数理解析研究所講究録
第 878 巻 1994 年 56-63

Indices of vectorfields and Nash blowup

Jean-Paul Brasselet (CIRM - SiGmA - CNRS)

Kyoto, November 11, 1994

Introduction

The aim of this lecture is to explicit the behaviour of the index of vectorfields through the Nash procedure. We will consider vectorfields tangent to the strata of a Whitney stratification of an analytic variety X, imbedded in a smooth one M. They will be suitable continuous sections of the tangent bundle of M restricted to X and will be called radial vectorfields. Their construction was given by M.H. Schwartz [S2]. The main property is a proportionality theorem which is an important step in the proof of equality of M.H. Schwartz and R. MacPherson classes of singular algebraic complex varieties. This result was given in [BS] as an intermediary result. Another application, given in [S2], is the Poincaré-Hopf theorem for (real) singular analytic varieties.

1. Stratified vectorfields.

Let X denote a real analytic variety of dimension n, stratified with respect to Whitney conditions. This means that :

(1) X is union of a locally finite collection of disjoint locally closed subsets V_i, such that $V_i \cap \overline{V_j} \neq \emptyset \Leftrightarrow V_i \subset \overline{V_j}$,

(2) the subsets $V_i = V_i^s$ are smooth manifolds of dimension s, called the strata,

(3) whenever $V_i \subset \overline{V_j}$ then the pair satisfies the Whitney conditions :

- (a) suppose (x_n) is a sequence of points in V_j converging to $y \in V_i$ and suppose that the sequence of tangent spaces $T_{x_n}(V_j)$ converges to some limit L. Then $T_y(V_i) \subset L$.
- (b) suppose (x_n) is a sequence of points in V_j converging to $y \in V_i$ and (y_n) is a sequence of points in V_i also converging to y, suppose that the sequence of tangent spaces $T_{x_n}(V_j)$ converges to some limit L and that the sequence of secant lines $\overline{x_n y_n}$ converges to some limit λ. Then $\lambda \subset L$.

We will assume that X is imbedded in an analytic manifold M of dimension m. Then, given a Whitney stratification of X, there is a Whitney stratification of M adding $M - X$ as supplementary stratum. Let us denote by $T(M)$ the tangent bundle to M and $\pi : T(M) \to M$ the natural projection. Let A be a subset of M, the subspace of $T(M)|_A$, union of all restrictions $T(V_i \cap A)|_A$ will be denoted by

$$E(A) = \bigcup_{x \in V_i \cap A} T_x(V_i)$$

Definition. A stratified vectorfield v on a subset $A \subset M$ is a section of $T(M)$ defined on A and such that if $x \in V_i \cap A$, then $v(x) \in E(x) = T_x(V_i)$.

Suppose a is an isolated singular point of the stratified vectorfield v defined on a neighborhood of a in M. There are two well defined indices : the index of v as a section of $T(V_i)$, denoted by $I(v|_{V_i}, a)$ and the index of v as a section of $T(M)$, denoted by $I(v, a)$. In general, these indices do not agree.

2. Radial vectorfields.

In [S2], M.H. Schwartz proved existence of so-called radial vectorfields. They are special cases of stratified vectorfields with isolated singular points, satisfying the following main property : If $a \in V_i$ is an isolated singular point of a radial vectorfield v, then the index of v as a section of $T(V_i)$ and the index of v as a section of $T(M)$ agree.

In this section, we recall the construction of radial vectorfields. They will be obtained as a sum of two extensions :

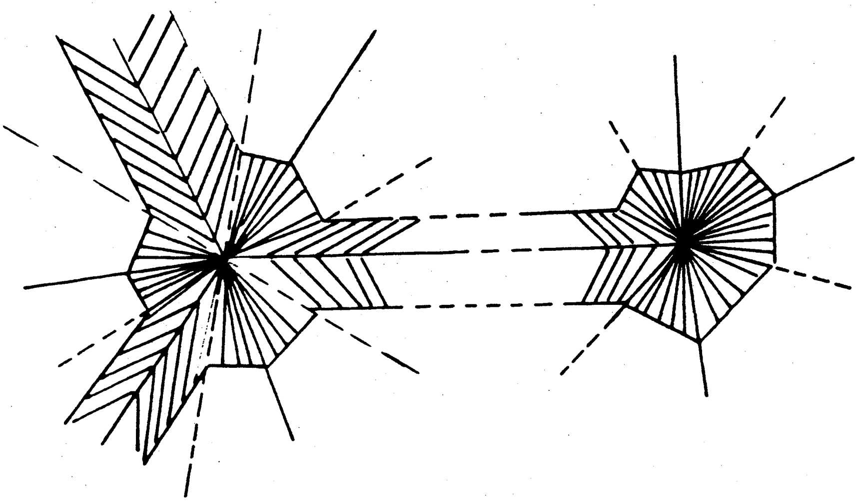
Using the Whitney condition (a) we can extend a stratified vectorfield v, given on a stratum V_i, in suitable neighborhoods $\mathcal{T}_\varepsilon(V_i)$ as a "parallel extension" of v.

