| Title | SOME EXAMPLES OF DEFORMATIONS OF COMPLEX
|       | MANIFOLDS(Singularities of Holomorphic Vector Fields and Related Topics) |
| Author(s) | GHYS, ETIENNE |
| Citation | 数理解析研究所講究録 | 1994, 878: 108-112 |
| Issue Date | 1994-06 |
| URL | http://hdl.handle.net/2433/84163 |
| Type | Departmental Bulletin Paper |
| Textversion | publisher |

Kyoto University
SOME EXAMPLES OF DEFORMATIONS
OF COMPLEX MANIFOLDS

Étienne Ghys

Let $\Gamma$ be a discrete cocompact subgroup in $SL(n, \mathbb{C})$. Recall that if $n \geq 2$, a (special case of) a theorem of A. Weil shows that any homomorphism from $\Gamma$ to $SL(n, \mathbb{C})$ close enough to the embedding is conjugate to this embedding. Moreover, Raghunathan has shown that if $n \geq 3$ then the compact complex manifold $SL(n, \mathbb{C})/\Gamma$ is rigid as a complex manifold. The purpose of this note is to describe explicit examples of non trivial deformations of the complex manifold $SL(2, \mathbb{C})/\Gamma$.

This note is extracted from [Gh] which will be published elsewhere: it corresponds to the talk I gave in the meeting 'Singularities of holomorphic vector fields and related topics' at RIMS, Kyoto, in November 1993.

Observe that, up to a $\mathbb{Z}/2\mathbb{Z}$-extension, $SL(2, \mathbb{C})$ is the isometry group of the real hyperbolic 3-dimensional space $\mathbb{H}^3$ so that $\Gamma$ is the fundamental group of a hyperbolic 3-dimensional orbifold. Many examples have nonvanishing first Betti number, i.e., are such that there exist nontrivial homomorphisms $u : \Gamma \rightarrow \mathbb{C}^*$ (see [Th]).

If $u$ is such a homomorphism, we consider the right action of $\Gamma$ on $SL(2, \mathbb{C})$ defined by:

$$(x, \gamma) \in SL(2, \mathbb{C}) \times \Gamma \mapsto x \bullet \gamma = \begin{pmatrix} u(\gamma) & 0 \\ 0 & u(\gamma)^{-1} \end{pmatrix} x \gamma.$$ 

If this action is free, proper and totally discontinuous, we denote by $SL(2, \mathbb{C})/\mathbb{C} \Gamma$ the quotient, and we say that $u$ is admissible. We noted that there is a natural $\mathbb{C}^*$-action on this quotient, coming from left translations by matrices $\begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}$.

Let $H^+$ and $H^-$ be the right invariant holomorphic vector fields in $SL(2, \mathbb{C})$ corresponding to the elements $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ of the Lie algebra of $SL(2, \mathbb{C})$ and denote by $\mathcal{H}^+$ and $\mathcal{H}^-$ the one-dimensional holomorphic foliations generated by $H^+$ and $H^-$. It is easy to check that the differential of the right action by $\gamma$ in $SL(2, \mathbb{C})$ maps $H^+$ and $H^-$ to $u(\gamma)^2 H^+$ and $u(\gamma)^{-2} H^-$ so that $H^+$ and $H^-$ are not invariant (unless $u^2$ is trivial) but $\mathcal{H}^+$ and $\mathcal{H}^-$ are invariant. In other words, on the compact manifold $SL(2, \mathbb{C})/\mathbb{C} \Gamma$, we have two natural foliations $\mathcal{H}^+$ and $\mathcal{H}^-$ which are invariant under the $\mathbb{C}^*$-action. When $u^2$ is trivial, $\mathcal{H}^+$ and $\mathcal{H}^-$ are parametrized by vector fields $H^+$ and $H^-$. In order to simplify our description of these examples, we shall assume that $\Gamma$ is torsion-free (this can always be achieved by replacing $\Gamma$ by a finite index subgroup by a theorem of Selberg). In particular, $\Gamma$ injects into $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm id\}$. Typeset by $\LaTeX$
ÉTIENNE GHYS

