On Regular Algebras

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Abstract

The notation of a (non-commutative) regular, graded algebra is introduced in [AS]. The results of that paper, combined with those in [ATV1], gives a complete description of the regular graded ring of (global) dimension three. Further M.Artin [A] defined Quantum Proj for non-commutative graded algebras and studied projective geometry of quantum proj.

In this paper, we shall explain those results.

1 Regular algebras

Let $k$ be an algebraically closed field of characteristic zero. A graded algebra $A$ will mean a (connected) $\mathbb{N}$-graded algebra, generated in degree one; thus $A = \bigoplus_{i \geq 0} A_i$, where $A_0 = k$ is central, $\dim_k A_i < \infty$ for all $i$, and $A$ is generated as an algebra by $A_1$. M.Artin and W.Schelter defined the regular graded algebra as follows.

Definition 1 A graded algebra $A$ is regular of dimension $d$ provided that

1. $A$ has global dimension $d$; that is every graded (left) $A$-module has projective dimension $\leq d$

2. $A$ has polynomial growth; that is there exists $\rho \in \mathbb{R}$ such that $\dim A_n \leq n^\rho$ for all $n$.

3. $A$ is Gorenstein; that is $\text{Ext}^n_A(k, A) = \delta_{d,n} k$

These conditions put strong restriction on $A$. For example, if $A$ is commutative, and regular, then $A$ must be a polynomial ring. If $d = 1$, the only such $A$ is the polynomial ring $k[x]$. If $d = 2$, then $A$ is of the form $k(x,y)$ (free algebra of rank two) with a single quadratic relation, which is either $yx - xy = x^2$, or $yx = \lambda xy$ for some $0 \neq \lambda \in k$. In particular, the quantum plane gives a regular algebra. If $d = 3$, then things begin to get interesting. There are 13 class of regular algebras (for detailed see [AS],[ATV1]), these algebras are of the forms $k(x, y)$ with two cubic relations, or $k(x, y, z)$ with three quadratic relations. However two such classes are
of particular interest.

Fix \((a, b, c) \in \mathbb{P}^2\), and let \(A = C(x, y, z)\) with defining relations

\[
ax^2 + byz + cxy = 0
\]
\[
ay^2 + bzx + cxz = 0
\]
\[
aZ^2 + bxy + cyx = 0
\]

This algebra is very closely related to the subvariety of \(\mathbb{P}^2\), \(E\) say, defied by the equation \((a^3 + b^3 + c^3)xyz - abc(x^3 + y^3 + z^3) = 0\). Usually \(E\) is an elliptic curve. If \((a, b, c) = (0, 1, -1)\), then \(E = \mathbb{P}^2\) and \(A\) is the polynomial ring. Suppose that \((a, b, c)\) is such that \(E\) is an elliptic curve. Then \(A\) is regular algebra, and noetherian domain. In general, let \(A\) be a graded algebra of the form

\[
A = k\langle x_1, \ldots, x_r \rangle / (f_1, \ldots, f_s)
\]

where \(f_i\) are homogeneous elements. Then multilinearization of \(\{f_1, \ldots, f_s\}\) defines a scheme \(E\) in \((\mathbb{P}^{r-1})^{s-1}\). Further projective scheme \(E\) define the homogeneous coordinate ring \(B\). This is isomorphic to \(\oplus_{n \geq 0} \Gamma(E, \varphi)\), where \(\varphi\) is the invertible sheaf \(\vartheta(1)\). Let \(\sigma\) be an automorphism of \(E\) and denote the pullback \(\sigma^* \varphi\) by \(\varphi^\sigma\), then we set

\[
B_n = \Gamma(E, \varphi \otimes \varphi^\sigma \otimes \cdots \otimes \varphi^\sigma^{n-1})
\]

for all \(n \geq 0\) and \(B = \oplus_{n \geq 0} B_n\). Multiplication of section is defined by the rule that if \(a \in B_m\) and \(b \in B_n\), then

\[
a \cdot b = a \otimes b^\sigma^m
\]

If \(E = \text{Spec}(R)\) and \(\sigma\) is an automorphism of \(E\), then \(B = R[t, t^{-1}; \sigma]\), where \(ta = a^\sigma t\). If \(A\) is a regular algebra, then the next theorem is proved in [ATV1].

**Theorem 1** If \(A\) is a regular algebra of dimension 3, then \(\dim E = 1, 2\). If \(\dim E = 1\), then \(A/gA \cong B^\sigma\), where \(g\) is an element of \(A\) such that \(gA = Ag\). If \(\dim E = 2\), then \(A \cong B\).

