{Ker}z_n(1) & j=p-1. \end{cases} \]

(3.1)

\[ E_{2}^{*,*} = 0. \]
We can prove

\[ (3.7) \quad E_{zp}^{*,z-j} = \begin{cases} S_{zn} \otimes_{zn}/(z_n(1), w_n(1), z_n(2) f^{n-1}) & j \equiv 0 \mod p \\ \mathbb{Z}/p\{f^n\} & 1 \leq j < p-1 \\ 0 & j = p-1. \end{cases} \]

\section{4. $E_{zp+z}$-term.

The next differential is (1.7)

\[ (4.1) \quad d_{zp+1}(u^p) = \mathcal{O} z_n(1) = z_n(2). \]

Let $E = S_{zn} \otimes_{zn}/(z_n(1), w_n(1))$. Then we get from (3.7)

\[ (4.2) \quad E_{zp+z-j} = \begin{cases} E/(z_n(2)) & \text{for } j \equiv 0 \mod p \\ H(E/z_n(2)f^{n-1}, z_n(2)) & 0 < j \leq p-2 \\ \ker(z_n(2)|E/z_n(2)f^{n-1}) & j = p-1. \end{cases} \]

\[ E_{zp+z-j} = \mathbb{Z}/p\{f^n\} \quad 0 < j \leq p-2. \]

\section{5. Homology of $H(E/E_{p+1})$.

Proposition 5.18. $H(E, z_n(2))^{odd} \subset H(E, z_n(2))^{even} - \mathbb{Z}/p\{f^n\}$

and $H(E, z_n(2))^{odd} \cong S_4(x_1', \ldots, x_n')/(y_{j_1}x_{j_2}, y_{j_3}x_{j_4} - y_{k_1}x_{k_2})$.

where we express $x_i' = x_i f^{n-1}$, $y_{j_1} = y_{j_1} f^{n-1}$, $y_{k_1} = y_{k_1} f^{n-1} - y_{j_1} f^{n-1} y_{j_2}$.

From (4.2) we get

Corollary 5.19. $H(E/z_n(2)f^{n-1}, z_n(2))$ is generated by $f^{n-1}$ as $S_{zn} \otimes_{zn}$-module and

\[ H(E/z_n(2)f^{n-1}, z_n(2))^{odd} = H(E, z_n(2))^{odd}, \]

\[ H(E/z_n(2)f^{n-1}, z_n(2))^{even} = S_{zn}/(y_{j_1} y_{j_1} f^{n-1}) \otimes \mathbb{Z}/p\{f^n\}. \]

\section{6. $E_{zp+(p-1)}$-term for $G'$.

Let $y_{j_1} = y_{j_1} f_{j_1} f_{j_1}'$ and $y_{j_1}' = (y_{j_2}^2 y_{j_1} - y_{j_1} y_{j_2}^2)/y_{j_1} y_{j_1}$.

Therefore we can prove
Proposition 6.14. For \( n=2 \) case
\[
E_{2, (p-1) + 2^{p^{j}} \cdot p^{j}} \cong \left\{ \begin{array}{ll}
S_{4} \otimes \Lambda_{4}/(z_{2}(1), z_{2}(2), w_{2}(1), w_{2}(2), (y_{z_{1}} + y_{z_{2}}) \otimes (x_{1} x_{2})) \\
\end{array} \right.
\]
\[
\text{mod } p
\]
\[
S_{4} \otimes \Lambda_{4}/(y_{1}, y_{j}, y_{j} x_{2} - y_{j} x_{1}, f, x_{3} x_{4}) ((h, k) = (1, 2), (3, 4))
\]
\[
E_{2, (p-1) + 2^{p^{j}} \cdot p^{j}} \cong \left\{ \begin{array}{ll}
\mathbb{Z}/p(x_{3}) \\
0
\end{array} \right.
\]
\[
\text{mod } p
\]
\[
0 < j < p - 1
\]
\[
\text{mod } p
\]

In the next section, we will prove also

Theorem 6.15. \( E_{2, (p-1) + 2^{p^{j}} \cdot p^{j}} \cong E_{\infty} \).

\[\text{§ 8. Ker } f \text{ in } H^{*}(E_{2}).\]