Using the Whitney condition (b), we can define a "transverse" vectorfield to every stratum V_i. This vectorfield is tangent to the geodesic arcs issued from V_i (relatively to a given Riemannian metric).

a) *The parallel extension.*

Let us denote by (K) a triangulation of M compatible with the stratification and denote by σ^q an (open) q-simplex in (K). Fix $\varepsilon > 0$, for every $y \in \sigma^q$, we will denote by $\mathcal{T}_\varepsilon(y)$ the set of points of the (open) star of σ^q whose barycentric coordinates relatively to vertices of σ^q are proportional to those of y with ratio $\geq 1 - \varepsilon$. The set $\mathcal{T}_\varepsilon(y)$ is $(m - q)$-dimensional and it admits a natural partition in radii.

If A is a closed subset of V_i^s, we will write $\mathcal{T}_\varepsilon(A) = \bigcup_{y \in A} \mathcal{T}_\varepsilon(y)$.



The tube $\mathcal{T}_\varepsilon(A)$ with radii.

Consider a stratified vectorfield v defined on A, we will construct an extension v' of v on $\mathcal{T}_\varepsilon(A)$ as following : Let us denote by U_k the open star of the vertex $a_k \in V_i^s$ relatively to (K) and by $\{\phi_k\}$ a partition of unity associated to the covering of V_i^s by the subsets $U_k \cap V_i^s$. The Whitney condition (a) shows that, for every k, there is a continuous map $\Psi_k : U_k \times \mathbf{R}^s \to E(U_k)$ whose restriction to $y \in U_k \cap V_i^s$ is a complex linear isomorphism $\Psi_k : \{y\} \times \mathbf{R}^s \to E(y)$.

Every point x of $\mathcal{T}_\varepsilon(A) - A$ belongs to an unique radius issued from a point $y \in A$. For every k, such that $y \in U_k \cap A$, there is an unique vector $v^{(k)}(x)$ in $E(x)$ such that $p_2 \circ \Psi_k^{-1}(v(y)) = p_2 \circ \Psi_k^{-1}(v^{(k)}(x))$ where $p_2 : U_k \times \mathbf{R}^s \to \mathbf{R}^s$ is the second projection.

The parallel extension v' of v is defined by :

$$v'(x) = \sum_k \phi_k(y) v^{(k)}(x)$$

It satisfies :

Lemma. [S2] Let v be a section of E over a closed subset $A \subset V_i$, then :

a) for every $\varepsilon > 0$, the parallel extension v' of v is a section of E on $\mathcal{T}_\varepsilon(A)$. If v is non zero, then v' is also non zero,

b) if two sections of E are homotopic on $A \subset V_i$, then their parallel extensions are also homotopic on $\mathcal{T}_\varepsilon(A)$.

b) *The transversal vectorfield.*

Fix a closed subset $A \subset V_i^s$, we choose a Riemannian metric and a function μ of class C^2 on an open neighborhood of A in M. Fix $\eta > 0$, for every point $y \in A$, we call $\Theta_\eta(y)$ the $(m-s)$-disc whose radii are the geodesic arcs issued from y, orthogonal to V_i and with common length $\mu(y) = \eta$. Write $\Theta_\eta(A) = \bigcup_{y \in A} \Theta_\eta(y)$ where η is so small that all the discs $\Theta_\eta(y)$ are disjoint.

The transversal vectorfield is constructed using the canonical vectorfield $g(x) = \overrightarrow{grad\ x}$ tangent to the radii of $\Theta_\eta(A)$. This vectorfield is not tangent to the strata and its projection $g'(x)$ on E does not define a continuous vectorfield.

