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Graded algebras associated with indecomposable vector bundles
over an elliptic curve

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§1. Introduction

Let $X$ be an elliptic curve over an algebraically closed field $k$ with $\text{char}(k) \neq 2$. Our object is to compute the graded algebra

$$ \bigoplus_{i \geq 0} \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{L}^\otimes i) $$

for a line bundle $\mathcal{L}$ and a vector bundle $\mathcal{E}$ over $X$ defined as follows. Choose a point $P \in X$ and let $\mathcal{L} = \mathcal{L}(P)$ be the line bundle associated to the divisor $P$. Vector bundles over $X$ were classified by Atiyah [1]. Among them we choose the following ones. For each positive integer $n$ there exists uniquely an indecomposable vector bundle $\mathcal{E}_n$ of rank $n$ which is a successive extension of the trivial bundle. That is,

$$ \mathbb{O}_X = \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots $$

$$ 0 \rightarrow \mathcal{E}_{n-1} \hookrightarrow \mathcal{E}_n \rightarrow \mathbb{O}_X \rightarrow 0 $$

exact, non split.

Now put

$$ \Lambda(n) = \bigoplus_{i \geq 0} \Gamma(X, \mathcal{E}nd(\mathcal{E}_n) \otimes \mathcal{L}^\otimes i) = \bigoplus_{i \geq 0} \text{Hom}(\mathcal{E}_n, \mathcal{E}_n \otimes \mathcal{L}^\otimes i). $$

We aim to give an explicit description of the algebra $\Lambda(n)$.

§2. Homogeneous coordinate ring

First of all, we look at the algebra

$$ S = \bigoplus_{i \geq 0} \Gamma(X, \mathcal{L}^\otimes i). $$

We know the following presentation of $S$ [2, p. 336].

generators: $t \in S_1$, $z \in S_2$, $y \in S_3$

relation: $y^2 = z(z - t^2)(z - \lambda t^2)$ with $\lambda \in k - \{0, 1\}$. 
Also we have $S_0 = k$, $\dim S_i = i$ for $i > 0$ and a $k$-basis of $S$ is given by $t^i x^j$, $t^i x^j y$ for $i, j \geq 0$. In addition, $X$ is determined by $\lambda$ as

\[ X \cong \{ x_2^2 z_2 = z_0 (z_0 - x_2)(x_0 - \lambda x_1) \} \subset \mathbb{P}^2 \]

\[ \begin{array}{c} \text{We fix } t, x, y, \lambda \text{ throughout.} \\
\end{array} \]

§3. First properties of $\Lambda(n)$

We collect here some properties of $\Lambda(n)$ which are easily proved.

• The functor

\[ \Gamma_* : \text{quasi-coherent } \mathcal{O}_X\text{-mod} \to \text{graded } S\text{-mod} \]

\[ \mathcal{F} \mapsto \bigoplus_{i \in \mathbb{Z}} \Gamma(X, \mathcal{F} \oplus \mathcal{O}^i) \]

is fully faithful, because $\mathcal{L}$ is ample. Hence we have an $S$-algebra isomorphism

\[ \Lambda(n) \cong \text{End}_S(\Gamma_*(\mathcal{E}_n)). \]

We shall describe the $S$-module $\Gamma_*(\mathcal{E}_n)$ in §6.

• $\Lambda(n)$ is a maximal order in $\Lambda(n) \otimes_S \text{Frac}(S) \cong M_n(\text{Frac}(S))$.

• The degree 0 part $\Lambda(n)_0 = \text{End}(\mathcal{E}_n)$ is generated by a single endomorphism $f$ defined by

\[ f : \mathcal{E}_n \to \mathcal{E}_n/\mathcal{E}_1 \cong \mathcal{E}_{n-1} \to \mathcal{E}_n. \]

We have $f^n = 0$ and $\dim \Lambda(n)_0 = n$. We shall construct $f$ explicitly in §7.

• The degree $i$ part $\Lambda(n)_i$ has dimension $n^2 i$ for $i > 0$.