Theorem 8.25. There is an additive isomorphism
\[
H^{*}(E_{2}) \cong (A/(f) \oplus (\ker(f)|A) \cdot A) \otimes_{1 \leq i \leq p^{2} - 2} (H^{*}(f_{i}) \oplus H^{*}(f_{j}))
\]
\[
\otimes_{1 \leq i \leq p^{2} - 3 \text{ and } i \neq 1, \neq p \text{ mod } p \text{ or } i = p - 1} (\mathbb{Z}/p(z_{i}) \oplus \mathbb{Z}/p(z_{j}))
\]
\[
\otimes \mathbb{Z}/p(u^{p^{j}})
\]

where
(i) \( A \cong S_{4} \otimes \Lambda_{4}/(z_{2}(1), z_{2}(2), w_{2}(1), w_{2}(2), (y_{z_{1}} + y_{z_{2}}) \beta(z_{1} x_{2}, z_{2}(3)) \)
with
\[
S_{4} \otimes \Lambda_{4} = \mathbb{Z}/p(y_{1}, \ldots, y_{4}) \otimes \Lambda(x_{1}, \ldots, x_{4}), z_{2}(1) = \beta f, z_{2}(2) = P \beta f, z_{2}(3) = P P \beta f \text{ and } w_{2}(1) = \beta z_{2}(2), w_{2}(2) = \beta z_{2}(3)
\]
\[
(6.5) y_{j} = y_{j}^{(p^{j})} + y_{j}^{(p^{j} - 1)} + \cdots + y_{j}^{(p^{1})},
\]
(ii) \( f = \{x_{1} x_{2} + x_{3} x_{4}\}, \)
(iii) (Proposition 8.2) \( \ker(f)|A \) is generated as an \( S_{4} \)-module by
\[
y_{j} y_{j}, y_{j} x_{i} - y_{i} x_{j}, y_{j} x_{i},
\]
\[
f, x_{k} x_{k} ((k, h) \neq (1, 2), (3, 4)),
\]
\[
x_{1} x_{2} x_{3} x_{4},
\]
where \( y_{j} = y_{j}^{p^{j}} - y_{j}^{p^{j} - 1} \).
(iv) \( z \) corresponds \( n \) zero element in \( H^{1}(S^{1}) = E_{0}^{1} \) in (2.4),
(v) (Proposition 5.18) \( H^{*}(E_{2}) \cong \mathbb{Z}/p(1) \) and \( H^{*}(E_{2}) \cong S_{4} \{x_{1}, \ldots, x_{4}\}/(y_{i} x_{j}, y_{i} x_{k} = y_{k} x_{i})
\)
\[
(f) f = \{f u^{p^{j}}\} \text{ in the spectral sequence (2.2)},
\]
\[
(vi) z_{i} \{f^{2} u^{i}\} = \{x_{1} x_{2} x_{3} x_{4} u^{i}\} \text{ in (2.2)}.\]
§2. Hochschild-Serre spectral sequence.

We consider the spectral sequence with $E_2$-term

$$(2.1) \quad E_2^{pq} = H^q(\Theta^pZ/p; H^*(BS^1)).$$

In this paper cohomology $H^*(-)$ always means the $Z/p$-coefficient $H^*(-; Z/p)$. Let us write

$$H^{*}(Z/p) = S_{\infty} \otimes \Lambda_{\infty}, \quad H^*(BS^1) \cong Z/p[u]$$

with $S_{\infty} = Z/p[y_1, \ldots, y_{\infty}]$, $\Lambda_{\infty} = \Lambda(x_1, \ldots, x_{\infty})$, $\beta x_i = y_i$.

We assume first non zero differential

$$(2.2) \quad d_2 u = \beta f \text{ with } f = \sum_{i=1}^{\infty} x_{2i-1} x_{2i}.$$

Then by Cartan-Serre and Kudo transgression theorems, we know

$$(2.3) \quad d_{3p}^{i-1, i+1}(u^{p^{i-1}}) = z(i), \quad d_{3p}^{i, i+1}(z(i) \otimes u^{(p-1)p^{i-1}}) = w(i)$$

with $z(i) = \sum_{j=1}^{i-1} \beta^j x_{2j} - y_{2j}$.

$$w(i) = \sum_{j=1}^{i-1} \beta^j y_{2j} - y_{2j}.$$

Let us write $S(i) = S_{\infty} \otimes (w(1), \ldots, w(i))$. Recall $(w(1), \ldots, w(n))$ is regular in $S_{\infty}$ [7].

Lemma 2.4. For $i \leq n-1$, we get

(i) $1$ is $S(i)$-free in $E_3^{i+1, 0}$.

(ii) $z(i+1)$ is $S(i)$-free in $E_3^{i+1, 0}$.