To obtain a continuous stratified vectorfield $w(x)$, we proceed by induction on the dimension of the strata, as following : Let V_j be a stratum such that $V_i \subset \overline{V}_j$. Suppose that $w = w^{(j)}$ is already known on V_j and suppose V_k is a stratum such that $V_i \subset \overline{V}_j \subset \overline{V}_k$. We may consider, in $\Theta_\eta(A)$, a neighborhood P of V_j in $\overline{V}_k$ and whose radii are those of $\mathcal{T}_\varepsilon(V_j)$ in a neighborhood of V_i. Fix a radius $]y, z]$ of P such that $y \in V_j$ and $z \in V_k$; for every point $x \in]y, z]$, we define :

$$w(x) = (1 - \lambda(x))w^{(j)}(x) + \lambda(x)g'(x)$$

where $\lambda(x) = \dfrac{\overline{yx}}{\overline{yz}}$ and $w^{(j)}$ is the field w already built on V_j. The vectorfield w is called transversal vectorfield. Obviously, $w(x)$ is $w^{(j)}(x)$ on V_j and $g'(x)$ on $V_k - P$.

The Whitney condition (b) shows that :

Lemma. [S2] If A is a compact subset of V_i^s, for every fixed $\varepsilon > 0$, there are a riemannian metric μ on M, a tubular neighborhood $\Theta_\eta(A)$ of A such that the transversal vectorfield w in $\Theta_\eta(A)$ satisfies :

(a) for every point $x \in \Theta_\eta(A)$, we have : angle $\langle w(x), g(x) \rangle < \varepsilon$,

(b) the transversal vectorfield w is a section of E defined as an extension of the zero section on A. It does not have zeroes in $\Theta_\eta(A) - A$ and it points outward of $\Theta_\eta(A)$ on $\partial\Theta_\eta(A) - \Theta_\eta(\partial A)$.

c) *The local radial extension of a vectorfield.*

Let A denote a closed subset of V_i and A' a closed neighborhood of A in V_i such that, for $\varepsilon > 0$ and $\eta' > 0$, we have $\mathcal{T}_\varepsilon(A) \subset \Theta_{\eta'}(A')$. Consider a section v of $T(V_i)$ over A, such that $\|v\| < 1$, relatively to the previous Riemannian metric. We can define a parallel extension v' of v in $\mathcal{T}_\varepsilon(A)$ and a transversal vectorfield w in $\Theta_{\eta'}(A')$.

Following MH Schwartz [S2], we denote by v^{rad} and we call radial vectorfield the extension $v^{rad} = v' + w$. The radial vectorfield satisfies the following properties :

Proposition. [S2] If $\eta > 0$ and $\varepsilon > 0$ are sufficiently small, the radial vectorfield v^{rad}defined in $\mathcal{T}_\varepsilon(A)$ satisfies :

i) if $B \subset A$, the vectorfield v^{rad} points outward of every tube $\Theta_\eta(B) \subset \mathcal{T}_\varepsilon(A)$ on $\partial\Theta_\eta(B) - \Theta_\eta(\partial B)$,

ii) if a point $a \in B \subset V_i$ is an isolated singularity of v, it is also an isolated singularity of v^{rad} and v^{rad} satisfies the "conservation of indices" property :

$$I(v^{rad}, a) = I(v^{rad}|_{V_i}, a) \qquad \text{with } v^{rad}|_{V_i} = v$$

iii) if two sections v_0 and v_1 of $T(V_i)|_B$ are homotopic, then their radial extensions v_0^{rad} and v_1^{rad} are also homotopic on $\Theta_\eta(B)$.

3. Poincaré-Hopf theorems.

The radial vectorfields are the good ones to recover a Poincaré-Hopf theorem for singular varieties. Let D be a compact subset in M such that ∂D is smooth and transverse to all strata V_i. Let w be a stratified vectorfield pointing outward of $D \cap X$ on $\partial D \cap X$, without singularity on $\partial D \cap X$ and with isolated singularities $a_k \in V_{i(a_k)} \cap D$, where $V_{i(a_k)}$ denotes the stratum containing the point a_k. For such a vectorfield, and using the usual definition of index, the Poincaré-Hopf theorem is false in general, i.e.

$$\sum_{a_k \in X \cap D} I(w|_{V_{i(a_k)}}, a_k) \neq \chi(X \cap D)$$

It is easy to construct such an example on the pinched torus, see also the example in [S2],6.2.1.

To recover the Poincaré-Hopf theorem, one has either to use the radial vectorfields with the usual definition of index, or to modify the definition of index.

Theorem. [S2] (Generalization of the Poincaré-Hopf theorem). Let X be a Whitney stratified analytic variety imbedded in an analytic manifold M. Let D be a compact subset of M with smooth boundary, transverse to the strata. Let v^{rad} be a radial vectorfield pointing outward of $D \cap X$ on $\partial D \cap X$ and with isolated singularities $a_k \in V_{i(a_k)} \cap D$, then :

$$\sum_{a_k \in X \cap D} I(v^{rad}|_{V_{i(a_k)}}, a_k) = \chi(X \cap D)$$

where, if $\dim V_{i(a_k)} = 0$ then $I(v^{rad}|_{V_{i(a_k)}}, a_k) = +1$.

