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On some results of the cohomology of extra special p-groups

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Extra special p-groups are central extensions of \( \mathbb{Z}/p \) by elementary abelian p-groups. These groups occupy a distinctive in the cohomology and representation theories of finite groups. Quillen decided mod 2 cohomology of the extra special 2-groups \([Q]\). However the corresponding calculation for odd \( p \) is still unknown. Tezuka-Yagita studied the varieties defined from its mod \( p \) cohomology \([T-Y]\). Extending these results, Benson-Carlson decided the mod \( p \) cohomology modulo Jacobson radical \([B-C]\). The radical parts seem very difficult. For the group of the order \( p^3 \), Lewis decided the integral cohomology and Leary wrote down the mod \( p \) cohomology completely \([Lw],[L2]\).

Minh computed the mod 3 cohomology of the group with the order \( 3^6 \) and of the exponent \( 3^8 \) \([M]\).

One of main results of this paper, is to give the additive structure of the mod \( p \) cohomology of the group with the order \( p^6 \) and its exponent \( p \) (Theorem 8.25). The another results are existence of groups and their modules which period is exactly \( 2p^n \) for each \( n \).

\section{1. Extra special p-group.}

An extra special p-group \( G \) is a group such that its center is \( \mathbb{Z}/p \) and there is the central extension

\[
1 \longrightarrow \mathbb{Z}/p \longrightarrow G \overset{\pi}{\longrightarrow} V \longrightarrow 1
\]

where \( V = \mathbb{Z}/2Z/p \).

Such group is isomorphic to the \( n \)-th central product \( E \ldots E = E_n \) or \( E_{n-1} M \) where \( E \) (resp. \( M \)) is the non abelian group of the order \( p^3 \) and exponent \( p \) (resp. \( p^2 \)). Hence we can explicitely write
The group $E_{n-1}M$ is written similarly except for $a_{2n}^p=c$.

Let us write by $x_i \in H^i(V) = \text{Hom}(V, Z/p)$ the dual of $t_i(a_i)$ and write $y_i = \phi x_i$.

Then the cohomology of $V$ is

$$H^*(V) \cong S_{2n} \otimes \Lambda_{2n} \text{ with } S_{2n} = Z/p[y_1, \ldots, y_{2n}] \text{ and } \Lambda_{2n} = \Lambda(x_1, \ldots, x_{2n}).$$

Proposition 1.3. The extension (1.1) represents element in $H^s(V)$

$$f = \sum x_{2i-1}x_i \text{ (resp. } \sum x_{2i-1}x_{2i}y_{2n}) \text{ for } G = E_n \text{ (resp. } E_{n-1}M).$$

We consider spectral sequence induced from (1.1)

$$(1.4) \quad E_2^{s,m} = H^s(V; H^m(Z/p)) \cong S_{2n} \otimes \Lambda_{2n} \otimes Z/p[u] \otimes \Lambda(z) \xrightarrow{d_2} H^4(G)$$

with $\phi z = u$. From Proposition 1.3, we know

$$(1.5) \quad d_2 z = f.$$

By transgression theorem,

$$d_2 z = \phi d_2 z = \sum y_{2i-1}x_{2i-1}y_{2i}x_{2i-1}.$$

By Kudo's transgression theorem

$$d_2 u^{s+1} = \phi d_2 z = \sum y_{2i-1}x_{2i-1}y_{2i}x_{2i-1}.$$

Let us write (1.6) = $z_n(1)$, (1.7) = $z_n(s+1)$, (1.8) = $w_n(s+1)$. Hence $E_\infty^{s,0}$ is a quotient of

$$E = S_{2n} \otimes \Lambda_{2n} / (f, z_n(1), \ldots, z_n(n), w_n(1), \ldots, w_n(n))$$

We also know that $u^{sH}$ is a permanent cycle because which represents $p^{s+1}$-th Chern class of induced representation from a maximal elementary abelian $p$-group. Write by $u'$ a corresponding element in $H^*(G)$. Then $H^*(G)$ is a $E\otimes Z/p[u']$-module. From Benson-Carlson [B-C].