Note that if $\epsilon : \Gamma \to \{\pm 1\}$ is a homomorphism, the map $\tau : \gamma \in \Gamma \mapsto \epsilon(\gamma)\gamma \in \text{SL}(2, \mathbb{C})$ is an injective homomorphism whose image is another discrete subgroup $\Gamma'$ of $\text{SL}(2, \mathbb{C})$. In such a situation, we shall write $\Gamma = \pm \Gamma'$. This happens precisely when $\Gamma$ and $\Gamma'$ have the same projection in $\text{PSL}(2, \mathbb{C})$. Of course, $u : \Gamma \to \text{SL}(2, \mathbb{C})$ is admissible if and only if $\epsilon, u \circ \tau^{-1} : \Gamma' \to \mathbb{C}^*$ is admissible and the corresponding actions of $\mathbb{C}^*$ are conjugate.

**Proposition.** Let $\Gamma$ be a discrete torsion-free cocompact subgroup of $\text{SL}(2, \mathbb{C})$. Then homomorphisms $u : \Gamma \to \mathbb{C}^*$ which are close enough to the trivial homomorphism are admissible.

Let $\Gamma_1$ and $\Gamma_2$ be two discrete torsion-free cocompact subgroups of $\text{SL}(2, \mathbb{C})$. Then $\text{SL}(2, \mathbb{C})//_{u_1} \Gamma_1$ and $\text{SL}(2, \mathbb{C})//_{u_2} \Gamma_2$ are homeomorphic if and only if there is a continuous automorphism $\theta$ of $\text{SL}(2, \mathbb{C})$ such that $\theta(\Gamma_1) = \pm \Gamma_2$. In such a case, there is a $C^\infty$-diffeomorphism between $\text{SL}(2, \mathbb{C})//_{u_1} \Gamma_1$ and $\text{SL}(2, \mathbb{C})//_{u_2} \Gamma_2$ sending orbits of the first $\mathbb{C}^*$-action to orbits of the second (without necessarily commuting with the actions).

**Proof.** The first property follows from a very general fact. Let $G$ be a Lie group acting analytically on a manifold $V$ and let $\Gamma \to G$ be a homeomorphism such that the induced action of $\Gamma$ on $V$ is free, proper and totally discontinuous. Then any perturbation of the homomorphism $\Gamma \to G$ has the same property (see, for instance [Th]). Assertion i) follows from the special case where $V = \text{SL}(2, \mathbb{C})$ and $G = \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})$ acting by left and right translations.

Assume $\text{SL}(2, \mathbb{C})//_{u_1} \Gamma_1$ and $\text{SL}(2, \mathbb{C})//_{u_2} \Gamma_2$ are homeomorphic. Then $\Gamma_1$ and $\Gamma_2$ are isomorphic as abstract groups and it follows from Mostow's rigidity theorem that there is a continuous automorphism $\theta$ of $\text{SL}(2, \mathbb{C})$ such that $\theta(\Gamma_1) = \pm \Gamma_2$. Note that, up to conjugacy, the only nontrivial continuous automorphism of $\text{SL}(2, \mathbb{C})$ is given by $\theta(x) = \gamma x \gamma^{-1}$.

We now show that if $\Gamma_2 = \pm \theta(\Gamma_1)$ then $\text{SL}(2, \mathbb{C})//_{u_1} \Gamma_1$ and $\text{SL}(2, \mathbb{C})//_{u_2} \Gamma_2$ are diffeomorphic. We can of course assume that $\theta = \text{id}$, and that $\Gamma_1 = \Gamma_2 = \Gamma$. Let us consider first of all the quotients $M_i = U(1) \setminus \text{SL}(2, \mathbb{C})//_{u_i} \Gamma_i$ ($i = 1, 2$). These are manifolds since we assumed that $\Gamma$ is torsion free. Note that if $u_i$ is trivial, then $\text{SL}(2, \mathbb{C})/\Gamma_i$ is the 2-fold (spin)-cover of the orthonormal frame bundle of the 3-manifold $V$ which is the quotient of the hyperbolic 3-space by the action of $\Gamma$ and $M_i$ is the unit tangent bundle of $V$.

