<table>
<thead>
<tr>
<th>Title</th>
<th>A POINCARE-BENDIXSON TYPE THEOREM FOR HOLOMORPHIC VECTOR FIELDS (Singularities of Holomorphic Vector Fields and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>ITO, TOSHIKAZU</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1994), 878: 1-9</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1994-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/84173">http://hdl.handle.net/2433/84173</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
A POINCARÉ–BENDIXSON TYPE THEOREM
FOR HOLOMORPHIC VECTOR FIELDS

TOSHIKAZU ITO

INTRODUCTION

Let $Z_1$ be a linear vector field on the two-dimensional complex space $C^2$:

$$Z_1 = \sum_{j=1}^{2} \lambda_j z_j \partial/\partial z_j, \quad \lambda_j \in C, \quad \lambda_j \neq 0.$$ 

We have the following well-known

Fact ([1]). If $\lambda_1/\lambda_2$ does not belong to $\mathbb{R}_{-}$, the set of negative real numbers, then the three-dimensional unit sphere $S^3(1) = S^3(1:0)$ centered at the origin 0 in $C^2$ is transverse to the foliation $\mathcal{F}(Z_1)$ defined by the solutions of $Z_1$.

If $\lambda_1/\lambda_2$ belongs to $\mathbb{R}_{-}$, $S^3(1)$ is not transverse to $\mathcal{F}(Z_1)$.

We carry $S^3(1:0)$ to the sphere $S^3(1:(2,2))$ centered at the point $(2,2)$ in $C^2$. Next we deform $S^3(1:(2,2))$ to $\tilde{S}^3(1:(2,2))$ as shown in Figures 5 and 6.

Intuitively it appears that $S^3(1:(2,2))$ and $\tilde{S}^3(1:(2,2))$ are not transverse to $\mathcal{F}(Z_1)$. The above figures suggest to us a topological property of the transversality between spheres and holomorphic vector fields. This observation leads us to the following Poincaré–Hopf type theorem for holomorphic vector fields.

This research was partially supported by the Brazilian Academy of Sciences.
Theorem 1. Let $M$ be a subset of $\mathbb{C}^n$, diffeomorphic to the $2n$–dimensional closed disk $\bar{D}^{2n}(1)$ consisting of all $z$ in $\mathbb{C}^n$ with $\|z\| \leq 1$. We write $\mathcal{F}(Z)$ for the foliation defined by solutions of a holomorphic vector field $Z$ in some neighborhood of $M$. If the boundary of $M$ is transverse to $\mathcal{F}(Z)$, then $Z$ has only one singular point, say $p$, in $M$. Furthermore, the index of $Z$ at $p$ is equal to one.

From Theorem 1, we get an answer to the problem suggested by Figures 5 and 6.

Corollary 2. Consider a linear vector field in $\mathbb{C}^n$: $Z = \sum_{j=1}^{n} \lambda_j z_j \partial / \partial z_j$, $\lambda_j \in \mathbb{C}$, $\lambda_j \neq 0$. If a smooth imbedding $\varphi$ of $(2n-1)$– sphere $S^{2n-1}$ in $\mathbb{C}^n-\{0\}$ belongs to the zero element of the homotopy group $\pi_{2n-1}(\mathbb{C}^n-\{0\})$, then $\varphi$ is not transverse to $\mathcal{F}(Z)$.

Since the distance function for solutions of a holomorphic vector field $Z$ with respect to the origin $0$ is subharmonic, each solution of $Z$ is unbounded except the singular set of $Z$. Therefore we have formulated a Poincaré–Bendixson type theorem for holomorphic vector fields.

Theorem 3. Let $M$ denote a subset of $\mathbb{C}^n$ holomorphic and diffeomorphic to the $2n$–dimensional closed disk $\bar{D}^{2n}(1)$. Let $Z$ be a holomorphic vector field in some neighborhood of $M$. If the boundary $\partial M$ of $M$ is transverse to the foliation $\mathcal{F}(Z)$, then each solution of $Z$ which crosses $\partial M$ tends to the unique singular point $p$ of $Z$ in $M$, that is, $p$ is in the closure
of L. Further, the restriction $\mathcal{F}(Z)|_{M-\{p\}}$ of $\mathcal{F}(Z)$ to $M-\{p\}$ is $C^\omega$-diffeomorphic to the foliation $\mathcal{F}(Z)|_{\partial M \times (0, 1]}$ of $M-\{p\}$, where $\mathcal{F}(Z)|_{\partial M}$ denotes the restriction of $\mathcal{F}(Z)$ to $\partial M$.