Note the regularity of the sequence $w_1(1), \ldots, w_n(n)$ in $S_{2n}$ is shown in Tezuka-Yagita [T-Y].
§2. \( \tilde{G} \); the central product of \( G \) and \( S^1 \).

The spectral sequence (1.4) is very complicated even for \( E \) (see [K-S-T-Y] (5.11)). Hence we consider another arguments which are used by Krogholler, Leary, Huebshmann and Moselle. Embed \(<c>\cong \mathbb{Z}/p\subset S^1\) and consider the central product

(2.1) \[ G = G \times_{<c>} S^1. \]

Note that \( \tilde{E}_n \cong \tilde{E}_{n-1} \), indeed, take \( a_{2n} c^{-1/p} \) as \( a_{2n} ^\text{th} \) if \( a_{2n} ^\text{th} = c \). Then we have the exact sequence

(2.2) \[ 1 \longrightarrow S^1 \longrightarrow \tilde{G} \longrightarrow V \longrightarrow 1 \]

and induced spectral sequence

(2.3) \[ E_2^{s,t} \cong H^s(V; H^t(BS^1)) \]
\[ \cong S_{2n} \otimes A_{2n} \otimes \mathbb{Z}/p[u] \longrightarrow H^*(\tilde{G}) \]

Then differentials (1.6)–(1.8) also hold but \( d_4 = 0 \) by the dimensional reason.

Given \( H^*(\tilde{G}) \), to see \( H^*(G) \) we use the following fibration induced from (2.1)

(2.4) \[ S^1 \cong \tilde{G}/G \longrightarrow BG \longrightarrow B\tilde{G}. \]

The induced spectral sequence is

(2.5) \[ E_3^{s,t} \cong H^s(\tilde{G}; H^t(S^1)) \]
\[ \cong H^s(\tilde{G}) \otimes A(z) \longrightarrow H^*(G) \]

and \( d_4 z = f \) (1.5). Therefore

Proposition 2.6. \[ H^*(G) \cong (\text{Ker}(f)|H^*(G)\{z\}G^*(G)/f). \]

§3. \( 2p \)-terms for \( E_n \) for \( n < p \).

In sections 3–5, we consider spectral sequence (2.3) for \( E_n \) for \( p < n \). Given graded algebra \( A \) and \( z \in A^\text{deg} \), we define homology \( H(A,z) \) with the differential \( d_z \)
\( (a) = za \). The first non zero differential in (2.3) is \( d_4 u = z_0(1) \). Hence

(3.1) \[ E_4^{s,t} \cong \begin{cases} S_{2n} \otimes A_{2n}/(z_0(1)) \text{ if } j \equiv 0 \mod p \\ H(S_{2n} \otimes A_{2n}, z_0(1)) \text{ if } j \equiv -2 \\ \text{Ker}z_0(1) \text{ if } j \equiv -1 \end{cases} \]

(3.1) \[ E_4^{s,t} = 0. \]
We can prove

\[(3.7)\quad E_{p+j} \cong \begin{cases} \mathbb{Z}/p[f^n] & \text{for } j=0 \pmod{p} \\ S_{\mathbb{Z}[\Lambda]}/(z_n(1),w_n(1),z_n(2)f^{n-1}) & \text{for } 1 \leq j < p-1 \\ 0 & j=p-1. \end{cases} \]

4. \(E_{p+z}\)-term.