§4. $\Lambda$ as an $R$-algebra

Write $\Lambda = \Lambda(n)$. Put $R = k[t, x]$, a polynomial subalgebra of $S$. Then $S = R \oplus R y$. $\Lambda$ is an $R$-free module of rank $2n^2$. We shall give an $R$-basis of $\Lambda$.

There exist $g \in \Lambda_1$, $h \in \Lambda_2$, $l \in \Lambda_3$ such that the following diagrams commute.

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\varrho} & \mathcal{E} \oplus \mathcal{L} \\
\uparrow & & \downarrow \\
\mathcal{O} & \xrightarrow{t} & \mathcal{L} \\
\end{array}
\]
Here the left vertical arrows are the inclusion map and the right ones are induced by the surjection $E \to O$. An explicit form of $g$ will be given in §7. Then the following monomials form an $R$-basis of $A$.

$$f^i, f^i g f^j, f^i h f^j, f^i l \quad 0 \leq i \leq n - 1, \quad 0 \leq j \leq n - 2.$$  

The quotient $\tilde{A} = A / R_+ A = A / (t, x) A$ is a symmetric graded $k$-algebra of dimension $2n^2$. We have the following isomorphisms of bimodules over $\tilde{A}_0 = A_0$.

$$\tilde{A}_1 \cong \tilde{A}_2 \cong \text{Ker}(\tilde{A}_0 \otimes \tilde{A}_0^\text{mult} \tilde{A}_0)$$  

$\tilde{A}_{i+1} = \tilde{A}_i \quad i > 3.$

§5. $\Lambda$ as a $k$-algebra

Let $n > 2$. Regard $\Lambda$ as a left $\Lambda_0 \otimes \Lambda_0$-module by $(a \otimes b) \cdot c = acb$.

**Proposition.** $\Lambda_+ = \Lambda_1 \oplus \Lambda_2 \oplus \cdots$ is a free $\Lambda_0 \otimes \Lambda_0$-module with basis

$$(gf^{n-1})^i g, \quad (gf^{n-1})^i (gf^{n-2})^j g f^{n-3} g \quad \text{for } i, j \geq 0.$$  

**Theorem.** The $k$-algebra $\Lambda$ is generated by $f$ and $g$. The relations between them are generated by the following ones.

*Case $n$ is even: $f^n = 0$ and $n - 2$ quadratic relations of the form*  

$$gf^k g = A_k \cdot gf^{n-3} g + B_k \cdot gf^{n-1} g$$  

*with $A_k, B_k \in \Lambda_0 \otimes \Lambda_0$ for $0 \leq k \leq n - 2, k \neq n - 3$.*

*Case $n$ is odd: $f^n = 0$ and $n - 2$ quadratic relations as above and one cubic relation of the form*  

$$gf^{n-3} g f^{n-3} g = C \cdot gf^{n-2} g f^{n-3} g + D \cdot gf^{n-1} g f^{n-3} g + E \cdot gf^{n-1} g f^{n-1} g$$  

*with $C, D, E \in \Lambda_0 \otimes \Lambda_0$.***
§6. $S$-module $\Gamma_*(\mathcal{E}_n)$

Put $v = x - (\lambda + 1)t^2$, $u = (x - t^2)(x - \lambda t^2)$. Define a graded $S$-module $M$ as follows. $M$ is $R$-free with basis $\alpha, \beta_i, \gamma_i$ for $i > 0$ with $\deg \alpha = 0$, $\deg \beta_i = 1$, $\deg \gamma_i = 2$. The action of $y$ on $M$ is given by

$$
\begin{align*}
y\alpha &= x\beta_1 + t\gamma_1 \\
y\beta_i &= -\lambda t^3 O_i \beta_{i-1} - t x \beta_{i+1} + v \gamma_{i-1} - t^2 \gamma_{i+1} \\
y\gamma_i &= x^3 \beta_{i+1} + \lambda t^3 E_i \gamma_{i-1} + tx \gamma_{i+1}
\end{align*}
$$

where $\beta_0 = -t \alpha$, $\gamma_0 = x \alpha$ and $O_i = 1$ for an odd $i$, $O_i = 0$ for an even $i$, $E_i = 1 - O_i$.