Adrien Douady proved Theorem 3 in the case $n = 2$.

From Theorem 3 we get an affirmative answer to a special case of the Seifert conjecture.

**Corollary 4.** Let $Z$ be a holomorphic vector field in some neighborhood of $\bar{D}^4(1) \subset C^2$. If the boundary $\partial \bar{D}^4(1) = S^3(1)$ is transverse to $\mathcal{F}(Z)$, then the restriction $\mathcal{F}(Z)|_{S^3(1)}$ to $S^3$ has at least one compact leaf.

The author wishes to thank César Camacho for valuable discussions.

### §1. Definition of Transversality Between Manifolds and Holomorphic Vector Fields

Let $Z = \sum_{j=1}^{n} f_j(z) \partial/\partial z_j$ be a holomorphic vector field in the complex space $C^n$ of dimension $n$. We identify $C^n$ with the real space $R^{2n}$ of dimension $2n$ by the natural correspondence. We have a real representation of $Z$:

$$Z = \sum_{j=1}^{n} f_j(z) \partial/\partial z_j$$

$$= \sum_{j=1}^{n} (g_j(x, y) + ih_j(x, y)) \frac{1}{2}(\partial/\partial x_j - i \partial/\partial y_j)$$

$$= \frac{1}{2} \left\{ \left[ \sum_{j=1}^{n} (g_j(x, y) \partial/\partial x_j + h_j(x, y) \partial/\partial y_j) \right]$$

$$- i \left[ \sum_{j=1}^{n} (-h_j(x, y) \partial/\partial x_j + g_j(x, y) \partial/\partial y_j) \right] \right\}$$

$$= \frac{1}{2}(X - iY), \quad (1.1)$$

where we set

$$X = \sum_{j=1}^{n} (g_j(x, y) \partial/\partial x_j + h_j(x, y) \partial/\partial y_j) \quad (1.2)$$

and

$$Y = \sum_{j=1}^{n} (-h_j(x, y) \partial/\partial x_j + g_j(x, y) \partial/\partial y_j). \quad (1.3)$$
Let $J$ be the natural almost complex structure of $\mathbb{C}^n$. The vector fields $X$ and $Y$ satisfy the following equations:

\[ JX = Y, \quad JY = -X \quad \text{and} \quad [X, Y] = 0. \quad (1.4) \]

Let $N$ be a smooth manifold of dimension $2n - 1$. We define below the transversality of a smooth map $\Phi : N \to \mathbb{C}^n$ to the foliation $\mathcal{F}(Z)$ determined by solutions of $Z$.

**Definition 1.1.** We say that the map $\Phi$ is transverse to the foliation $\mathcal{F}(Z)$ or the holomorphic vector field $Z$ if the following equation is satisfied for each point $p \in N$:

\[ \Phi^*(T_p N) + \{X, Y\}_{\Phi(p)} = T_{\Phi(p)} \mathbb{R}^{2n}, \]

where $T_p N$ and $T_{\Phi(p)} \mathbb{R}^{2n}$ are the tangent space of $N$ at $p$ and the tangent space of $\mathbb{R}^{2n}$ at $\Phi(p)$ respectively, and $\{X, Y\}_{\Phi(p)}$ is the vector space generated by $X_{\Phi(p)}$ and $Y_{\Phi(p)}$. In particular, if $N$ is a submanifold in $\mathbb{C}^n$, we say that $N$ is transverse to $\mathcal{F}(Z)$.