The next differential is (1.7)

\[(4.1)\quad d_{2p+i}(u^p) = \mathcal{O}z_n(1) = z_n(2).\]

Let \(E = S_{\mathbb{Z}[\Lambda]}/(z_n(1),w_n(1))\). Then we get from (3.7)

\[(4.2)\quad E_{p+z} = \begin{cases} E/(z_n(2)) & \text{for } j=0 \pmod{p} \\ H(E/z_n(2)f^{n-1},z_n(2)) & 0 < j \leq p-2 \\ \ker (z_n(2)|E/z_n(2)f^{n-1}) & j=p-1 \end{cases} \]

\[E_{p+z} = \mathbb{Z}/p[f^n] \quad 0 < i \leq p-2 \]

5. Homology of \(H(E_i/E_{i+1})\).

Proposition 5.18. \(H(E,z_n(2))^{\text{odd}} \cong H(E,z_n(2))^{\text{even}} = \mathbb{Z}/p\{f^n\}\)

and \(H(E,z_n(2))^{\text{odd}} \cong \{y_{i,j} = (y_{i,j} - y_{i,j} f^{n-1})/y_{i,j}\}\)

where we express \(x_i' = x_i f^{n-1}, x_i'' = y_{i,j} f^{n-1}, y_{i,j} = y_{i,j} - y_{i,j} f^{n-1}\).

From (4.2) we get

Corollary 5.19. \(H(E/z_n(2)f^{n-1},z_n(2))\) is generated by \(f^{n-1}\) as \(S_{\mathbb{Z}[\Lambda]}\)-module and

\[H(E/z_n(2)f^{n-1},z_n(2))^{\text{odd}} = H(E,z_n(2))^{\text{odd}},\]

\[H(E/z_n(2)f^{n-1},z_n(2))^{\text{even}} = \mathbb{Z}/p\{f^n\}\).

6. \(E_{p+1}\)-term for \(\mathcal{G}\).

Let \(y_{i,j} = y_{i,j} - y_{i,j} f^{n-1}/y_{i,j}\) and \(y_{i,j}' = (y_{i,j} - y_{i,j} f^{n-1})/y_{i,j}\).

Therefore we can prove
Proposition 6.14. For $n=2$ case
\[ E_{2(p-1)p+2}^+ \cong \begin{cases} S_4 \otimes \Lambda_4 / (z_2(1), z_2(2), w_2(1), w_2(2), (y_{21} + y_{43}) \beta(x_1 x_3)) \\ j=0 \mod p \\ S_4 \otimes \Lambda_4 / (y_{ij}, y_{ij}, y_{ij}y_{ij} - y_{ij}x_1, x_nx_3((h, k) = (1, 2), (3, 4))) \\ 0 < j < p-1 \mod p \end{cases} \]

In the next section, we will prove also

Theorem 6.15. $E_{2(p-1)p+2}^+ \cong E_{\infty}^+$. 

§ 8. $\text{Ker } f$ in $H^*(E_2)$.

Theorem 8.25. There is an additive isomorphism
\[ H^*(E_2) \cong (A^j(f) \oplus (\text{Ker}(f)A)^z) \oplus \langle 1 \leq i \leq p-2 \rangle (H^i(f_1) \oplus H^i(f_2)) \]
\[ \oplus \langle 1 \leq i \leq p-3 \text{ and } i \neq 1, \neq 0 \mod p \text{ or } \equiv (p-1) \rangle (Z/p(z_i) \oplus Z/p^2(z_i))) \]
\[ \otimes Z/p^{u^2} \]

where

(i) $A \cong S_4 \otimes \Lambda_4 / (z_2(1), z_2(2), w_2(1), w_2(2), (y_{21} + y_{43}) \beta(x_1 x_3), z_2(3))$

with $S_4 \otimes \Lambda_4 = z_2(p), \beta \otimes \Lambda(x_1, \ldots, x_4), z_2(1) = \beta f, z_2(2) = P \beta f, z_2(3) = P^p \beta f$ and $w_2(1) = \beta z_2(2), w_2(2) = \beta z_2(3)$

