A direct approach to the planar graph presentations of the braid group

(Singularities of Holomorphic Vector Fields and Related Topics)

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数理解析研究所講究録 1994, 878: 103-107

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0. Introduction

Recall that the classical braid group on $n$ strings $B_n$ can be considered as the fundamental group of the configuration space of unordered $n$ points in the plane.

Given a planar finite graph whose vertices are $n$ given points, one can define for each edge $\sigma$ a braid, also denoted $\sigma$ like in figure 1:

One just turns half around $\sigma$ in a neighbourhood, the other strings being vertical.

If the graph is

one obtains the Artin generators of the braid group $B_n$, see [B].

Let us now suppose that the graph $\Gamma$ is connected and without loops. In [S] we noted that the braids $\{\sigma\}$ corresponding to the edges verify the following relations:

(i) disjointness: if $\sigma_1 \cap \sigma_2 = \emptyset$ then $\sigma_1 \sigma_2 = \sigma_2 \sigma_1$.

(ii) adjacency: if $\sigma_1 \cap \sigma_2 = \text{one vertex}$ then $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$.
(iii) nodal: if $\sigma_1, \sigma_2, \sigma_3$ have one common vertex like in figure 3; then $\sigma_1\sigma_2\sigma_3\sigma_1 = \sigma_2\sigma_3\sigma_1\sigma_2 = \sigma_3\sigma_1\sigma_2\sigma_3$

(fig. 3)

(iv) cyclic: if $\sigma_1 \cdots \sigma_n$ is a cycle such that $\sigma_1 \cdots \sigma_n$ bounds a disc without interior vertices, then $\sigma_1\sigma_2 \cdots \sigma_{n-1} = \sigma_2 \cdots \sigma_n = \sigma_n \sigma_1 \cdots \sigma_{n-2}$

(fig. 4)

Moreover, we proved in [S] the

0.1. Theorem. — The braid group $B_\Gamma$ on the vertex set $v(\Gamma)$ has a presentation $(X_\Gamma, R_\Gamma)$ where $X_\Gamma$ is the set of edges $\{\sigma\}$ and $R_\Gamma$ the set of relations (i) – (iv).

0.2. Remark. — The above statement, which appears in [S] in a slightly more general context, was chosen here in order to keep notations simpler.

This theorem was presented at the Kyoto meeting together with some corollaries. The proof given in [S] used a recursive device using Artin’s presentation as the starting point. Here I shall sketch a direct argument suggested by Fadell-Van Buskirt’s proof, see [B], as modified by J. Morita [M].

I am grateful to Professors Suwa and Ito for the opportunity they gave me to participate to the R.I.M.S. meeting and for their warm hospitality.
1. The geometric argument

Let $\Gamma$ be a finite tree, $v \in \Gamma$ an end vertex and $\Gamma' = \Gamma - \{v\}$ and $v'$ the neighbour of $v$. Let $P_{\Gamma}$ the kernel of the natural map $B_{\Gamma} \xrightarrow{\pi} \Sigma_{\Gamma}$, i.e. the pure braid group, where $\Sigma_{\Gamma}$ is the permutation group of $v(\Gamma)$.

Forgetting the last string from $v$ to $v'$, one gets a natural map $P_{\Gamma} \rightarrow P_{\Gamma'}$. Think about this map as coming from the natural projection between configuration spaces. One easily sees that its kernel is the free group $\pi_{1}(C - v(\Gamma'))$ with $|v(\Gamma)| - 2$ generators.

Consider the subgroup $B_{\Gamma}^{0} = \pi^{-1}(\Sigma_{\Gamma'})$ of $B_{\Gamma}$. Then $P_{\Gamma} \subset B_{\Gamma}^{0}$ and there is a natural map

$$\theta : B_{\Gamma}^{0} \rightarrow B_{\Gamma'}$$

which "forgets" the last string. The diagram

$$\begin{array}{ccc}
P_{\Gamma} & \rightarrow & P_{\Gamma'} \\ \downarrow & & \downarrow \\ B_{\Gamma}^{0} & \rightarrow & B_{\Gamma'}
\end{array}$$

is commutative and the kernel of the horizontal maps is the same. One gets the

1.1. Proposition. — The kernel of the map $\theta : B_{\Gamma}^{0} \rightarrow B_{\Gamma'}$ is a free group of rang $|v(\Gamma)| - 2$.

2. The inductive assertion

In this paragraph we will formulate the statement needed to prove theorem 0.1 for a tree $\Gamma$.

Let $\tilde{B}_{\Gamma}$ be the group given by a presentation $(X_{\Gamma}, R_{\Gamma})$ as in theorem 0.1. Our task is to prove that the natural map $\tilde{B}_{\Gamma} \rightarrow B_{\Gamma}$ is an isomorphism. We use induction on $|v(\Gamma)|$.