On $M_i$, we have a real one-parameter flow $f^t_i$ coming from the complex one-parameter flow on $\text{SL}(2, \mathbb{C})//_{u_i} \Gamma$. Of course when $u_i$ is trivial the flow $f^t_i$ is nothing but the geodesic flow of $V$.

The quotient $\mathbb{C}^* \setminus \text{SL}(2, \mathbb{C})$ of $\text{SL}(2, \mathbb{C})$ by the diagonal subgroup is isomorphic to the complement of the diagonal in $\mathbb{CP}^1 \times \mathbb{CP}^1$. The universal cover $\tilde{M}_i$ of $M_i$, naturally identified with $U(1) \setminus \text{SL}(2, \mathbb{C})$, fibres over the complement of the diagonal $\Delta$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$:

$$D_i : \tilde{M}_i \to \mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta$$

and this fibration is equivariant under the diagonal embedding:

$$H_i : \gamma \in \Gamma \mapsto (\gamma, \gamma) \in \text{PSL}(2, \mathbb{C}) \times \text{PSL}(2, \mathbb{C})$$
The fibres of $D_1$ are the orbits of the lifted flow $\tilde{f}_1^t$. We therefore observe that both flows $f_1^t$ and $f_2^t$ have the same transverse structure, i.e., equivalent holonomy pseudogroups. It follows from [Ha] (see also [Gr], [Ba]) that there is a $C^\infty$-diffeomorphism between $M_1$ and $M_2$ sending orbits of $f_1^t$ to orbits of $f_2^t$ and, in particular, that $M_1$ and $M_2$ are diffeomorphic.

We claim that the circle fibrations $\text{SL}(2, \mathbb{C})//u_1 \Gamma \to M_t$ are trivial fibrations. This follows from the fact that orientable closed 3-manifolds are parallelizable and from the fact that the space of homomorphisms from $\Gamma$ to $\mathbb{C}^*$ is connected. Indeed, choose a path $u_t$ ($t \in [0,1]$) connecting the trivial holomorphism to $u_1$ and consider the right action of $\Gamma$ on $\text{SL}(2, \mathbb{C}) \times \mathbb{H}^3$ (where $\mathbb{H}^3$ is the hyperbolic 3-space) given by:

$$(x, p) \cdot \gamma = \left( \begin{array}{ll} u_t(\gamma) & 0 \\ 0 & u_t(\gamma)^{-1} \end{array} \right) x \gamma^{-1}(p) \in \text{SL}(2, \mathbb{C}) \times \mathbb{H}^3.$$

The second factor has been introduced in such a way that the action is free, proper, and totally discontinuous for each $t \in [0,1]$. The quotient spaces are homotopy equivalent to $\text{SL}(2, \mathbb{C})/\Gamma$ and $\text{SL}(2, \mathbb{C})//u_1 \Gamma$ for $t = 0$ and $t = 1$. Moreover, for each $t$, the right-action of $\Gamma$ commutes with left translations by $U(1)$ so that each quotient space is the total space of circle bundle. Since we noticed that this circle bundle is trivial of $t = 0$, we deduce that it is also trivial for $t = 1$. Hence the circle bundles $\text{SL}(2, \mathbb{C})//u_1 \Gamma \to M_t$ are trivial and the diffeomorphism between $M_1$ and $M_2$ sending orbits of $f_1^t$ to orbits of $f_2^t$ can be lifted to a diffeomorphism between $\text{SL}(2, \mathbb{C})//u_1 \Gamma$ and $\text{SL}(2, \mathbb{C})//u_1 \Gamma$ sending orbits of the first $\mathbb{C}^*$-action to orbits of the second one.