(ii) $f = \{x_1 x_2 + x_3 x_4\}$

(iii) (Proposition 8.2) $\text{Ker}(f)A$ is generated as an $S_4$-module by

\[ y_i y_{ji}, y_j x_i - y_i x_j, y_{ji} x_i, \]

\[ f_i x_k x_i((k, h) \neq (1, 2), \neq (3, 4)), \]

\[ x_i x_j x_k, x_1 x_2 x_3 x_4, \]

where $y_{ji} = y_{ji} - y_{ij} y_{ji}^{-1}$

(iv) $z$ corresponds non zero element in $H^1(S^1) = B^2_{0, 1}$ in (2.4)

(v) (Proposition 5.18) $H^1 \text{odd} \cong H^1 \text{even} / (Z/p(1))$ and $H^1 \text{odd} \cong S_4 \{x_1, \ldots, x_4\} / (y_{ji} x_j, y_{ji} x_k = y_k x_i)$

(vi) $f = \{f \text{a}^1\}$ in the spectral sequence (2.2)

(vii) $z_i = \{f \text{a}^1\} = \{x_1 x_2 x_3 x_4 u^1\}$ in (2.2).
§2. Hochschild-Serre spectral sequence.

We consider the spectral sequence with $E_2$-term

$$E_2^{s,t} = H^s(\Theta_t^*Z/p; H^t(\text{BS}^1)).$$

In this paper cohomology $H^*(-)$ always means the $Z/p$-coefficient $H^*(-;Z/p)$. Let us write

$$H^* (\mathbb{Z}/p) = S_{\mathbb{Z}/p} \otimes_{\mathbb{Z}/p} H^* (\text{BS}^1) \cong \mathbb{Z}/p[u]$$

with $S_{\mathbb{Z}/p} = \mathbb{Z}/p[y_1, \ldots, y_n]$, $\Lambda_{\mathbb{Z}/p} = \Lambda(x_1, \ldots, x_n)$, $\beta x_i = y_i$.

We assume first non zero differential

$$(2.2) \quad d_{\mathbb{Z}/p} \beta f \text{ with } f = \sum_{i=1}^n x_{i-1} x_i.$$ 

Then by Cartan-Serre and Kudo transgression theorems, we know

$$(2.3) \quad d_{\mathbb{Z}/p}^i \beta f = z(i), \quad d_{\mathbb{Z}/p} (\beta f \beta f) = w(i)$$

with

$$z(i) = \beta \beta f \Delta^2 \ldots \beta \beta^t f = \sum_{j=1}^i x_j - y_j, \quad w(i) = \beta \beta f \Delta^2 \ldots \beta \beta^t f = \sum_{j=1}^i x_j - y_j.$$

Let us write $S(i) = S_{\mathbb{Z}/p}/(w(1), \ldots, w(i))$. Recall $(w(1), \ldots, w(n))$ is regular in $S_n$ [7].

Lemma 2.4. For $i \leq n - 1$, we get

(i) 1 is $S(i)$-free in $E_{\mathbb{Z}/p}^{i+1,i}$.

(ii) $z(i+1)$ is $S(i)$-free in $E_{\mathbb{Z}/p}^{i+1,i}$.

(iii) if $x \in E_{\mathbb{Z}/p}^{i+1,i}$ is higher $w(i+1)$-torsion, then $x$ is higher $w(j)$-torsion for all $j > n$ (i.e., $w(j)^* x = 0$ for some $s$ and all $j > n$).

(iv) $E_{\mathbb{Z}/p}^{i+1,i}$ is higher $w(j)$-torsion for all $j > n$.

For the proof of this lemma, we recall the base wise reduced powers defined by Araki.

Theorem 2.5. (Araki [2]) There are cohomology operations

$$s \Theta^* : E_r^{a-b} \rightarrow E_r^{(r-2)+a^* (2s-b)(s-1) + 1},$$

$$s \beta F^* : E_r^{a-b} \rightarrow E_r^{(r-2)+a^* (2s-b)(s-1) + 1},$$

which satisfy the naturality and Cartan formula.
Proof of Lemma 2.4. We use induction on $i$. Suppose (i)-(iv) for $i-1$. First we will prove (iv) i.e.,

(1) $H^*(E_{p,i+1}, z(i+1))$ is higher $w(j)$-torsion.