For example consider the $(2n-1)$-dimensional sphere $S^{2n-1}(r)$, consisting of all $z \in \mathbb{C}^n$ with $||z|| = r$. $S^{2n-1}(r)$ is tangent to $\mathcal{F}(Z)$ at $p \in S^{2n-1}(r)$ if and only if the following equation is satisfied at $p$:

\[ \sum_{j=1}^{n} f_j(z) \bar{z}_j = (X, N) - i(Y, N) = 0, \quad (1.6) \]

where we denote by $N = \sum_{j=1}^{n} (z_j \partial / \partial x_j + y_j \partial / \partial y_j)$ the usual normal vector field on $S^{2n-1}(r)$. We set $\Sigma = \{ z \in \mathbb{C}^n | \sum_{j=1}^{n} f_j(z) \bar{z}_j = 0 \}$ and say that $\Sigma$ is the total contact set of spheres and $\mathcal{F}(Z)$. We denote by $R(z) = \sum_{j=1}^{n} |z_j|^2$ the distance function between $z \in \mathbb{C}^n$ and the origin 0 in $\mathbb{C}^n$. A critical point of the restriction $R|_L$ of $R$ to a solution $L$ of $Z$ is a contact point of $L$ and the sphere.

We will conclude this section by giving some examples of the contact set $\Sigma \cap S^{2n-1}(r)$ of $S^{2n-1}(r)$ and $\mathcal{F}(Z)$.

**Example 1.2.** Consider $Z = z_1(2 + z_1 + z_2) \partial / \partial z_1 + z_2(1 + z_1) \partial / \partial z_2$ defined in $\mathbb{C}^2$. The set $\text{Sing}(Z)$ of singular points of $Z$ consists of three points: $(0, 0), (-2, 0)$ and $(-1, -1)$. Now $\text{Sing}(Z) \cap \bar{D}^4(1)$ consists of $(0, 0)$ only, where $\bar{D}^4(1)$ is the four-dimensional closed disk centered at the origin in $\mathbb{C}^2$ with radius 1. For any $r$, $0 < r < 1$, the contact set $S^3(r) \cap \Sigma$ is empty; that is, $S^3(r)$ is transverse to $\mathcal{F}(Z)$. Therefore, each solution of $Z$ which crosses $S^3(1)$ tends to the origin in $\mathbb{C}^2$. 
Example 1.3. Let $a$ be a complex number different from zero. Define $Z$ on $C^2$ by $Z = (2z_1 + az_2^2) \partial/\partial z_1 + az_2 \partial/\partial z_2$. We mention here that one can find in [3] one of the normal forms of holomorphic vector fields in $C^2$:

$$\tilde{Z} = (\lambda_1 z_1 + az_2^2) \partial/\partial z_1 + \lambda_2 z_2 \partial/\partial z_2, \quad \lambda_1 = n\lambda_2.$$ 

The singular set $\text{Sing}(Z)$ consists of a single point $(0,0)$. There exists a number $r_0 > 0$ such that
(i) if $0 < r < r_0$, $\Sigma \cap S^3(r)$ is empty;
(ii) if $r = r_0$, $\Sigma \cap S^3(r_0)$ is diffeomorphic to the circle $S^1$;
(iii) if $r_0 < r$, $\Sigma \cap S^3(r)$ is diffeomorphic to the disjoint union $S^1 \sqcup S^1$ of two copies of the circle $S^1$.

In the case (ii), the circle $\Sigma \cap S^3(r_0)$ consists of degenerate critical points. If $L_p$ is the solution of $Z$ passing through $p \in \Sigma \cap S^3(r_0)$, then $L_p \cap \Sigma$ is a singleton set \{p\}.

In the case (iii), one circle of $\Sigma \cap S^3(r)$ consists of minimal points and the other consists of saddle points. In particular, for $p \in \Sigma \cap S^3(r)$ the set $L_p \cap \Sigma$ consists of two points $p$ and $q$, $p \neq q$. More precisely, one of these two points is a saddle point of $R|_{L_p}$ and the other a minimal point of $R|_{L_p}$.