This completes the proof of proposition 6.1. □

**Proposition.** If $u : \Gamma \to \mathbb{C}^*$ is an admissible homomorphism such that $u^2$ is non trivial, then the space of holomorphic vector fields on $\text{SL}(2, \mathbb{C})//u_1 \Gamma$ has complex dimension 1 and is generated by the vector field corresponding to the $\mathbb{C}^*$-action.

**Proof.** We have already noticed that there are two holomorphic one dimensional foliations $\mathcal{H}^+$ and $\mathcal{H}^-$ on $V = \text{SL}(2, \mathbb{C})//\Gamma$ which are invariant under the $\mathbb{C}^*$-action and which provide, together with the tangent bundle to the orbits of $\mathbb{C}^*$, a splitting of $T_C V$ as a sum of three line-bundles. In order to show the proposition, it is enough to show that there is no nonzero holomorphic vector field in $V$ tangent to $\mathcal{H}^+$ (or to $\mathcal{H}^-$) if $u$ is nontrivial. Assume there is such a vector field $\xi$. Using the fact that the $\mathbb{C}^*$-action preserves $\mathcal{H}^+$ and that the space of holomorphic vector fields is finite dimensional, one can choose $\xi$ such that the $\mathbb{C}^*$-action $\phi(T)$ ($T \in \mathbb{C}^*$) satisfies, for some $k \in \mathbb{Z}$:

$$d\phi(T)(\xi) = T^k \xi \text{ for all } T \in \mathbb{C}^*.$$ 

If one lifts $\xi$ to $\text{SL}(2, \mathbb{C})$, one gets a vector field $\tilde{\xi}$ which is of the form $f \cdot H^+$ where $f$ is holomorphic on $\text{SL}(2, \mathbb{C})$. Taking into account the invariance of $\tilde{\xi}$ under the action of $\Gamma$ and the non-invariance of $H^+$ already observed, we get:

1. $$f(x \cdot \gamma) = u(\gamma)^{-2} f(x) \text{ for } \gamma \in \Gamma \text{ and } x \in \text{SL}(2, \mathbb{C}).$$

Moreover, we have:

2. $$f \left( \begin{array}{ll} T & 0 \\ 0 & T^{-1} \end{array} \right) \cdot x = T^k f(x) \text{ for all } T \in \mathbb{C}^* \text{ and } x \in \text{SL}(2, \mathbb{C}).$$
Assume first that \( k = 0 \) so that \( f \) actually defines a function \( \bar{f} \) on
\[
\mathbb{C}^* \backslash \text{SL}(2, \mathbb{C}) \cong \mathbb{C}P^1 \times \mathbb{C}P^1 - \Delta.
\]

Then, by (1), \( \bar{f} \) is invariant under the action of the first commutator group \( \Gamma' \) of \( \Gamma \)
on which \( u \) is obviously trivial. Now this action of \( \Gamma' \) on \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) is topologically transitive. This is equivalent to the fact that the geodesic flow of the homology cover of a compact hyperbolic manifold is topologically transitive. Indeed all non trivial normal subgroups of a discrete group of isometries of a hyperbolic space have the same limit set and all non elementary groups act topologically transitively on the square of their limit set (see [Th] and [G-H] page 123). Therefore \( f \) is constant—but this is impossible if \( u \) and \( f \) are not trivial.