(iii) if $x \in E_3^{i+2, 0}$ is higher $w(i+1)$-torsion, then $x$ is higher $w(j)$-torsion for all $j \geq n$ (i.e., $w(j)^s x = 0$ for some $s$ and all $j \geq n$).

(iv) $E_3^{i+2, 0}$ is higher $w(j)$-torsion for all $j \geq n$.

For the proof of this lemma, we recall the base wise reduced powers defined by Araki.

Theorem 2.5. (Araki [2]) There are cohomology operations

$$s \Theta^*: \quad E_r^{a, b} \rightarrow E_r^{(r-2) \cdot a, (2s-2) \cdot (p-1) \cdot b}$$

$$s \beta F^*: \quad E_r^{a, b} \rightarrow E_r^{(r-2) \cdot a, (2s-2) \cdot (p-1) \cdot b}$$

which satisfy the naturality and Cartan formula.
Proof of Lemma 2.4. We use induction on \( i \). Suppose (i)-(iv) for \( i-1 \). First we will prove (iv) i.e.,

\[ (1) \ H^*(E_{*i-1}l^s\cdot\cdot\cdot, z(i+1)) \text{ is higher } w(j) \text{ torsion.} \]

Here \( H(A, z) \) means the homology with the differential \( d=za \) for \( a \in A \). Let us write by \( TC E_{*i-1}l^s\cdot\cdot\cdot \) the higher \( w(j) \)-torsion parts and \( F=E_{*i-1}l^s\cdot\cdot\cdot \). By the inductive assumption, \( H(E_{*i-1}l^s\cdot\cdot\cdot, z(i)) \cong E_{*i-1}l^s\cdot\cdot\cdot \) is higher \( w(j) \)-torsion. Hence for \( 2p^{i-1}+2 < r \leq 2(p-1)p^{i-1} \), we see \( \text{im} d \subset F \). Therefore

\[ (2) \ E_{*i-1}l^s\cdot\cdot\cdot \subset \text{higher } w(j) - \text{torsion} \subset F. \]

Next we consider the Kudo transgression \( d_{*i-l(i-1)+1} \). Let us write simply \( q=2(p-1)p^{i-1} \). Recall that \( E_{*i-1}l^s\cdot\cdot\cdot \) contains \( z(i) \) and is a submodule of \( \text{Ker}(z(i)) \cong H(E_{*i-1}l^s\cdot\cdot\cdot, z(i)) \Omega \text{Im} z(i) \).

Since \( \text{Im} z(i) \) is \( S(i-1) \)-free from (ii), if \( \text{Ker}(d_{*i-1}) \cap \text{Im} z(i) \neq 0 \), then it is a contradiction because \( E_{*i-1}l^s\cdot\cdot\cdot \) is \( w(i) \)-torsion since so is \( 1 \). Therefore \( \text{Ker}(d_{*i}) \cap \text{Im} z(i) = 0 \). Since \( H(E_{*i-1}l^s, z(i)) \) is higher \( w(j) \)-torsion, given \( a \in E_{*i-1}l^s\cdot\cdot\cdot \) we get \( w(i+1)a \in \text{Im} z(i) \) for some large \( s \). Hence \( E_{*i-1}l^s\cdot\cdot\cdot \) is higher \( w(j) \)-torsion. Then we also show, for \( 2(p-1)p^{i-1}+1 < r \leq 2p^i \),

\[ (3) \ E_{*i-1}l^s\cdot\cdot\cdot \subset \text{higher } w(i) - \text{torsion} \subset F/(w(i))/(\text{higher } w(j) - \text{torsion}). \]

Let \( x \in E_{*i-1}l^s\cdot\cdot\cdot \) and \( x \in \text{Ker}(z(i+1)) \). From (3) we can write in \( E_{*i-1}l^s\cdot\cdot\cdot \)

\[ (4) \ z(i+1)x = w(i+a) \text{ with } t \text{; higher } w(j) - \text{torsion mod }(w(i)). \]

Therefore for large \( s \), we have

\[ (5) \ z(i+1)w(i+1)x = w(i)a' \]

We consider Araki's reduced powers

\[ s\rho^s, s\rho^s' : E_{*i-1}l^s\cdot\cdot\cdot \rightarrow E_{*i-1}l^s\cdot\cdot\cdot. \]

Act \( s\rho^s \) to (5). Since \( w(i)z(i+1) = 0 \) in \( E_{*i-1}l^s\cdot\cdot\cdot \) and \( s\rho^s'z(i+1) = w(i+1) \), we get in \( E_{*i-1}l^s\cdot\cdot\cdot \)

\[ (6) \ w(i+1)x = w(i+1)a'. \]

