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Decompositions of the Trivial Module

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The following report is a survey of some of the recent results on the structure of modules, particularly as revealed in the quotient category. This work has been joint with many people including Dave Benson, Peter Donovan, Jeremy Rickard, Geoff Robinson and especially Wayne Wheeler. Even though the study is still in its early stages, it has turned up some very interesting phenomena. It is my expectation that much more can be done in the area.

Throughout the paper we assume that G is a finite group and k is an algebraically closed field of characteristic $p > 0$. All kG-modules are assumed to be finitely generated. We begin in the first theorem with a simple exact sequence. First some notation. Let E be an elementary abelian p-subgroup of G. Let $D_G(E) = D$ be the subgroup of the normalizer $N = N_G(E)$ consisting of all elements whose conjugation action on $E \cong \mathbb{F}_p^r$ is given by a scalar matrix. That is, if $d \in D$ then there exists an element $m \in \mathbb{F}_p$ such that

$$dyd^{-1} = y^m$$

for all $y \in E$. Notice that $D_G(E) = C_G(E)$ if either G is a p-group or $p = 2$.

THEOREM 1. **[C2]** Suppose that E is a maximal elementary abelian p-subgroup of G. Let $m = |N_G(E) : D_G(E)|$. There exists an integer $n > 0$ such that for any $\ell > 0$ there is a projective module P and an exact sequence

$$0 \longrightarrow L \longrightarrow (\Omega^{n\ell}(k))^m \oplus P \longrightarrow k_{D_G(E)}^{\uparrow G} \longrightarrow 0$$

with the property that the variety, $V_E(L_E)$, of the restriction of L to E is a proper subvariety of $V_E(k)$.

Some explanation of the notation is in order. Let

$$\cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} k \longrightarrow 0$$

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be a minimal projective kG-resolution of the trivial module k. Then $\Omega^n(k)$ is the kernel of $\partial_{n-1}, \Omega^n(k) = \ker\partial_{n-1} = \partial_n(P_n)$. It should be noted here that $H^n(G,k) = \mathrm{Ext}^n_{kG}(k,k) \cong \mathrm{Hom}_{kG}(\Omega^n(k),k)$. The statement about varieties is more technicial(see **[B]**, **[E]**), but a simplified version can be given as follows. Suppose that $E = \langle x_1,\ldots,x_n\rangle$ has order p^n. For $\alpha = (\alpha_1,\ldots,\alpha_n) \in k^n$ let $u_\alpha = 1+\sum \alpha_i(x_i-1) \in kE$. Notice that u_α is a unit of order p (assuming $\alpha \neq 0$) in kG. Then the variety $V_E(L_E)$ is isomorphic to

$$\{\alpha \in k^n | L_{\langle u_\alpha\rangle} \text{ is not a free } k\langle u_\alpha\rangle\text{-module }\} \cup \{0\}.$$

So the following are equivalent:
(A) $V_E(L_E)$ is a proper subvariety of $V_E(k) \cong k^n$ and
(B) There exists $\alpha \in k^n$, $\alpha \neq 0$ such that L is free as a $k\langle u_\alpha\rangle$-module.

From a more technical standpoint $V_G(k)$ is the maximal ideal spectrum of the cohomology ring $H^*(G,k) = \mathrm{Ext}^*_{kG}(k,k)$. The variety of a kG-module M is the subvariety of $V_G(k)$ consisting of all maximal ideals which contain the ideal $J(M)$ which is the annihilator in $H^*(G,k)$ of the cohomology ring $\mathrm{Ext}^*_{kG}(M,M)$ of M. An important point is that the dimension of the variety $V_G(k)$ is the p-rank of G, i.e. the maximal of the ranks of the elementary abelian p-subgroups of G. We should mention also that $\dim V_G(M)$ is the complexity of the module M. The original definition of complexity by Alperin said that it is the polynomial rate of growth of a minimal projective resolution

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of M. That is, the complexity of M is the least integer $s \geq 0$ such that

$$\lim_{n\to\infty} (\mathrm{Dim}P_n)/n^s = 0.$$

The details of complexity as well as those of varieties and cohomology rings can be found in the texts **[B]** or **[E]**.

Before stating more general results, let us focus on a few applications of Theorem 1. First an easy one.

Applications 1 **[C2]**. Suppose that M is an indecomposable kG-module and E is a maximal elementary abelian p-subgroup of G. Let $D = D_G(E)$ and suppose that U is a direct summand of $M_D^{\uparrow G}$ such that U is not in the same block as M. Then $V_E(U_E)$ is a proper subvariety of $V_E(k)$.