Here $H(A,z)$ means the homology with the differential $da = za$ for $a \in A$. Let us write by $TC E_{p,i+1}^*, z$ the higher $w(j)$-torsion parts and $F = E_{p,i+1}^*, z$. By the inductive assumption, $H(E_{p,i+1}^*, z(i)) \cong E_{p,i+1}^*, z(i)$ is higher $w(j)$-torsion.

Hence for $2p^{i-1} + 2 \leq r \leq 2(p-1)p^{i-1}$, we see $im_d \subset T$. Therefore

(2) $E_{p,i+1}^*, z(i)$ is higher $w(j)$-torsion, $\supseteq F$.

Next we consider the Kudo transgression $d_{p,i+1} = 1$. Let us write simply $q = 2(p-1)p^{i-1}$. Recall that $E_{p,i+1}^*, z$ contains $z(i)$ and is a submodule of $H(E_{p,i+1}^*, z(i)) \cong \text{Im}_z(i)$.

Since $\text{Im}_z(i)$ in $E_{p,i+1}^*, z$ is $S(i-1)$-free from (ii), if $\text{Ker}(d_{p,i+1}) \cap \text{Im}_z(i) \neq 0$, then it is a contradiction because $E_{p,i+1}^*, z(i)$ is $w(i)$-torsion since so is 1. Therefore $\text{Ker}(d_{p,i+1}) \cap \text{Im}_z(i) = 0$. Since $H(E_{p,i+1}^*, z(i))$ is higher $w(j)$-torsion, given $a \in E_{p,i+1}^*, z$ we get $w(i+1)a \in \text{Im}_z(i)$ for some large $s$. Hence $E_{p,i+1}^*, z$ is higher $w(j)$-torsion. Then we also show, for $2(p-1)p^{i-1} + 1 \leq r \leq 2p^i$,

(3) $E_{p,i+1}^*, z(i)$ is higher $w(j)$-torsion $\supseteq F/(w(i))/(w(j)$-torsion).

Let $x \in E_{p,i+1}^*, z$ and $x \in \text{Ker}(z(i+1))$. From (3) we can write in $E_{p,i+1}^*, z$.

(4) $z(i+1)x = w(i)a + t$ with $t$; higher $w(j)$-torsion mod $(w(i))$.

Therefore for large $s$, we have

(5) $z(i+1)w(i+1)a = w(i)a$.

We consider Araki's reduced powers

$$s \triangleright E_{p,i+1}^*, z \rightarrow E_{p,i+1}^*, z$$

Act $s \triangleright E^*$ to (5). Since $w(i) = z(i+1) = 0$ in $E_{p,i+1}^*, z$ and $z(i+1)w(i+1) = w(i+1)$, we get in $E_{p,i+1}^*, z$.

(6) $w(i+1)a'x = w(i+1)a$.

Multiply $z(i+1)$ to (5), we know $w(i)z(i+1)a = 0$. Act $s \triangleright E^*$ to this, and we have

(7) $w(i+1)a = 0$ in $E_{p,i+1}^*, z$.

From (6)
From (3), this means
\[ w(i+1)^* x = w(i)a'' + t' \quad \text{in} \quad E_{Z^+ Z*0}. \]
as (4). Multiply \( w(i+2)^* \) to this for large \( s' \), we get \( w(i+1)^* x = w(i)a'' \). Operate \( sP^{i*1} * \) on this. Thus we prove
\[ \text{as (8)} \]
\[ w(i+2)^* x = 0 \quad \text{in} \quad E_{Z^+ Z*0}. \]
Continuing this argument and we show (1), i.e., (iv). The arguments (7) to (8) implies (iii).