Example 1.4. One finds in [4] the following example of a one-form $\omega$ on $C^2$: $\omega = z_2(1 - i - z_1 z_2)dz_1 - z_1(1 + i - z_1 z_2)dz_2$. We consider here $Z = z_1(1 + i - z_1 z_2) \partial/\partial z_1 + z_2(1 - i - z_1 z_2) \partial/\partial z_2$ on $C^2$. The singular set $\text{Sing}(Z)$ consists of a single point, namely $(0,0)$. If $0 < r < \sqrt{2}$, $\Sigma \cap S^3(r)$ is empty. If $r = \sqrt{2}$, $\Sigma \cap S^3(\sqrt{2})$ is diffeomorphic to the circle $S^1$. Indeed $\Sigma \cap S^3(\sqrt{2})$ belongs to the solution $z_1z_2 = 1$ of $Z$. If $r > \sqrt{2}$, $\Sigma \cap S^3(r)$ is diffeomorphic to the disjoint union $S^1 \sqcup S^1$ of two copies of the circle $S^1$, and consists of saddle points.

§2. PROOF OF THEOREM 1

In this section we shall use the same notation as in the previous sections.

First, we note that the following property of analytic sets in $C^n$: the set of singular points of $Z$ in $M$ consists of isolated finite points. Since the boundary $\partial M$ of $M$ is transverse to $\mathcal{F}(Z)$, there exists a smooth vector field $\xi$ in some neighborhood of $\partial M$ such that
(i) $\xi$ is represented by $aX + bY \neq 0$, where $a$ and $b$ are smooth functions defined in some neighborhood of $\partial M$;
(ii) $\xi$ is required to point outward at each point of $\partial M$.

We obtain a smooth map $(a,b)$ of some neighborhood of $\partial M$ to $R^2 - \{0\}$. When $n \geq 2$ using obstruction theory (see [9]), we can extend the map $(a,b)$ to a smooth map $(\alpha, \beta)$ of some neighborhood of $M$ to $R^2 - \{0\}$ such that the restriction of $(\alpha, \beta)$ to some neighborhood of $\partial M$ is the map $(a,b)$. 
There should be no confusion if we use $\xi$ for the extended smooth vector field $\xi = \alpha X + \beta Y$. By the definition of $\xi$ on a neighborhood of $M$, the set $\text{Sing}(Z)$ of the singular points of $Z$ coincides with that of $\xi$.

In order to calculate the index of $\xi$ at $p \in \text{Sing}(Z)$, we may think of the vector field $\xi$ as a map $\xi : M \to \mathbb{R}^{2n}$. Similarly we may think of the holomorphic vector field $Z$ as a map $Z : M \subset \mathbb{C}^{n} \to \mathbb{C}^{n}$ or as a map $Z : M \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. We say that the vector field $Z$ is non-degenerate at $p \in \text{Sing}(Z)$ if the Jacobian $\det(D(Z)(p))$ of $Z$ at $p$ is different from zero. By a direct calculation we obtain the following:

$$
\det(D(\xi)(p)) = \det \begin{pmatrix} \alpha(p) & -\beta(p) \\ \beta(p) & \alpha(p) \end{pmatrix} \det(D(Z)(p))
$$

$$= |\det((\alpha(p) + i\beta(p))I_{n})|^{2} |\det\left(\frac{\partial g_{j}}{\partial x_{k}}(p) + i\frac{\partial g_{j}}{\partial y_{k}}(p)\right)|^{2}, \tag{2.1}
$$

where $\det A$ denotes the determinant of a matrix $A$ and $I_{n}$ is the identity matrix of $\text{GL}(n, \mathbb{R})$. In particular, since $\det(D(Z)(p))$ is positive at a non-degenerate singular point $p \in \text{Sing}(Z)$, the index of $\xi$ at $p$ is one (see [6]).

In order to calculate the index of $\xi$ at a degenerate singular point $p \in \text{Sing}(Z)$, we recall the following

**Proper mapping theorem ([5]).** Let $F : \mathbb{C}^{n} \to \mathbb{C}^{n}$ be a holomorphic map such that $F(0)$ is equal to 0. Assume that 0 is an isolated point in $F^{-1}(0)$ and $\det(D(F)(0))$ is 0. Then there exists a number $\epsilon > 0$ together with a neighborhood $W$ of 0 such that $F|_{W} : W \to \Delta(0 : \epsilon) = \{z \in \mathbb{C}^{n}||z|| < \epsilon\}$ is surjective.