The proof here is very straightfoward. Take any sequence as in Theorem 1 and tensor it with M. We get an exact sequence of the form

$$E : 0 \longrightarrow M\otimes L \longrightarrow (M\otimes\Omega^n(k))^m \oplus (M\otimes P) \longrightarrow M\otimes k_D^{\uparrow G} \longrightarrow 0.$$

Now notice that $M\otimes k_D^{\uparrow G} \cong M_D^{\uparrow G}$ by Frobenius reciprocity. Also $M\otimes\Omega^n(k) \cong \Omega^n(M)\oplus Q$ for some projective module Q. Hence the middle term has only one isomorphism type of nonprojective direct summand, $\Omega^n(M)$, and that summand is in the same block as M. Next we notice that the sequence E splits into a direct sum of exact sequences, one for each block. Of course some of these sequences might be zero. But the sequence for the block of the module U has the form

$$0 \longrightarrow W \longrightarrow R \longrightarrow U \oplus U' \longrightarrow 0$$

where R is projective and W is a direct summand of $M \otimes L$. It follows that $V_E(W_E) \subseteq V_E(L \otimes M) \subset V_E(L_E)$. But also $V_E(U) \subseteq V_E(U \oplus U') = V_E(W_E)$. So the theorem proves the statement.

Application 2 [**CR**]. Suppose that $p > 2$. Let M be an indecomposable module in the principal kG-block such that $H^*(G, M) = 0$. Then for any maximal elementary abelian p-subgroup E of G, $V_E(M_E)$ is a proper subvariety of $V_E(k)$. In particular, the complexity of M is less than the complexity of the trivial module k.

The proof of Application 2 is fairly complicated. One of the key points is that if $p > 2$, then the centralizer of a maximal elementary abelian p-subgroup is p-nilpotent. Hence if $C = C_G(E)$ is such a centralizer then $H^*(C, M_0) = H^*(C/H, M_0) = H^*(C, M)$ where M_0 is the sum of the summands of M_C is the principal block and $H = O_{p'}(C)$. That is, H is the kernel of the principal block. But because C/H is a p-group the variety of the cohomology $H^*(C/H, M_0)$ is the same as the variety of M_0. The theorem allows us to relate the variety $V_G(M)$ to that of M_D ($V_D(M_D)$). With some care we derive a statement about the variety of M_C, and in particular about the part M_0 of M_C which lies in the principal kC-block. The theorem in [**CR**] answered a question left open in earlier work of the authors with Dave Benson [**BCRo**].

Applications 3 [**C2**]. Suppose that G is a p-group and that E is a maximal elementary abelian p-subgroup of G. Let $m = |N_G(E) : C_G(E)|$. Then the cohomology ring $\mathrm{Ext}^*_{kG}(k_C^{\uparrow G}, k_C^{\uparrow G})$ has an irreducible module of dimension m.

In particular, the result indicates that $\mathrm{Ext}^*_{kG}(k_C^{\uparrow G}, k_C^{\uparrow G})$ fails to be commutative in an essential way. For there must be a maximal two sided ideal J such that

$$\mathrm{Ext}^*_{kG}(k_C^{\uparrow G}, k_C^{\uparrow G})/J \cong \mathrm{Mat}_m(k)$$

the ring of $m \times m$ matrices over k. The result was proved by direct computation in [**C1**]. The proof using Theorem 1 is also direct but still requires some details. Notice that because G is a p-group, $C_G(E) = D_G(E)$. Now choose a unit $u_\alpha \in kE$, as before where $u_\alpha^p = 1$ and $u_\alpha - 1 \notin (\mathrm{Rad}\ kE)^2$. Further, we wish to choose u_α in such a way that the module L in one of the sequences of the theorem is free as a kU-module, $U = \langle u_\alpha \rangle$. Then because U is cyclic $\Omega^n(k) \downarrow_U \cong k \oplus (\mathrm{proj})$. Hence

$$k_U^m \cong (\Omega^n(k))^m \downarrow_U \cong (k_C^{\uparrow G})_U$$

modulo projective modules. Thus we have

$$\begin{aligned} \mathrm{Ext}^*_{kU}(k_C^{\uparrow G}, k_C^{\uparrow G}) &\cong \mathrm{Ext}^*_{kU}(k^m, k^m) \\ &\cong \mathrm{Mat}_m(\mathrm{Ext}^*_{kU}(k, k)), \end{aligned}$$

the ring of $m \times m$ matrices in $\mathrm{Ext}^*_{kU}(k,k)$. Now choose a suitable homomorphism $\mathrm{Ext}^*_{kU}(k,k) \to k$ to get a homomorphism

$$\mathrm{Ext}^*_{kU}(k_C^{\uparrow G}, k_C^{\uparrow G}) \to \mathrm{Mat}_m(k).$$