Multiply \( z(i+1) \) to (5), we know \( w(i)z(i+1)a'=0 \). Act \( s\rho^s \) to this, and we have

\[ w(i+1)a'=0 \text{ in } E_{*i-1}l^s\cdot\cdot\cdot. \]

From (6)

\[ (7) \ w(i+1)x = 0 \text{ in } E_{*i-1}l^s\cdot\cdot\cdot. \]
From (3), this means
\[ w(i+1)^{s+1}x = w(i)a^{s+1} + t' \quad \text{in } E_{2p^{s+1}}}.

as (4). Multiply \( w(i+2)^{s} \) to this for large \( s' \), we get \( w(i+1)^{s+1}x = w(i)a^{s+1} \).

Operate \( sP^{s} \cdots \) on this. Thus we prove
\[ (8) \quad w(i+2)^{s+1}x = 0 \quad \text{in } E_{2p^{s+1}}. \]

Continue this argument and we show (1), i.e., (iv). The arguments (7) to (8) implies (iii).

We already know, for \( q=2(p-1)p^{-1} \), \( d_{s+1} : S(i-1)\{z(i)\} \cong \text{Ideal}(w(i)) \) in \( S(i-1) \), by the arguments before (3). Suppose \( d_{s+1}(au^{q/2}) = w^{\parallel}0 \) mod \( w(i) \) in \( S(i-1) \). Suppose \( w(i+1)^{s}w^{\parallel}0 \) for all \( s \) in \( S(i-1)/(w(i)) = S(i) \) since \( w(i+1) \) is non zero divizer in \( S(i) \). On the other hand \( H(E_{2p^{s+1}}) \) is higher \( w(j) \)-torsion, we get \( w(i+1)^{s}a \in \text{Im}(i) \) for large \( s \).

This means \( w(i+1)^{s}w^{\parallel}0 \) mod \( w(i) \) and this is a contradiction. Hence \( 1 \) and \( z(i+1) \) are \( S(i)- \) free in \( E_{2p^{s+1}} \). From (3), so are in \( E_{2p^{s+1}} \). Therefore we show (i) and (ii). q.e.d.

From (3) in the above proof, we also get:

Corollary 2.6. With modulo higher \( w(j) \)-torsion, there is the isomorphism

\[ E_{2p^{s+1}} \cong S_{\infty} \otimes \Delta_{\infty}/(z(1), \ldots, z(i), w(1), \ldots, w(i)). \]

§ 3. Extra special \( p \)-groups

Let \( E_{n} \) be the extra special \( p \)-group of the order \( 2p^{n+1} \) and the exponent \( p \)

\[ E_{n} = \langle a_{1}, \ldots, a_{2n}, c | a_{i}^{p} = c^{p} = 1, c \in \text{Center} \rangle \]

\[ [a_{i}, a_{j}] = \begin{cases} c & \text{if } i=2k-1, j=2k \\ 1 & \text{other } i<j \end{cases} \]

Consider central products \( \hat{E}_{n} = E_{n} \times_{c} S^{1} \) and \( \hat{E}(s)_{n} = E_{n} \times_{c} Z/p^{s} \). Then there are central extensions

\[ (3.2) \quad 1 \rightarrow S^{1} \rightarrow \hat{E}_{n} \rightarrow \otimes^{\ast}Z/p \rightarrow 1 \]

\[ (3.3) \quad 1 \rightarrow Z/p^{s} \rightarrow \hat{E}(s)_{n} \rightarrow \otimes^{\ast}Z/p \rightarrow 1 \]
and induced spectral sequence $E_r^{*,*}$ and $E(s)^{*,*}$ from (3.2) and (3.3) respectively. The spectral sequence $E_r^{*,*}$ satisfies (2.2) and hence Lemma 2.4.

Let $H^*(Z/p^s) \cong Z/p[u] \Theta \Lambda(z)$. If $s \geq 2$, then $d_z z = 0$ by $\Theta z = 0$ and the symmetry of $E(s)_n$. Thus

$$E(s)^{*,*} \cong E_r^{*,*} \Theta \Lambda(z) \text{ for } s \geq 2.$$  
Therefore (i), (ii) in Lemma 2.4 satisfies for (3.3).