We already know, for \( q=2(p-1)p^{-1} \), \( d_s : S(i-1)/\{ z(i) \} \cong \text{Ideal}(w(i)) \) in \( S(i-1) \), by the arguments before (3). Suppose \( d_{s,s}(au^{s/z}) = w \not\equiv 0 \mod (w(i)) \) in \( S(i-1) \) (or \( \not\equiv 0 \mod (w(i)z(i)) \) in \( S(i-1)/\{ z(i) \} \)). Then \( w(i+1)^* w \not\equiv 0 \) for all \( s \) in \( S(i-1)/(w(i)) = S(i) \) since \( w(i+1) \) is non zero divisor in \( S(i) \). On the other hand \( H(E_{Z^+ Z*0}) \) is higher \( w(j) \)-torsion, we get \( w(i+1)^* \not\equiv 0 \mod (w(i)) \) and this is a contradiction. Hence \( 1 \) and \( z(i+1) \) are \( S(i) \)-free in \( E_{Z^+ Z*0} \). From (3), so are in \( E_{Z^+ Z*0} \). Therefore we show (i) and (ii). q.e.d.

From (3) in the above proof, we also get:

Corollary 2.6. With modulo higher \( w(j) \)-torsion, there is the isomorphism
\[ E_{Z^+ Z*0} \cong S_{2n_0} \otimes \Lambda_{2n}/(z(1), \ldots, z(i), w(1), \ldots, w(i)). \]

§ 3. Extra special \( p \)-groups

let \( E_n \) be the extra special \( p \)-group of the order \( 2p^{n+1} \) and the exponent \( p \)

\[ E_n = \langle a_1, \ldots, a_{2n}, c | a_i^{p^n} = c^p = 1, c \epsilon \text{Center} \]

\[ [a_i, a_j] = \begin{cases} c & i=2k-1, j=2k \\ 1 & \text{other} \quad i<j \end{cases} \]

Consider central products \( \tilde{E}_n = E_n \rtimes \epsilon \times S^1 \) and \( \tilde{E}(s)_s = E_n \rtimes \epsilon \times Z/p^s \). Then there are central extensions
\[ 1 \longrightarrow S^1 \longrightarrow \tilde{E}_n \longrightarrow \otimes Z/p^s \longrightarrow 1 \]

\[ 1 \longrightarrow Z/p^s \longrightarrow \tilde{E}(s)_s \longrightarrow \otimes Z/p^s \longrightarrow 1 \]
and induced spectral sequence $E_r^{*,*}$ and $E(s)_r^{*,*}$ from (3.2) and (3.3) respectively. The spectral sequence $E_r^{*,*}$ satisfies (2.2) and hence Lemma 2.4.

Let $H^*(\mathbb{Z}/p^s)$ be $\mathbb{Z}/p[u] \otimes \langle z \rangle$. If $s \geq 2$, then $d_z = 0$ by $z = 0$ and the symmetry of $\mathcal{E}(s)_n$. Thus

$$E(s)_r^{*,*} \cong E_r^{*,*} \otimes \langle z \rangle$$

for $s \geq 2$.

Therefore (i), (ii) in Lemma 2.4 satisfies for (3.3).

Corollary 3.5. ([7]) In $H^*(\mathcal{E}_n)$ or $H^*(\mathcal{E}(s)_n)$, $s \geq 2$, the $S_n$-submodule generated by 1 is $S_n/(w(1), \ldots, w(n))$.

Moreover for $n = 2$, the spectral sequence $E_r^{*,*}$ is given completely in [8].

§ 4. Periodic modules with large period.