Using the proper mapping theorem we find a sufficiently small number $\epsilon > 0$ and a neighborhood $W$ of $p \in \text{Sing}(Z)$ such that $W \cap \text{Sing}(Z)$ is a singleton set. Since there exist regular values $y$ of $Z$ in $\Delta(0 : \epsilon_{1})$, by (2.1), we may select a regular value $y$ of $\xi$ in $\Delta(0 : \epsilon_{1}) = \{y \in \mathbb{R}^{2n}||y|| < \epsilon_{1}\}$, $0 < \epsilon_{1} < \epsilon$. The set $N_{1} = \xi^{-1}(\Delta(0 : \epsilon_{1})) \cap W$ is compact. We then choose a compact set $N$ with $W \supset N \supset N_{1}$ and a smooth function $\lambda$ which takes on the value one at $x \in N_{1}$ and zero at $x \notin N$. Define $\bar{\xi}$ by $\bar{\xi}(x) = \xi(x) - \lambda(x)y$. Then $\bar{\xi}$ is different from zero at each point $x \in N - N_{1}$; hence $\bar{\xi}^{-1}(0) \cap W$ is compact and each point $\bar{p} \in \bar{\xi}^{-1}(0) \cap W$ is non-degenerate. Now we are ready to calculate the index of the vector field $\xi$ at a degenerate point $p \in \text{Sing}(Z)$:

$$\text{index}_{\bar{p}} \bar{\xi} = \sum_{\bar{p} \in \bar{\xi}^{-1}(0) \cap W} \text{index}_{\bar{p}} \bar{\xi}
$$

$$= \text{the number of elements of } \xi^{-1}(0) \cap W \geq 1, \tag{2.2}
$$

where $\text{index}_{p} \xi$ denotes the index of $\xi$ at $p$. 
On the other hand, by the Poincaré–Hopf theorem we have the following:

$$1 = \chi(M) = \sum_{p \in \text{Sing}(Z) \cap M} \text{index}_p \xi,$$  \hspace{1cm} (2.3)

where $\chi(M)$ denotes the Euler number of $M$. From (2.2) and (2.3) we conclude that the number of elements of $\text{Sing}(Z)$ in $M$ is one. This completes the proof of Theorem 1.

§3. PROOF OF THEOREM 3

We continue to use the same notation.

Since $M$ is holomorphic, diffeomorphic to the $2n$–dimensional closed disk $\bar{D}^{2n}(1)$, we give a proof of Theorem 3 for $\bar{D}^{2n}(1)$. Using a Möbius transformation, we can assume that the sole singular point of $Z$ in $\bar{D}^{2n}(1)$ is the origin 0. We define a function $F$ in some neighborhood of $\bar{D}^{2n}$ minus the origin 0 by

$$F(z) = \frac{\sum_{j=1}^{n}f_{j}(z)\bar{z}_{j}}{\sum_{j=1}^{n}|z_{j}|^{2}}.$$

Since the boundary $S^{2n-1}(1)$ of $\bar{D}^{2n}(1)$ is transverse to $\mathcal{F}(Z)$, the restriction $F|_{S^{2n-1}(1)}$ of $F$ to $S^{2n-1}(1)$ takes on the values in $C \setminus \{0\}$. Consider a complex line $l_{z}$ through a point $z \in S^{2n-1}(1)$: $l_{z} = \{tz \in C^n | t \in C\}$.