On the same way, we can construct a radial vectorfield v^{-rad} pointing inward of $D \cap X$ on $\partial D \cap X$ and with isolated singularities $a_h \in V_{i(a_h)} \cap D$.

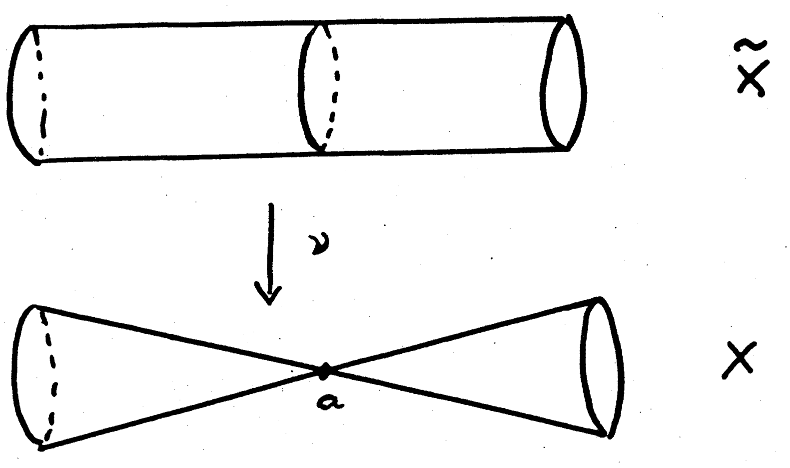
Theorem. [S2] Under the same hypotheses but if v^{-rad} points inward of $D \cap X$ on $\partial D \cap X$ then :

$$\begin{aligned}\sum_{a_h \in X \cap D} I(v^{-rad}|_{V_{i(a_h)}}, a_h) &= \chi(X \cap \mathrm{int}D) \\ &= \chi(X \cap D) - \chi(X \cap \partial D)\end{aligned}$$

We now consider the complex case. The previous constructions of radial vectorfields are also valid in the complex case, relatively to Whitney complex analytic stratifications (see [BS] and [S1]). From now on, let us denote by n, m and s the complex dimensions of X, M and V_i respectively. The tangent spaces will be complex tangent spaces.

4. Nash construction.

Let $G_n(TM)$ denote the Grassmannian bundle associated to the complex tangent bundle $\pi : TM \to M$. The fiber of $G_n(TM)$ over $x \in M$ is the set of n-complex planes in T_xM. The projection $\nu : G_n(TM) \to M$ has a canonical section σ over the regular part X_{reg} of X, defined by $\sigma(x) = T_x(X_{\text{reg}})$. The Nash blowup of X, denoted by $\widetilde{X}$, is, by definition, the closure of $\mathrm{Im}(\sigma)$ in $G_n(TM)$. We denote also by ν the analytic map $\widetilde{X} \to X$, restriction of ν (see [McP]).



The Nash blowup of a cone

The Nash blowup of a cone is a cylinder, but in general the Nash blowup is not a smooth manifold (see [McP]).

Consider the tautological bundle ξ over $G_n(TM)$. The fiber ξ_P over an n-plane $P \in G_n(TM)$ is the set of vectors $v \in P$. We will denote by $\widetilde{\xi}$ the restriction $\xi|_{\widetilde{X}}$. It is a subspace of

$$\Lambda = \{(v, P) \,:\, v \in TM|_X,\ P \in \widetilde{X} \subset G_n(TM),\ \pi(v) = \nu(P)\,\}\,,$$

and we will use the symbol $\nu_* : \widetilde{\xi} \to T(M)|_X$ to denote the restriction to $\widetilde{\xi}$ of the canonical projection $\Lambda \to T(M)$.

Proposition. [BS] a) Suppose $x \in V_i \subset X$ and $v(x) \in T_x(V_i)$. For every point $\widetilde{x}$ in $\nu^{-1}(x)$, there is an unique vector $\widetilde{v}(\widetilde{x})$ in $\widetilde{\xi}(\widetilde{x})$ such that $\nu_*(\widetilde{v}(\widetilde{x})) = v(x)$.

b) If v is a stratified vectorfield on $A \subset X$, then $\widetilde{v}$ defines a section of $\widetilde{\pi}: \widetilde{\xi} \to \widetilde{X}$ over $\widetilde{A} = \nu^{-1}(A)$, called the lifting of v.