Let $k$ be an algebraic closure of $F_p$. Let $\Omega_0^r(M)$ be the $r$-th kernel in the minimal resolution of $k(G)$-module $M$, i.e., if

$$0 \rightarrow M_r \rightarrow M_{r-1} \rightarrow \ldots \rightarrow M_0 \rightarrow M \rightarrow 0$$

is exact and if each $M_i$ is projective, then $M_r \cong \Omega_0^r(M) \otimes Q$ for some projective module $Q$. A $G$-module $M$ is said to be periodic if $\Omega_0^m(M) \cong M$ for some $m \geq 0$. The smallest of such $m$ is called the period of $M$.

We denote by $V_0(k)$, the variety defined by commutative ring $H^*(G;k)/\Gamma_0$. For a $G$-module $M$, let $I_0(M)$ be the annihilator in $H^*(G;k)$ of $\text{Ext}_{k[G]}^{*,*}(M,M) \cong H^*(G,\text{Hom}_k(M,M))$. Let $V_0(M)$ be the subvariety of $V_0(k)$ associated to $I_0(M)$.

Remark that if $V$ is a closed homogeneous subvariety of $V_0(k)$, then there is a $K(G)$-module $M$ with $V_0(M) = V$ (Proposition 2.1 (vii) in [3]).

We recall arguments of Andrews and Benson-Carlson [3]. Consider a central extension of a finite group

$$1 \rightarrow \mathbb{Z}/p \rightarrow G \rightarrow E \rightarrow 1.$$}

Let $\mathbb{Z}/p$ denote the sum $\sum_{r \in \mathbb{Z}/p} g$ as an element of the group ring $k(\mathbb{Z}/p)$. Then for $r > 0$, $\mathbb{Z}/p \Omega_0^{r,r}(k)$ is a $k(G)$-module with $\mathbb{Z}/p$-acting trivially, so we may regard it as a $k(E)$-module. We set
(4.3) \[ V_r = V_{\tilde{\tau}}(\tilde{\tau}/p \cup \alpha^{-r}(k)) \subseteq V_{\tilde{\tau}}(k). \]

Theorem 4.4. (Andrews) Let \( M \) be an indecomposable \( k(E) \)-module regard as a \( k(G) \)-module by inflation. Then \( M \) is a periodic \( k(G) \)-module of periodic dividing \( 2r \) if and only if \( V_{\tilde{\tau}}(M) \cap V_{\tilde{\tau}} = (0) \).

Theorem 4.4 (Benson-Carlson [3]) Let \( E_r^{*,*} \) be the spectral sequence induced from (4.2). Let \( I_p^a \subseteq H^*(E) \) be the Kernel of the induced map \( E_2^{*,0} \to E_{2p}^{*,*} \). Then \( V_p^a = V_{\tilde{\tau}}(I_p^a) \).

Lemma 4.5. ([3] Proposition 2.2.) If \( M \) is a periodic \( k(G) \)-module, then the period of \( M \) divides \( 2[G;E] \) where \( E \) is a maximal elementary abelian \( p \)-groups of \( G \).

Theorem 4.6. Let \( G \) be the \( p \)-group \( \tilde{\sigma}(s) \), \( s \geq 2 \). Then there are periodic \( k(G) \)-modules of period \( 2^s \) for \( a \neq n \), and no higher period.

Proof. (See the proof of Corollary 6.2 in [3].) From above lemma, the only possible periods are \( 2p^s \) for \( a \neq n \). By Lemma 2.4 in section 2 and Theorem 4.4, for \( a \neq n \) we may find a closed homogeneous subvariety \( V \) of \( V_{\tilde{\tau}}(k) \) with \( V \cap V_{\tilde{\tau}}^{-1} = (0) \) and \( V \cap V_{\tilde{\tau}} = (0) \). By the remark after the definition of \( V_0(M) \), we may find a \( k(E) \)-module \( M \) with \( V_{\tilde{\tau}}(M) = V \). Then by the Andrews theorem \( \cup_{\tilde{\tau}}^{-1}(M) \downarrow M \) but \( \cup_{\tilde{\tau}}^{-1}(M) \downarrow M \), so \( M \) has period exactly \( 2p^s \). q.e.d.
References


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