For $n \geq 1$ define a graded $S$-submodule $M(n)$ of $M$ to be the free $R$-submodule generated by $\alpha, \beta_i, \gamma_i$ for $1 \leq i \leq n-1$ and $x \beta_n + t \gamma_n$.

**Proposition.** $\Gamma_*(\mathcal{E}_n) \cong M(n)$ as graded $S$-modules.

So we may identify $\Lambda(n) = \text{End}_S(M(n))$.

Though the $S$-module $M$ is not free, the $S[\frac{1}{y}]$-module $M[\frac{1}{y}] = S[\frac{1}{y}] \otimes_S M$ is free with basis $\alpha_i$, $i \geq 0$, given by

$$
\begin{align*}
\alpha_i &= \frac{1}{x} \gamma_i & \text{i: odd} \\
&= -\frac{1}{u} (\lambda t^3 \beta_i - v \gamma_i) & \text{i: even}
\end{align*}
$$

§7. Generators

Let us construct $f, g \in \Lambda$ as endomorphisms of the $S$-module $M(n)$. Define an $S[\frac{1}{y}]$-linear map $f: M[\frac{1}{y}] \to M[\frac{1}{y}]$ by

$$
\begin{align*}
f(\alpha_i) &= \alpha_{i-1} - \frac{\lambda t^3 y}{ux} \alpha_{i-2} + \frac{(\lambda + 1)v + \lambda t^2}{u} \alpha_{i-3} \\
&\quad - \frac{\lambda ty}{u} \alpha_{i-4} + \frac{\lambda vz}{u} \alpha_{i-5} & \text{if } i \text{ is even} \\
f(\alpha_i) &= \alpha_{i-1} + \frac{\lambda t^3 y}{ux} \alpha_{i-2} \\
&\quad + \frac{(\lambda + 1)x - \lambda t^2}{z} \alpha_{i-3} + \frac{\lambda ty}{u} \alpha_{i-4} & \text{if } i \text{ is odd}
\end{align*}
$$
where we understand $\alpha_i = 0$ for $i < 0$. Then

$$f(\alpha) = 0$$

$$f(\beta_i) = \beta_{i-1} + (\lambda + 1)\beta_{i-3} \quad \text{i: even}$$

$$= \beta_{i-1} + (\lambda + 1)\beta_{i-3} + \lambda\beta_{i-5} \quad \text{i: odd}$$

$$f(\gamma_i) = \gamma_{i-1} + (\lambda + 1)\gamma_{i-3} + \lambda\gamma_{i-5} - \lambda t\beta_{i-3} \quad \text{i: even}$$

$$= \gamma_{i-1} + (\lambda + 1)\gamma_{i-3} + \lambda t\beta_{i-3} \quad \text{i: odd}$$

So $M$ and $M(n)$ are stable under $f$. We denote also by $f$ the restrictions of $f$ to $M$ and $M(n)$. Thus $f \in \Lambda(n)_0$ for all $n$.

Secondly, define an $S[y^{-1}]$-linear map $g: M[y^{-1}] \to M(n)[y^{-1}]$ as follows. When $n$ is even,

$$g(\alpha_0) = t\alpha_{n-1} - \frac{y}{z}\alpha_{n-2}$$

$$g(\alpha_1) = \frac{y}{z}\alpha_{n-1} + \frac{t((\lambda + 1)x - \lambda t^2)}{z}\alpha_{n-2} + \frac{\lambda t^2y}{u}\alpha_{n-3}$$

$$g(\alpha_2) = -\frac{\lambda t^2y}{u}\alpha_{n-2} + \frac{\lambda tvx}{u}\alpha_{n-3}$$

$$g(\alpha_i) = 0 \quad \text{for i > 2},$$

and when $n$ is odd,

$$g(\alpha_0) = t\alpha_{n-1} - \frac{vy}{u}\alpha_{n-2}$$

$$g(\alpha_1) = \frac{y}{z}\alpha_{n-1} + (\lambda + 1)t\alpha_{n-2}$$

$$g(\alpha_2) = -\frac{\lambda t^2y}{u}\alpha_{n-2} + \sum_{i \geq 3, \text{odd}} \lambda(-\lambda - 1)^{(i-3)/2}(\lambda\alpha_{n-i} - \frac{vy}{u}\alpha_{n-i-1})$$

$$g(\alpha_i) = 0 \quad \text{for i > 2}.$$
§8. Explicit equations in case $n$ even