Theorem 1 actually guarantees that there is such a homomorphism whose composition with the restriction from $\mathrm{Ext}^*_{kG}(k_C^{\uparrow G}, k_C^{\uparrow G})$ to kU is a surjective ring homomorphism.

Next we notice that there is a global version of Theorem 1, a version which accounts for all of the abelian p-subgroups at once. Before stating it, let's recall a couple of facts. A theorem of Quillen (see **[B]** or **[E]**) tells us that $V_G(k) = \bigcup_{i=1}^{t} V_{E_i}$ where $E_1, \ldots, E_t$ is a complete set of representatives of the conjugacy classes of maximal elementary abelian p-subgroups of G, $V_{E_1}, \ldots, V_{E_t}$ are the components of $V_G(k)$, and

$$V_{E_i} = \mathrm{res}^*_{G,E_i}(V_{E_i}(k))$$

is the image of the map induced by restriction on varieties. It is a fact that res^*_{G,E_i} is finite-to-one and so $\dim V_{E_i} = \dim V_{E_i}(k) = p$-rank (E_i).

THEOREM 2. *Let* $m = lcm\{|N_G(E_i) : C_G(E_i)|\}$ *and for each* i, *let* $m_i = m/|N_G(E_i) : C_G(E_i)|$. *There exists a positive integer* n *with the property that for every* $\ell > 0$ *there is a projective module* P *and an exact sequence*

$$0 \longrightarrow L \longrightarrow (\Omega^{n\ell}(k))^m \oplus P \longrightarrow \sum_{i=1}^{t} (k_{D_G(E_i)}^{\uparrow G})^{m_i} \longrightarrow 0$$

such that $V_G(L)$ *does not contain any of the components of* V_{E_i}.

In particular, the theorem says that for any $i = 1, \ldots t$ there is a unit $u_i \in kE_i$ of order p such that L is free as a $k\langle u_i \rangle$-module. Now observe that if M is any kG-module then

$$M \otimes k_{D_G(E_i)}^{\uparrow G} \cong M_{D_G(E_i)}^{\uparrow G}$$

by Frobenius reciprocity. Hence the theorem tells us that, except for some complexity factor (represented by the module L), the module theory of kG is determined at the level of the "diagonalizers", $D_G(E_i)$, of the maximal elementary abelian p-subgroups of G. But the "complexity factor" is something of a mystery.

There is a context in which all of the above results seem natural and make very good sense. This is the context of quotient categories of modules. To explain the construction we begin with the stable category, *stmod-kG*, of kG-modules modulo projective. The objects in *stmod-kG* are finitely generated kG-modules, but the morphisms are given by (for M and N kG-modules)

$$\mathrm{Hom}_{stmod\text{-}kG}(M,N) = \mathrm{Hom}_{kG}(M,N)/\mathrm{PHom}_{kG}(M,N).$$

Here $\mathrm{PHom}_{kG}(M, N)$ is the set of all kG-homomorphisms from M to N which factor through a projective module.

The important point is that *stmod-kG* is not an abelian category, rather it is triangulated (see [**H**]). The triangulation says that any morphism $\alpha : M \longrightarrow N$ in the stable category fits into a triangle

$$L \xrightarrow{\beta} M \xrightarrow{\alpha} N \xrightarrow{\gamma} \Omega^{-1}(L).$$

That is to say, there are projective modules P and Q such that there exist exact sequences

$$0 \longrightarrow L \longrightarrow P \oplus M \xrightarrow{\alpha'} N \longrightarrow 0$$

and
$$0 \longrightarrow M \xrightarrow{\alpha''} N \oplus Q \longrightarrow \Omega^{-1}(L) \longrightarrow 0$$

with α' and α'' in the equivalence class α modulo $\mathrm{PHom}_{kG}(M, N)$. Here $\Omega^{-1}(L)$ is the cokernel of a minimal embedding $L \xrightarrow{\sigma} I$ where I is an injective module. Thus Ω^{-1} is an automorphism of the stable category. In the language of triangulated categories Ω^{-1} is called the translation functor.

