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On some periodic modules for group algebras of finite groups

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1.Introduction

Let G be a finite group and let k be a field of characteristic $p>0$. Let M be a finitely generated kG-module. Let $\phi : P \longrightarrow M$ be the projective cover of M and let $\Omega(M)$ be the kernel of ϕ. We define inductively as $\Omega^{n+1}(M) = \Omega(\Omega^n(M))$ for any positive integer n. Similarly we define $\Omega^n(M)$ for a negative integer n using the injective hull. We say that M is periodic if $\Omega^n(M) \cong M$ for some $n \geq 1$. If n is the smallest such integer then n is called the period of M.

Let

$$G = G(m,n) = < s,t | s^{p^n} = t^{p^m} = 1, s^{-1}ts = t^{p^{m-n}+1} >$$

be a metacyclic p-group of order p^{m+n} where p is an odd prime and $m-n>0, n>0$. The cohomology ring $H^*(G,k)$ was determined by Diethelm [7]. We shall follow the notation in [7]. By [7,Theorem 2],

$$\begin{aligned}
(1.1) \qquad & H^*(G,k) = k[a_1, \ldots, a_{p-1}, b, y, v, w] \\
& a_i a_j = a_i y = a_i v = b^2 = v^2 = 0, \\
& deg\ a_i = 2i-1, deg\ b = 1, deg\ y = 2, \\
& deg\ v = 2p-1, deg\ w = 2p, \\
& b, y \in Im(inf : H^*(G/<t>, k) \longrightarrow H^*(G,k)), \\
& res^G_{<t>}(a_1) \neq 0.
\end{aligned}$$

(We omit all relations which are consequences of the skew commutative relation.)

Let $\hat{y^i} : \Omega^{2i}(k) \longrightarrow k$ be the cocycle which represents y^i and let L_i be the kernel of $\hat{y^i}$ for $i \geq 1$. Then w generates the periodicity of L_i since $H^*(G,k)$ is finitely generated over $k[y,w]$ as a module(cf.[1,5.10]). Moreover, by [2,Lemma 4.4] and [4,Lemma 4.1], we have the following.

(1.2) *For every $i \geq 1$, L_i is an indecomposable periodic kG-module with period 2 or 2p.*

In [8], Okuyama and Sasaki showed that the period of L_p is exactly $2p$. The following is our main result.

THEOREM. *The period of L_i is $2p$ if $i \geq 2$ and 2 if $i = 1$.*

Let G be an arbitrary finite group. For a kG-module M, we set

$$\begin{aligned}\hat{H}^i(G, M) &= \underline{Hom}_{kG}(\Omega^i(k), M)\\ &= Hom_{kG}(\Omega^i(k), M)/PHom_{kG}(\Omega^i(k), M)\end{aligned}$$

where $PHom_{kG}(\Omega^i(k), M)$ is a subspace of $Hom_{kG}(\Omega^i(k), M)$ generated by projective homomorphisms. If N is a kG-module, there exists a product

$$\hat{H}^i(G, M) \otimes \hat{H}^j(G, N) \longrightarrow \hat{H}^{i+j}(G, M \otimes N).$$

In particular we have the Tate duality, namely,

$$\hat{H}^i(G, k) \otimes \hat{H}^{-(i+1)}(G, k) \longrightarrow \hat{H}^{-1}(G, k) = k$$

is non-degenerate for any i (cf.[6,XII]).

Let $\zeta(\neq 0) \in H^i(G, k)(i > 0)$. Then ζ is represented by $\hat{\zeta} : \Omega^i(k) \longrightarrow k$. We set $L_\zeta = Ker\ \hat{\zeta}$. By definition of L_ζ there exists an exact sequence

$$0 \longrightarrow L_\zeta \longrightarrow \Omega^i(k) \xrightarrow{\hat{\zeta}} k \longrightarrow 0.$$

Hence we have a long exact sequence

$$\text{(1.3)}\qquad \begin{aligned}&\longrightarrow \hat{H}^{j-1}(G, k) \xrightarrow{\delta} \hat{H}^j(G, L_\zeta) \longrightarrow\\ &\hat{H}^j(G, \Omega^i(k)) \cong \hat{H}^{j-i}(G, k) \xrightarrow{\zeta} \hat{H}^j(G, k) \longrightarrow .\end{aligned}$$

Remark 1.4. If H is a subgroup of G and if $p||H|$, then the transfer map

$$t_H^G : \hat{H}^{-1}(H, k) \longrightarrow \hat{H}^{-1}(G, k)$$

is not zero. Indeed, consider the exact sequence

$$0 \longrightarrow \Omega(k) \xrightarrow{\iota} P_0 \xrightarrow{\epsilon} k \longrightarrow 0$$

where $P_0 \xrightarrow{\epsilon} k$ is the projective cover of k as a kG-module. Let $f(\neq 0) \in \hat{H}^{-1}(G, k)$ $= Hom_{kG}(k, \Omega(k))$. Since P_0 is projective there exists $g \in Hom_{kH}(k, P_0)$ such that $\iota \circ f = Tr_H^G(g)$. Since $\epsilon \circ g = 0$, $g = \iota \circ g'$ for some $g' \in Hom_{kH}(k, \Omega(k))$. Then $\iota \circ Tr_H^G(g') = Tr(g) = \iota \circ f$ and so $Tr_H^G(g') = f$.