When $n$ is even, we can give explicit defining equations for $\Lambda$, using additional generators. We define $e \in \Lambda_0$ and $g_+ \in \Lambda_1$ by

\[
\begin{align*}
e(\alpha_i) &= \alpha_{i-2} \quad \text{for all } i \\
g_+(\alpha_0) &= t\alpha_{n-2} - \frac{vy}{u}\alpha_{n-3} \\
g_+(\alpha_1) &= t\alpha_{n-1} + (\lambda + 1)t\alpha_{n-3} \\
g_+(\alpha_2) &= \frac{vy}{u}\alpha_{n-1} + (\lambda + 1)t\alpha_{n-2} \\
g_+(\alpha_i) &= 0 \quad \text{for } i > 2.
\end{align*}
\]

**Theorem.** If $n$ is even and $n > 2$, the $k$-algebra $\Lambda$ has the following presentation. The generators are $f, e, g, g_+$. The relations are

\[
e^3 = 0
\]

\[
f^2 = (1 + (\lambda + 1)e)(1 + \lambda e)(1 + e)e
\]

\[
fg(1 + (\lambda + 1)e) + (1 + (\lambda + 1)e)gf
\]

\[
= g_+ + (\lambda + 1)eg_+ + (\lambda + 1)g_+e + \lambda e^2g_+ + ((\lambda + 1)^2 + \lambda)eg_+e + \lambda g_+e^2
\]

\[
+ \lambda(\lambda + 1)e^2g_+e + \lambda(\lambda + 1)eg_+e^2
\]

\[
ge\frac{n-4}{2}g = \lambda g_+e\frac{n-2}{2}g_+
\]

\[
g_+e\frac{n-4}{2}g_+ = (\lambda + 1)g_+e\frac{n-2}{2}g_+
\]

\[
ge^i g = ge^j g_+ = 0 \quad \text{for } 0 \leq j \leq \frac{n-6}{2}.
\]

Finally we give another presentation of $\Lambda$ in line with the theorem of §5. Put

\[
c = e \otimes 1, d = 1 \otimes e, p = f \otimes 1, q = 1 \otimes f \in \Lambda_0 \otimes \Lambda_0
\]

and

\[
\alpha = (1 + (\lambda + 1)c)(1 + (\lambda + 1)d) - \lambda^2 c^2 d^2
\]

\[
\gamma = (\lambda + 1)(1 + \lambda c)(1 + c)(1 + \lambda d)(1 + d)
\]

\[
+ \lambda d(1 + \lambda c)(1 + c) + \lambda c(1 + \lambda d)(1 + d)
\]

\[
\beta = (1 + \lambda cd)\alpha - (\lambda + 1)cd\gamma
\]

\[
= 1 + (\lambda + 1)(c + d) + \lambda cd - (\lambda + 1)^2(c^3 d + cd^2)
\]

\[
- ((\lambda + 1)^4 + \lambda(\lambda + 1)^2 + \lambda^2)c^2 d^2 - \lambda(\lambda + 1)^2(c^3 d + cd^2)
\]

\[
- \lambda(\lambda + 1)((\lambda + 1)^2 + \lambda)(c^3 d^2 + c^2 d^2) - \lambda^2((\lambda + 1)^2 + \lambda)c^3 d^2.
\]