There are several other axioms of triangulated categories. Among other things,

$$0 \longrightarrow M \xrightarrow{Id} M \longrightarrow 0$$

is a triangle. The sequence

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \Omega^{-1}(L)$$

is a triangle if and only if

$$M \xrightarrow{\beta} N \xrightarrow{\gamma} L \xrightarrow{-\Omega^{-1}(\alpha)} \Omega^{-1}(M)$$

is a triangle. Given two triangles $(L, M, N, \alpha, \beta, \gamma)$ and $(L', M', N', \alpha', \beta', \gamma')$ and homomorphisms $\zeta : M \longrightarrow M'$, $\eta : N \longrightarrow N'$ with $\eta\beta = \beta'\zeta$, then there exists $\theta : L \longrightarrow L'$ such that the diagram

$$\begin{array}{ccccccc}
L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & \Omega^{-1}(L) \\
\downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\zeta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\Omega^{-1}(\theta)} \\
L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \xrightarrow{\gamma'} & \Omega^{-1}(L')
\end{array}$$

commutes in *stmod-kG*. Another axiom, the octahedral axiom, is basically the third isomorphism theorem for kG-modules. All of this is consistent with the definition of the triangles in terms of short exact sequences.

Consider the subcategory $\mathcal{M}_c$ of all kG-modules of complexity at most c. If

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is an exact sequence and if two of the modules L, M and N have complexity c or less then so does the third. Hence if two objects in a triangle in *stmod-kG* are in $\mathcal{M}_c$ then so also is the third. This means that $\mathcal{M}_c$ is a triangulated subcategory of *stmod-kG*. Of course, if r is the p-rank of G then $\mathcal{M}_r = \textit{stmod-kG}$.

Now we can define the quotient categories $\mathcal{Q}_c \cong \mathcal{M}_c/\mathcal{M}_{c-1}$. The objects in $\mathcal{Q}_c$ are the same as those in $\mathcal{M}_c$, but the morphisms are obtained by inverting any morphism in $\mathcal{M}_c$ if the third object in the triangle of that morphisms is in $\mathcal{M}_{c-1}$. Thus the typical morphism from M to N in $\mathcal{Q}_c$ is a diagram

$$M \xleftarrow{s} U \xrightarrow{f} N$$

for some f, s and U in $\mathcal{M}_c$ with s invertible in $\mathcal{Q}_c$. That is, the third object W in the triangle

$$W \longrightarrow U \xrightarrow{s} M \longrightarrow \Omega^{-1}(W)$$

is in $\mathcal{M}_{c-1}$. So we can think of the morphism as having the form $f \circ s^{-1}$, though of course, s^{-1} might not exist in *stmod-kG*. We see then that the quotient category construction is a localization process. One of the most basic theorems in the subject tells us exactly what we must localize. In essence it states that if $s : U \longrightarrow M$ is invertible in $\mathcal{Q}_c$ then there exists a positive integer n and a homomorphism $t : \Omega^n(M) \longrightarrow U$ which is also invertible in $\mathcal{Q}_c$. Specifically we have the following.

THEOREM 3. [**CDW**]. Let M and N be kG-modules in $\mathcal{Q}_c$. Then

$$\mathrm{Hom}_{\mathcal{Q}_c}(M, N) \cong [\mathrm{Ext}^*_{kG}(M, N) \cdot S^{-1}]^0$$

where

$$S = \{\zeta \cdot \mathrm{Id}_M | \zeta \in H^*(G, k) \text{ is homogeneous }, \dim V_G(\zeta) \cap V_G(M) < c\}$$

Here $\mathrm{Ext}^*_{kG}(M, N) \cdot S^{-1}$ is graded by the rule $\deg(\eta \cdot (\zeta \mathrm{Id}_M)^{-1}) = \deg(\eta) - \deg(\zeta)$ the symbol $[\,]^0$ indicates the zeroth grading. One part of the relationship is expressed by the fact that

$$\mathrm{Hom}_{\textit{stmod-kG}}(\Omega^n(M), N) \cong \mathrm{Ext}^n_{kG}(M, N),$$

since any cocycle $\zeta : P_n \to N$ $((P_*, \partial)$ a projective resolution of $M)$ must factor through $\Omega^n(M) \cong \partial(P_n)$.