We define a holomorphic function $\tilde{F}(t : z)$ in some neighborhood of $\bar{D}^{2n}(1 : 0) = \{t \in C | |t| \leq 1\}$ by

$$\tilde{F}(t : z) = \begin{cases} \frac{\sum_{j=1}^{n}f_{j}(tz)\bar{t}\bar{z}_{j}}{t\bar{t}}, & \text{if } t \neq 0 \\ \sum_{j,k=1}^{n} \frac{\partial f_{j}}{\partial z_{k}}(0)z_{k}\bar{z}_{j}, & \text{if } t = 0. \end{cases}$$

Then the degree of $\tilde{F}|_{|t|=1}$ is zero, because $F|_{S^{2n-1}(1)}$ is homotopic to a constant map. Hence, for any $z \in S^{2n-1}(1)$, $\tilde{F}(t : z)$ is not zero; that is, the only element of $\Sigma \cap \bar{D}^{2n}(1)$ is the origin 0 in $C^n$. In other words, $S^{2n-1}(r), 0 < r \leq 1$, are transverse to $\mathcal{F}(Z)$. Let $\tilde{N} \in T\mathcal{F}(Z)$ be the vector field of the projection of $N$ to $T\mathcal{F}(Z)$. The set of singular points of $\tilde{N}$ in $\bar{D}^{2n}(1)$ is the singleton set $\{0\}$ in $C^n$. Then each solution of $Z$ which crosses $S^{2n-1}(1)$ tends to 0 along the orbit of $\tilde{N}$. Furthermore, the restricted foliation $\mathcal{F}(Z)|_{S^{2n-1}(r)}$ of $S^{2n-1}(r)$ is $C^{\infty}$–diffeomorphic to the foliation $\mathcal{F}(Z)|_{S^{2n-1}(1)}$ of $S^{2n-1}(1)$ by the correspondence along orbits of $\tilde{N}$. This completes the proof of Theorem 3.
§4. A SPECIAL CASE OF SEIFERT CONJECTURE

The notation used in the Introduction, §1 and §3 carries over in the present section.

We first recall the Seifert conjecture. Consider the vector field $e = z_1 \partial/\partial z_1 + z_2 \partial/\partial z_2$ on $C^2$. All leaves of the restricted foliation $\mathcal{F}(e)|_{S^3(1)}$ of $S^3(1)$ are fibres of the Hopf fibration $S^3 \rightarrow S^2$. On the other hand, consider the vector field $e_\epsilon = (z_1 + \epsilon z_2) \partial/\partial z_1 + z_2 \partial/\partial z_2$, where the number $\epsilon$ is sufficiently small. Then the restricted foliation $\mathcal{F}(e_\epsilon)|_{S^3(1)}$ of $S^3(1)$ has one closed orbit $|z_1| = 1$ but all other leaves are diffeomorphic to $R^1$. In [8] H. Seifert proved the following

Theorem (H. Seifert). A continuous vector field on the three-sphere which differs sufficiently little from $\mathcal{F}(e)|_{S^3(1)}$ and which sends through every point exactly one integral curve, has at least one closed integral curve.

The Seifert conjecture says "every non-singular vector field on the three-dimensional sphere $S^3$ has a closed integral curve".

In [7] Paul Schweitzer constructed a counterexample to the Seifert conjecture: there exists a non-singular $C^1$ vector field on $S^3$ which has no closed integral curves.

In this section we investigate a certain property of a non-singular vector field on $S^3$ induced by a holomorphic vector field in some neighborhood of $\overline{D}^4(1)$ which is transverse to $S^3(1)$. This will prove Corollary 4.

Proof of Corollary 4. Using a M"obius transformation, we can assume that the only singular point of $Z$ in $\overline{D}^4(1)$ is the origin. First, we note that the existence of a separatrix of $Z$ at 0 was proved by C. Camacho and P. Sad [2]. Let $L$ be a separatrix of $Z$ at 0. There is a sufficiently small number $\epsilon > 0$ together with a holomorphic function $f$ defined in $D^4(\epsilon)$ such that $D^4(\epsilon) \cap \tilde{L} = \{f = 0\}$. Then for each $\epsilon_1$, $0 < \epsilon_1 < \epsilon$, $S^3(\epsilon_1) \cap L$ is a circle. Since $\mathcal{F}(F)|_{S^3(\epsilon_1)}$ is $C^\omega$-diffeomorphic to $\mathcal{F}(F)|_{S^3(1)}$, the latter has at least one compact leaf. This completes the proof of Corollary 4.

REFERENCES


DEPARTMENT OF NATURAL SCIENCES, RYUKOKU UNIVERSITY, FUSHIMI-KU KYOTO 612 JAPAN