Then $\alpha, \beta, \gamma \in \Lambda_0 \otimes \Lambda_0$ and $\beta$ is invertible.
THEOREM. If \( n \) is even and \( n > 2 \), the \( k \)-algebra \( \Lambda \) has the following presentation. The generators are \( f, e, g \). The relations are

\[
e^\frac{\beta}{2} = 0
\]

\[
f^2 = (1 + (\lambda + 1)e)(1 + \lambda e)(1 + e)e
\]

\[
ge^\frac{n-2}{2}g = (\coprod_1 p + \coprod_2 q)ge^\frac{n-2}{2}fg + (\coprod_3 p + \coprod_4 q)ge^\frac{n-2}{2}fg
\]

\[
\coprod_1 = -\frac{1}{\beta}(1 + \lambda d)(1 + d)(1 + (\lambda + 1)d + \lambda cd)
\]

\[
\coprod_3 = \frac{1}{\beta}(1 + \lambda d)(1 + d)\left[(\lambda + 1)(1 + (\lambda + 1)d) + \frac{\lambda c}{(1 + \lambda c)(1 + c)}(1 + (\lambda + 1)d + \lambda cd)\right]
\]

\[
\coprod_1 \leftrightarrow \coprod_2, \quad \coprod_3 \leftrightarrow \coprod_4 \quad \text{by interchange } c \leftrightarrow d
\]

\[
ge^\frac{n-4}{2}g = (\coprod_1 p + \coprod_2 q)ge^\frac{n-4}{2}fg + (\coprod_3 p + \coprod_4 q)ge^\frac{n-4}{2}fg
\]

\[
\coprod_1 = -\frac{1}{\beta}d(1 + (\lambda + 1)d)(1 + (\lambda + 1)c + \lambda cd)
\]

\[
\coprod_3 = \frac{1}{\beta}(1 + (\lambda + 1)d)\left[\lambda(1 + (\lambda + 1)c)(1 + (\lambda + 1)d) - (\lambda + 1)^2\beta + \lambda(\lambda + 1)\gamma cd\right]
\]

\[
\coprod_1 \leftrightarrow \coprod_2, \quad \coprod_3 \leftrightarrow \coprod_4 \quad \text{by interchange } c \leftrightarrow d
\]

\[
ge^\frac{n-k}{2}g = 0 \quad \text{for } k > 4, \text{ even}
\]

\[
ge^\frac{n-k}{2}fg = (\coprod_1 + \coprod_2 pq)ge^\frac{n-k}{2}fg + (\coprod_3 + \coprod_4 pq)ge^\frac{n-k}{2}fg
\]

\[
\coprod_1 = \frac{1}{\beta}((\lambda + 1)\beta - \lambda\gamma cd)
\]

\[
\coprod_3 = \frac{1}{\beta}\lambda(1 + \lambda cd)
\]

\[
\coprod_4 = \frac{1}{\beta}\left(\frac{\lambda\gamma}{(1 + \lambda c)(1 + c)(1 + \lambda d)(1 + d)} + \lambda\lambda(1 + \lambda cd)\right)
\]

\[
ge^\frac{n-k}{2}fg = (\coprod_1 + \coprod_2 pq)ge^\frac{n-k}{2}fg + (\coprod_3 + \coprod_4 pq)ge^\frac{n-k}{2}fg
\]
\[ \square_1 = \frac{1}{\beta}(1 + (\lambda + 1)c)(1 + (\lambda + 1)d) \]
\[ \times (1 - (\lambda + 1)^2 cd - \lambda(\lambda + 1)(c^2 d + cd^2) - \lambda^2 c^2 d^2) \]
\[ \square_2 = -\frac{1}{\beta} \lambda(\lambda + 1)cd \]
\[ \square_3 = -\frac{1}{\beta} (\lambda + 1)(1 + (\lambda + 1)c)(1 + (\lambda + 1)d) \]
\[ \times (1 - ((\lambda + 1)^2 + \lambda)cd - \lambda(\lambda + 1)(c^2 d + cd^2) - \lambda^2 c^2 d^2) \]
\[ \square_4 = \frac{1}{\beta} \left( \frac{\lambda \alpha}{(1 + \lambda c)(1 + c)(1 + \lambda d)(1 + d)} + \lambda(\lambda + 1)^2 cd \right) \]

\( ge^{\frac{-b}{2}} fg = 0 \) for \( k > 8 \), even.

References