Now, assume that \( k \neq 0 \). Consider the function \( f : V' = \text{SL}(2, \mathbb{C})/\Gamma' \to \mathbb{C} \).
According to (2), \( f \) has to vanish on periodic orbits of the \( \mathbb{C}^* \)-action on \( V' \). But, on any compact hyperbolic manifold the union of closed geodesics homologous to zero is dense (as follows also from [G-H]). This shows that the union of closed orbits of the \( \mathbb{C}^* \)-action on \( V' \) is dense \( V' \). It follows that \( f \) is zero. \( \square \)

**Corollary.** Let \( \Gamma \) be a discrete torsion free cocompact subgroup of \( \text{SL}(2, \mathbb{C}) \) and \( u_1, u_2 : \Gamma \to \mathbb{C}^* \) be two admissible homomorphisms. Then the compact complex manifolds \( \text{SL}(2, \mathbb{C})/\mu_i \Gamma \) \( (i = 1, 2) \) are holomorphically diffeomorphic if and only if there is an automorphism \( \theta \) of \( \Gamma \) such that \( u_2^{\pm 1} = u_1 \circ \theta \).

**Proof.** If \( u_2^2 \) is trivial, then \( \text{SL}(2, \mathbb{C})/\mu_i \Gamma \) is a homogeneous space of \( \text{SL}(2, \mathbb{C}) \) and therefore admits three linearly independent holomorphic vector fields. According to 6.2, on deduces that \( u_2^2 \) is also trivial if \( \text{SL}(2, \mathbb{C})/\mu_2 \Gamma \) is holomorphically diffeomorphic to \( \text{SL}(2, \mathbb{C})/\mu_1 \Gamma \). The corresponding complex manifolds are therefore of the form \( \text{SL}(2, \mathbb{C})/\Gamma_1 \) \( (i = 1, 2) \) and \( \Gamma_1 = \pm \Gamma_2 \). Any holomorphic diffeomorphism between these two homogeneous spaces induces an isomorphism between the Lie algebras of holomorphic vector fields which are themselves isomorphic to the Lie algebra of \( \text{SL}(2, \mathbb{C}) \). The corollary follows in this special case.

Now, assume that \( u_1 \) and \( u_2 \) are nontrivial and that there is a holomorphic diffeomorphism \( F \) between the corresponding compact complex manifolds. Proposition 6.2 implies that \( F \) conjugates the \( \mathbb{C}^* \)-actions or one with the inverse of the other. Let \( \gamma \) be a nontrivial element of \( \Gamma \) and denote by \( \lambda(\gamma), \lambda(\gamma)^{-1} \) its two eigenvalues. The \( \mathbb{C}^* \)-action on \( \text{SL}(2, \mathbb{C})/\mu_i \Gamma \) contains precisely two closed orbits containing a loop freely homotopic to \( \gamma^{\pm 1} \), whose “periods” are \( \lambda(\gamma)u_i(\gamma) \) and \( \lambda^{-1}(\gamma)u_i(\gamma) \). Note that periods of closed orbits related under \( F \) should be equal or inverse. If \( \theta \) denotes the automorphism of \( \Gamma \) (defined up to conjugacy) induced by \( F \), it follows that either \( u_2 = u_1 \circ \theta \) or \( u_2^{-1} = u_1 \circ \theta \). \( \square \)

**Corollary.** Let \( \Gamma \) be a discrete torsion free cocompact subgroup of \( \text{SL}(2, \mathbb{C}) \) and \( u_1, u_2 : \Gamma \to \mathbb{C}^* \) be two admissible homomorphisms. Then the \( \mathbb{C}^* \)-actions on \( \text{SL}(2, \mathbb{C})/\mu_i \Gamma \) are conjugate by a homeomorphism if and only if there is an automorphism \( \theta \) of \( \Gamma \) such that \( u_2 = u_1 \circ \theta \).

**Proof.** This is the same proof as that of the previous Corollary since we only used preservation of periods of closed orbits. \( \square \)
SOME EXAMPLES OF DEFORMATIONS OF COMPLEX MANIFOLDS

REFERENCES


ÉCOLE NORMALE SUPÉRIEURE DE LYON, UMR 128 CNRS, 46 ALLÉE D’ITALIE 69364 LYON, FRANCE