Suppose that $E_1, \ldots, E_s$ is a complete set of representatives of the conjugacy classes of elementary abelian p-subgroups of maximal rank r in G. Recall that $V_G(k) = \bigcup_{i=1}^{s} V_{E_i} \cup W$ where $V_{E_i} = \mathrm{res}^*_{G,E_i}(V_{E_i}(k))$ and W is the union of all components of dimension less than r.

Using standard tricks of communtative algebra we may choose $\zeta_1, \ldots, \zeta_s \in H^*(G, k)$ such that $V_G(\zeta_i)$ contains V_{E_j} if and only if $i \neq j$ and such that the degrees of $\zeta_1, \ldots, \zeta_s$ are all the same. If we let $\zeta = \zeta_1 + \cdots + \zeta_s$ and $e_i = \zeta_i \zeta^{-1}$, then $e_1, \ldots, e_s$ are central orthogonal idempotents in $\mathrm{Hom}_{\mathcal{Q}_r}(k, k)$. This implies the next theorem.

THEOREM 4. [**CDW**]. Let r be the p-rank of G. Then

$$\mathrm{Hom}_{\mathcal{Q}_r}(k,k) = R_1 \oplus \cdots \oplus R_s$$

where $R_i = \mathrm{Hom}_{\mathcal{Q}_r}(k,k) \cdot e_i$ is a local k-algebra with $R_i/\mathrm{Rad}R_i$ having transcendence degree $r-1$ over k.

It was this result which predicted the decomposition of Theorem 2. Normally idempotents in an endomorphism ring would seem to lead to an actual decomposition of the module. However there is a surprise here. To make the point clear we should first notice that in $\mathcal{Q}_r$, or any $\mathcal{Q}_c$, all of the objects are periodic. That is, as in Theorem 3, it is always possible to find a cocycle ζ representing a cohomology class in $\mathrm{Ext}^n_{kG}(M,M)$ for some n, such that $\zeta : \Omega^n(M) \longrightarrow M$ is invertible in the quotient category. So in this context we have the following as a direct consequence of Theorem 2.

THEOREM 5. *Assume the notation of Theorem 2. In* $\mathcal{Q}_r$

$$k^m \cong \sum_{i=1}^{s} (k^{\uparrow G}_{D_G(E_i)})^{m_i}.$$

The surprise is that in spite of the above decomposition, the trivial module k is indecomposable in $\mathcal{Q}_r$. Hence we are forced to conclude that $\mathcal{Q}_r$ has no Krull-Schmidt Theorem, no uniqueness of decompositions. Strangely, it is possible to recover the Krull-Schmidt Theorem if we can enlarge the category. In particular we must allow infinite direct sums and hence some infinitely generated modules. This requires some new definitions of the complexity and the variety of a module [**BCRi**].

Different sorts of decompositions for modules can be obtianed in general. In $\mathcal{Q}_c$ it is always true that $M \oplus \Omega(M)$ is a direct sum of modules each of which has a variety with only a single component in dimension c [**CW**]. In a sense the quotients decompose as unions of subcategories in correspondence with the subvarities of the prescribed dimension.

REFERENCES

[**B**] D. J. Benson, "Representations and Cohomology II: Cohomology of Groups and Modules," Cambridge University Press, Cambridge, 1991.

[**BCRi**] D. J. Benson, J. F. Carlson and J. Rickard, *Complexity and varieties for infinitely generated modules.*

[**BCRo**] D. J. Benson, J. F. Carlson and G. R. Robinson, *On the Vanishing of Group Cohomology*, J. Algebra **131** (1990), 40-73.

[**C1**] J. F. Carlson, *Cohomology rings of induced modules*, J. Pure and Appl. Algebra **44** (1987), 85-97.

[**C2**] J. F. Carlson, *Decomposition of the trivial module in the complexity quotient category.*

[**CDW**] J. F. Carlson, P. W. Donovan and W. W. Wheeler, *Complexity and quotient categories for group algebras*, J. Pure and Appl. Algebra (to appear).

[**CR**] J. F. Carlson and G. R. Robinson, *Varieties and modules with vanishing cohomology*, Math. Proc. Cambridge Phil. Soc. (to appear).

[**CW**] J. F. Carlson, and W. W. Wheeler, *Varieties and localization of module categories.*

[**E**] L. Evens, "The Cohomology of Groups," Oxford University Press, New York, 1991.

[**H**] D. Happel, "Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras," Cambridge University Press, Cambridge, 1988.