Proof. We first consider the case $x \in X_{\text{reg}}$, then $\widetilde{x} = T_x(X_{\text{reg}})$ is the unique point of $\nu^{-1}(x)$ and the pair $\widetilde{v}(\widetilde{x}) = (v(x), \widetilde{x})$ is a well defined element of $\widetilde{\xi}(\widetilde{x})$. Suppose now $x \in V_i$ and $v(x) \in T_x(V_i)$. For every point $\widetilde{x} \in \nu^{-1}(x)$, there exists a sequence $(\widetilde{x}_n) \in \widetilde{X}$ converging to $\widetilde{x}$ and such that $\nu(\widetilde{x}_n) = x_n$ is a regular point of X. The sequence (x_n) converges to x and the limit $L = \lim T_{x_n}(X_{\text{reg}})$ exists and is identified with $\widetilde{x}$. The Whitney condition (a) implies that $T_x(V_i) \subset L$ and then $v(x) \in T_x(V_i) \subset \widetilde{x}$. The pair $\widetilde{v}(\widetilde{x}) = (v(x), \widetilde{x})$ is a well defined element of $\widetilde{\xi}(\widetilde{x})$. This proves the part a). The part b) is obvious.

5. Euler obstruction.

Fix a point a in a stratum V_i, we can suppose that the restriction of M to a neighborhood U_a of a is an open subset in $\mathbf{C}^m$ and $V_i \cap U_a$ is an open subset in $\mathbf{C}^s$. Let us denote by b the (Euclidean) ball with center a and (sufficiently small) radius r. The geodesic tube $\Theta = \Theta_\eta(b)$ defined above is transverse to all strata V_j such that $V_i \subset \overline{V_j}$.

Definition. ([McP],[BS]) Let v_0 be a vectorfield tangent to V_i, pointing outward on ∂b, such that a is an unique isolated singular point of v inside b. Let v_0^{rad} be the restriction to $\partial\Theta(b)$ of the radial extension of v_0 and $\widetilde{v_0^{\text{rad}}}$ the lifting of v_0^{rad} as a section of $\widetilde{\xi}$ over $\nu^{-1}(\partial\Theta)$. The obstruction to the extension of $\widetilde{v_0^{\text{rad}}}$ inside $\nu^{-1}(\Theta)$ as a non zero section of $\widetilde{\xi}$ is called local Euler obstruction of X in a and is denoted $\mathrm{Eu}_a(X)$. It is independent of the choices of v_0, b, η.

If a is a regular point of X then $\mathrm{Eu}_a(X) = 1$.

Proposition. [BS] The local Euler obstruction is constant along each stratum of a Whitney stratification of X. We say that the local Euler obstruction is a constructible fonction.

Proposition. [BS] (Multiplicativity property) Let v be a radial vectorfield with index $I(v,a)$ in an isolated singularity $a \in V_i$. Let b be a ball, with center a and radius so small that ∂b is transverse to all strata V_j such that $V_i \subset \overline{V_j}$. If v is non zero on ∂b, then the lifting $\widetilde{v}$ is a well defined section of $\widetilde{\xi}$ on $\nu^{-1}(\partial b \cap X)$. The obstruction to the extension of $\widetilde{v}$ as a non zero section of $\widetilde{\xi}$ on $\nu^{-1}(b \cap X)$ is given by :

$$\mathrm{Obs}(\widetilde{v}, \widetilde{\xi}, b) = \mathrm{Eu}_a(X) \times I(v,a).$$

The local Euler obstruction is one of the main tool in the construction of Chern classes of algebraic varieties (with singularities), due to R. MacPherson [McP]. These classes are the same, via an Alexander isomorphism, than the classes previously defined by M.H. Schwartz in 1965, using the obstruction theory [S1]. The multiplicativity property is one of the main steps for the demonstration of the equality of these classes (see [BS]).

References.

[BS] **J.P.Brasselet et M.H.Schwartz** *Sur les classes de Chern d'un ensemble analytique complexe,* Astérisque 82-83, 93-146, (1981).

[McP] **R.MacPherson** *Chern classes for singular algebraic varieties,* Annals of Maths., 100 n°2, pp 423 – 432 (1974).

[S1] **M.H.Schwartz** *Classes caractéristiques définies par une stratification d'une variété analytique complexe,* CRAS t.260 (1965), 3262-3264 et 3535-3537.

[S2] **M.H.Schwartz** *Champs radiaux sur une stratification analytique,* Travaux en cours 39 (1991).

Jean-Paul Brasselet, CIRM - SiGmA - CNRS,

CIRM Luminy Case 916, F-13288 Marseille Cedex 9

e-mail : jpb@cirm.univ-mrs.fr