2.Proof of Theorem

In this section, we assume that $G = G(m, n)$ where p is odd and $m - n > 0, n > 0$. We take the following k-basis of $\hat{H}^i(G, k)(i = 1, 2))$ (cf.(1.1))

$$\begin{aligned} \hat{H}^1(G,k) &: a_1, b \\ \hat{H}^2(G,k) &: a_1 b, y \end{aligned}$$

and the dual basis with respect to the Tate duality,

$$\begin{aligned} \hat{H}^{-2}(G,k) &: (a_1)^*, b^* \\ \hat{H}^{-3}(G,k) &: (a_1 b)^*, y^*. \end{aligned}$$

First we consider the period of L_1. We set $H =< s, z = [s,t] > \triangleleft G$. Then by [7,Theorem 1],

(2.1)
$$\begin{aligned} &H^*(H,k) = k[a', b', x', y'] \\ &deg\ a' = deg\ b' = 1, deg\ x' = deg\ y' = 2, \\ &res^H_{<z>}(a') \neq 0, b' = res^G_H(b), y' = res^G_H(y). \end{aligned}$$

LEMMA 2.2. *$res^G_H(a_1^*) \neq 0$. In particular, $res^G_H((a_1 b)^*) \neq 0$.*

PROOF: Since $res^G_{<t>}(a_1) \neq 0$, $a_1^* = t^G_{<t>}(c)$ for some $c \in \hat{H}^{-2}(<t>, k)$ where $t^G_{<t>}$ is the transfer map (cf.Remark 1.4). Hence $res^G_H(a_1^*) = t^H_{<z>}(res^{<t>}_{<z>}(c)) \neq 0$. Since $b(a_1 b)^* = a_1^*$, it follows that $res^G_H((a_1 b)^*) \neq 0$.

LEMMA 2.3. *There exists $\zeta \in H^2(H,k)$ such that $\zeta res^G_H((a_1 b)^*) = 0$ and $L_1 \otimes L_\zeta$ is a projective kH-module.*

PROOF: Since $y(a_1 b)^* = 0$ and $res^G_H(b(a_1 b)^*) \neq 0$, some k-linear combination of $a'b'$ and x' satisfies the condition of Lemma.

Now consider the following commutative diagram (cf.(1.3))

$$\begin{array}{ccccccc} & & \hat{H}^{-1}(G,L_1) & \longrightarrow & \hat{H}^{-1}(G,\Omega^2(k)) & \xrightarrow{\ y\ } & \hat{H}^{-1}(G,k) \\ & & {\scriptstyle res}\downarrow & & \downarrow{\scriptstyle res} & & \\ & & \hat{H}^{-1}(H,L_1) & \longrightarrow & \hat{H}^{-1}(H,\Omega^2(k)) & & \\ & & {\scriptstyle \zeta}\downarrow & & {\scriptstyle \zeta}\downarrow & & \\ \hat{H}^0(H,k) & \xrightarrow{\ \delta\ } & \hat{H}^1(H,L_1) & \longrightarrow & \hat{H}^1(H,\Omega^2(k)). & & \end{array}$$

Then there exists $e \in \hat{H}^{-1}(G, L_1)$ such that $\zeta res^G_H(e) = \delta(1)$. Note that $\zeta : \hat{H}^{-1}(H, L_1) \longrightarrow \hat{H}^1(H, L_1)$ is an isomorphism. Let $\theta = \delta(1) \in \hat{H}^1(G, L_1) \cong Hom_{kG}(k, \Omega^{-1}(L_1))$.