Next suppose that \(d = 4\). Not all the regular algebras are known for \(d = 4\), however there is one class that has been studied to some extent. This is a family of algebras defined by E.Sklyanin [Sk1],[Sk2]. Let \((\alpha, \beta, \gamma) \in \mathbb{P}^3\) lie on the surface \(\alpha + \beta + \gamma + \alpha \beta \gamma = 0\). Let \(A = C(a, x, y, z)\) with defining relations

\[
ax - xa = \alpha(yz + zy) \quad xy - yx = az + xa
\]
\[
ay - ya = \beta(xz + zx) \quad yz - zy = ax + xa
\]
\begin{align*}
az - za &= \gamma(xy + yx) \\
zx - xz &= ay + ya
\end{align*}

If \(\{\alpha, \beta, \gamma\} \cap \{0, +1, -1\} = \emptyset\), then \(A\) is a regular algebra of dimension 4, and has the same Hilbert series as the polynomial ring. Further if \((\alpha, \beta, \gamma) = (0, \delta, -\delta)\) \(\delta \neq 0, -1\), then \(A\) is a quotient of \(U_q(sl(2))\) (quantum group of \(sl(2)\)).

2 Quantum Proj

Let \(A\) be a finitely generated commutative graded \(k\)-algebra which is generated in degree 1. Let \(X = \text{Proj}(A)\), and denote by \(C\) the quotient category \((\text{gr}-A)/\tau\), where \((\text{gr}-A)\) is the category of finite graded \(A\)-modules and \(\tau\) is its full subcategory of modules of finite length. Serre's theorem (cf. [Se]) asserts that there is a natural equivalence of categories

\[\tau \to (\text{mod} - \vartheta)\]

between the quotient category \(\vartheta\) and the category \((\text{mod} - \vartheta)\) of coherent sheaves on \(\text{Proj}(A)\). The shift \(M(\mu)\) of module \(M\), defined by \(M(\mu)_n = M_{n+\mu}\), correspond to the tensor product by the polarizing invertible sheaf:

\[M \sim M(1) = M \otimes \vartheta(1)\]

This shift operation defines an autoequivalence of \(C\). The class of \(A\)-modules which corresponds to a coherent sheaf \(M\) on \(X\) is represented by the module

\[\Gamma(M) := \bigotimes_{n=0}^{\infty} \Gamma(X, M(n))\]

In particular, \(\Gamma(\vartheta) = \bigotimes_{n} \Gamma(X, \varphi^{\otimes n})\) agree with in a sufficient high degree, where \(\varphi\) is a invertible sheaf. Thus \(\text{Proj}(A)\) can recovered from category \(C\).

M.Artin (cf.[A],[ATV1],[AV]) has used this correspondence to define quantum Proj.

**Definition 2** Let \(A\) be a non-commutative graded algebra, generated in degree 1. Then \(\text{Proj}(A)\) is the triple \((C, \vartheta, s)\), where \(C = (\text{gr}-A)/\tau\), \(\vartheta\) is the object of \(C\) which is represented by the right module \(A\), and \(s\) is the operation \(M \sim M(1)\) on \(C\) induced by the shift of degree on an \(A\)-modules.

Suppose that \(R = C[x_0, \cdots, x_n]/J\) is a graded quotient ring of the commutative polynomial ring endowed with its usual graded structure. Let \(V(J) \subset \mathbb{P}^n\) be the projective variety cut out by \(J\). To each point \(p \in V(J)\) we may associate the
graded $R$ - module $M(p) = R/I(p) \cong C[X]$, where $I(p)$ is the ideal generated by the homogeneous polynomials vanishing at $p$. Since $C[X]$ is a domain, every proper quotient of $M(p)$ is finite dimensional, whence $M(p)$ is an irreducible object in $Proj(R)$. This motivates the following definition.

**Definition 3 ([A], [ATV2])** A point module is a graded cyclic $A$ - module $M$ with Hilbert series $(1 - t)^{-1}$.

A line module is a graded cyclic $A$ - module $M$ with Hilbert series $(1 - t)^{-2}$.

A plane module is a graded cyclic $A$ - module $M$ with Hilbert series $(1 - t)^{-3}$.

By using these modules, projective geometry over graded regular algebras of dimension 3 (quantum plane) is expanded (cf. [A]). In the case of dimension 4, projective geometry of regular algebra which obtained by homogenization of $sl(2)$ ([LBS]).

### 3 Remark and Problem

(1) In the definition of regular algebras, can the Gorenstein condition be changed to domain ? This is true in the case that $gl.dim A \leq 2$ (cf. [K1]) and it is known that regular algebras of dimension $\leq 4$ are Noetherian domain (cf. [SS]).

(2) In the non-graded case, is it possible to define a quantum algebraic geometry ? One direction has suggested by Manin ([M1],[M2]).

### References


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