Corollary 3.5. ([7]) In $H^*(\tilde{E}_n)$ or $H^*(\tilde{E}(s)_n)$, $s \geq 2$, the $S_n$-submodule generated by $1$ is $S_n/(w(1), \ldots, w(n))$.

Moreover for $n = 2$, the spectral sequence $E_r^{*,*}$ is given completely in [8].

§4. Periodic modules with large period.

Let $k$ be an algebraic closure of $F_p$. Let $\Omega_0^r(M)$ be the $r$-th kernel in the minimal resolution of $k(G)$-module $M$, i.e., if

$$0 \rightarrow M_r \rightarrow Q_{r-1} \rightarrow \ldots \rightarrow Q_0 \rightarrow M \rightarrow 0$$

is exact and if each $Q_i$ is projective, then $M \cong \Omega_0^r(M) \otimes Q$ for some projective module $Q$. A $G$-module $M$ is said to be periodic if $\Omega_0^m(M) \cong M$ for some $m \geq 0$. The smallest of such $m$ is called the period of $M$.

We denote by $V_0(k)$, the variety defined by commutative ring $H^*(G;k) / \Theta$. For a $G$-module $M$, let $I_0(M)$ be the annihilator in $H^*(G;k)$ of $\text{Ext}_{k(G)}^*(M,M) \cong H^*(G, \text{Hom}_k(M,M))$. Let $V_0(M)$ be the subvariety of $V_0(k)$ associated to $I_0(M)$.

Remark that if $V$ is a closed homogeneous subvariety of $V_0(k)$, then there is a $k(G)$-module $M$ with $V_0(M) = V$ (Proposition 2.1 (vii) in [3]).

We recall arguments of Andrews and Benson-Carlson [3]. Consider a central extension of a finite group

$$1 \rightarrow \mathbb{Z}/p \rightarrow G \rightarrow E \rightarrow 1.$$  

Let $\mathbb{Z}/p$ denote the sum $\sum_{z \in \mathbb{Z}/p} z$ as an element of the group ring $k(\mathbb{Z}/p)$. Then for $r > 0$, $\mathbb{Z}/p \Omega_0^{2r}(k)$ is a $k(G)$-module with $\mathbb{Z}/p$-acting trivially, so we may regard it as a $k(E)$-module. We set
(4.3) \( V_r = V_k(\tilde{\mathbb{Z}}/p \tilde{\mathbb{Q}}^{\mathfrak{p}}(k)) \subset V_k(k) \).

Theorem 4.4. (Andrews) Let \( \mathcal{M} \) be an indecomposable \( k(E) \)-module regard as a \( k(G) \)-module by inflation. Then \( \mathcal{M} \) is a periodic \( k(G) \)-module of periodic dividing \( 2r \) if and only if \( V_k(\mathcal{M}) \cap V_r = \{0\} \).

Theorem 4.4 (Benson-Carlson [3]) Let \( E^{*,*} \) be the spectral sequence induced from (4.2). Let \( I_p^a \subset H^*(E) \) be the Kernel of the induced map \( E^{*,*}_2 \rightarrow E^{*,*}_{2,1} \). Then \( V_p^a = V_k(I_p^a) \).

Lemma 4.5. ([3] Proposition 2.2.) If \( \mathcal{M} \) is a periodic \( k(G) \)-module, then the period of \( \mathcal{M} \) divides \( 2\langle G;E \rangle \) where \( E \) is a maximal elementary abelian \( p \)-groups of \( G \).

Theorem 4.6. Let \( G \) be the \( p \)-group \( \tilde{\mathbb{E}}(s)_a, s \not\equiv 2 \). Then there are periodic \( k(G) \)-modules of period \( 2^s \) for \( a \not\equiv n \), and no higher period.

Proof. (See the proof of Corollary 6.2 in [3].) From above lemma, the only possible periods are \( 2p^s \) for \( a \not\equiv n \). By Lemma 2.4 in section 2 and Theorem 4.4, for \( a \not\equiv n \) we may find a closed homogeneous subvariety \( V \) of \( V_k(k) \) with \( \forall V \cap V_{p^{s-1}} = \{0\} \) and \( V \cap V_{p^s} = \{0\} \). By the remark after the definition of \( V_0(\mathcal{M}) \), we may find a \( k(E) \)-module \( \mathcal{M} \) with \( V_k(\mathcal{M}) = V \). Then by the Andrews theorem \( \mathcal{M} \) \( V_{p^{s-1}}(\mathcal{M}) \mathcal{M} \) but \( \mathcal{M} \) has period exactly \( 2p^s \). q.e.d.
References


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