Then we have the following commutative diagram,

$$\begin{array}{ccc}
Hom_{kG}(\Omega^{-1}(L_1),\Omega(L_1)) & \xrightarrow{\theta^*} & Hom_{kG}(k,\Omega(L_1)) = \hat{H}^{-1}(G,L_1) \\
{\scriptstyle res}\downarrow & & \downarrow{\scriptstyle res} \\
\underline{Hom}_{kH}(\Omega^{-1}(L_1),\Omega(L_1)) & \longrightarrow & \underline{Hom}_{kH}(k,\Omega(L_1)) = \hat{H}^{-1}(H,L_1) \\
{\scriptstyle \zeta}\downarrow & & {\scriptstyle \zeta}\downarrow \\
\underline{Hom}_{kH}(\Omega^{-1}(L_1),\Omega^{-1}(L_1)) & \longrightarrow & \underline{Hom}_{kH}(k,\Omega^{-1}(L_1)) = \hat{H}^{1}(H,L_1).
\end{array}$$

Since y annihilates $Ext_{kG}(L_1,L_1)$ by [5,Theorem 4.1] (or [1,Propostion 5.9.6]) θ^* is onto. Hence $e=\theta^*(f)$ for some $f \in Hom_{kG}(\Omega^{-1}(L_1),\Omega(L_1))$ and $\zeta res_H^G(f)$ is an isomorphism (modulo projective). Hence f is an isomorphism.

Next we consider the case $i \geq 2$. We shall show that

$$res_H^G(y^{i-1}\hat{H}^{-1}(G,L_i)) = 0 \tag{2.4}$$

but

$$res_H^G(y^{i-1}\hat{H}^{1}(G,L_i)) \neq 0.$$

LEMMA 2.5. *Suppose that* $i \geq 2$. *If* $c \in \hat{H}^{-(2i+1)}(G,k)$ *and* $y^i c = 0$ *then* $y^{i-1}c = 0$.

Hence in the following commutative diagram

$$\begin{array}{ccccc}
 & & \hat{H}^{-1}(G,L_i) & \longrightarrow & \hat{H}^{-(2i+1)}(G,k) \xrightarrow{y^i} \\
 & & {\scriptstyle y^{i-1}}\downarrow & & {\scriptstyle y^{i-1}}\downarrow \\
 & \xrightarrow{\delta} & \hat{H}^{2i-3}(G,L_i) & \longrightarrow & \hat{H}^{-3}(G,k)
\end{array}$$

we have $y^{i-1}\hat{H}^{-1}(G,L_i) \subseteq Im\ \delta$. Now consider the commutative diagram,

$$\begin{array}{ccccc}
 & & \hat{H}^{2i-4}(G,k) & \xrightarrow{\delta} & H^{2i-3}(G,L_i) \\
 & & {\scriptstyle y}\downarrow & & {\scriptstyle y}\downarrow \\
\hat{H}^{-2}(G,k) & \xrightarrow{y^i} & \hat{H}^{2i-2}(G,k) & \xrightarrow{\delta'} & H^{2i-1}(G,L_i).
\end{array}$$

Since $y^i\hat{H}^{-2}(G,k) = 0$ (cf.[3,Lemma 2.2]) δ' is monomorphism. If $c = \delta(h) \in y^{i-1}\hat{H}^{-1}(G,L_i)(h \in \hat{H}^{2i-4}(G,k))$ then $yc=0$ since y^i annihilates $Ext^*_{kG}(L_i,L_i)$ ([5,Theorem 4.1] or [2,Proposition 5.9.6]). Hence we have $yh=0$. Since $res_H^G(y)$ is not a zero divisor in $H^*(H,k)$ (cf.(2.1)) we have that $res_H^G(h)=0$ and $res_H^G(c)=0$.

Next consider the following commutative diagram,

$$\begin{array}{ccccc}
 & & H^0(G,k) & \xrightarrow{\delta} & H^1(G,L_i) \\
 & & {\scriptstyle res}\downarrow & & {\scriptstyle res}\downarrow \\
 & & H^0(H,k) & \longrightarrow & H^1(H,L_i) \\
 & & {\scriptstyle y'^{i-1}}\downarrow & & {\scriptstyle y'^{i-1}}\downarrow \\
 & \xrightarrow{y'^i} & H^{2(i-1)}(H,k) & \xrightarrow{\delta''} & H^{2i-1}(H,L_i)
\end{array}$$

Since δ'' is monomorphism we have $res(y^{i-1}\delta(1)) \neq 0$.

Actually, to prove that $\Omega^2(L_i) \not\cong L_i$ for $i \geq 2$, it suffices to consider only the case $i = 2$ by the following Proposition.

PROPOSITION 2.7. *Let M be a non-projective indecomposable kG-module. Suppose that y^i annihilates $Ext^*_{kG}(M, M)$. If $\Omega^2(L_i) \cong L_i$ then $\Omega^2(M) \cong M$.*

PROOF: By[5,Lemma 4.4] (or[1,Proposition 5.9.5]), $L_i \otimes M \cong \Omega(M) \oplus \Omega^{2i}(M) \oplus (proj)$. So the result follows.

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