For each vertex $\omega \in \Gamma'$ let $\sigma_{1} \cdots \sigma_{\kappa_{\omega}}$ be the simple path from $\omega$ to $v$, $\rho_{\omega} = \sigma_{1} \cdots \sigma_{\kappa_{\omega}}$ the corresponding braid and $\tau_{\omega} = \sigma_{\kappa_{\omega}} \cdots \sigma_{2} \sigma_{1}^{-1} \cdots \sigma_{\omega}^{-1}$ if $\omega \neq v'$.
and \( \tau_\omega = \sigma_1^2 \) if \( \omega = v' \). Note that \( \rho_\omega \) and \( \tau_\omega \) make sense in \( B_\Gamma \) and in \( \tilde{B}_\Gamma \).

Let \( \tilde{B}_\Gamma^0 \) be the subgroup of \( \tilde{B}_\Gamma \) generated by \( \{ \sigma | \sigma \in \Gamma' \} \cup \{ \tau_\omega | \omega \in \Gamma' \} \).

One has a natural diagram:

\[
\begin{array}{ccc}
\tilde{B}_\Gamma^0 & \xrightarrow{\tilde{\theta}} & \tilde{B}_\Gamma' \\
\downarrow & & \downarrow \\
B_\Gamma^0 & \xrightarrow{\theta} & B_{\Gamma'} \\
\end{array}
\]

Note that the map \( \tilde{\theta} \) is well defined because the right map is an isomorphism by the inductive assumption.

In the next paragraph we shall prove that the left side map \( \tilde{B}_\Gamma^0 \rightarrow B_\Gamma^0 \) is an isomorphism and show how this implies that the map \( \tilde{B}_\Gamma \rightarrow B_\Gamma \) is an isomorphism.

### 3. Proof of the inductive step

The map \( \tilde{\theta} : \tilde{B}_\Gamma^0 \rightarrow \tilde{B}_\Gamma' \) has an obvious section. The kernel of \( \tilde{\theta} \) is the subgroup generated by the \( \{ \tau_\omega \} \) : this follows using the section and the fact that the \( \tau_\omega \)‘s generate a normal subgroup.

Direct checking shows that the \( \tau_\omega \)‘s, when considered in \( B_\Gamma^0 \) freely generate the kernel of \( \theta \) (see 1.1). This implies that the map from \( \text{ker} \tilde{\theta} \) to \( \text{ker} \theta \) is an isomorphism and by the five lemma and the inductive assumption the same is true for the map from \( \tilde{B}_\Gamma^0 \) to \( B_\Gamma^0 \).

In order to deduce that the map from \( \tilde{B}_\Gamma \) to \( B_\Gamma \) is an isomorphism we first note that it is surjective : it’s image contains \( P_\Gamma \subset B_\Gamma^0 \) and it obviously surjects onto \( \Sigma_\Gamma \).

\[
\begin{array}{ccc}
\tilde{B}_\Gamma & \rightarrow & B_\Gamma \\
\downarrow & & \downarrow \\
P_\Gamma & \rightarrow & \Sigma_\Gamma \\
\end{array}
\]
As $B^0_r$ is a subgroup of index $|v(\Gamma)|$ of $B_\Gamma$ by its very definition, it will be sufficient to show the same thing about the index of $\tilde{B}_\Gamma^0$ in $\tilde{B}_\Gamma$.

Consider the set $\tilde{X} = \bigcup_{\omega \in v(\Gamma)} \rho_\omega \tilde{B}_r^0$ (where we put $\rho_v = e$). We leave to the reader to prove that $\tilde{X}$ is a subgroup of $\tilde{B}_\Gamma$. One then deduces that the index of $\tilde{B}_\Gamma^0$ in $\tilde{X}$ is $|v(\Gamma)|$ as $\rho_{\omega_1}^{-1}\rho_{\omega_2} \notin \tilde{B}_\Gamma^0$ if $\omega_1 \neq \omega_2$. Finally, as $\tilde{B}_\Gamma$ is generated by $\tilde{B}_\Gamma^0$ together with any $\rho_\omega$, $\omega \neq v$, one has $\tilde{B}_\Gamma = \tilde{X}$ and so the index of $\tilde{B}_\Gamma^0$ in $\tilde{B}_\Gamma$ is $|v(\Gamma)|$. This completes the argument when $\Gamma$ is a tree.

4. End of the proof

We now take $\Gamma$ to be any graph like in theorem 0.1 and $b(\Gamma)$ it's first Betti number. If $b(\Gamma) = 0$, $\Gamma$ is a tree on the result is true.

Let us suppose that the theorem is true for all graphs whose first Betti number is less than $b(\Gamma)$. We chose an edge $\alpha$ on a cycle of $\Gamma$ which does not bound a second cycle on the other side. The theorem is then true for the graph $\Gamma - \alpha$ and it is easily seen that this implies it is true for $\Gamma$: any cyclic relation is true in $B_{\Gamma-\{\alpha\}} = B_{\Gamma}$ and it defines implicitly the element $\alpha \in B_{\Gamma}$ (see [S] for more details).

References


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(28 mars 1994)