A POINCARÉ–BENDIXSON TYPE THEOREM
FOR HOLOMORPHIC VECTOR FIELDS

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INTRODUCTION
Let $Z_1$ be a linear vector field on the two-dimensional complex space $C^2$:

$$Z_1 = \sum_{j=1}^{2} \lambda_j z_j \partial/\partial z_j, \quad \lambda_j \in \mathbb{C}, \quad \lambda_j \neq 0.$$ 

We have the following well-known

Fact ([1]). If $\lambda_1/\lambda_2$ does not belong to $R_-$, the set of negative real numbers, then the three-dimensional unit sphere $S^3(1) = S^3(1:0)$ centered at the origin 0 in $C^2$ is transverse to the foliation $\mathcal{F}(Z_1)$ defined by the solutions of $Z_1$.

If $\lambda_1/\lambda_2$ belongs to $R_-$, $S^3(1)$ is not transverse to $\mathcal{F}(Z_1)$.

We carry $S^3(1:0)$ to the sphere $S^3(1:(2,2))$ centered at the point $(2,2)$ in $C^2$. Next we deform $S^3(1:(2,2))$ to $\tilde{S}^3(1:(2,2))$ as shown in Figures 5 and 6.

Intuitively it appears that $S^3(1:(2,2))$ and $\tilde{S}^3(1:(2,2))$ are not transverse to $\mathcal{F}(Z_1)$. The above figures suggest to us a topological property of the transversality between spheres and holomorphic vector fields. This observation leads us to the following Poincaré–Hopf type theorem for holomorphic vector fields.

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Theorem 1. Let $M$ be a subset of $C^n$, diffeomorphic to the $2n$-dimensional closed disk $\overline{D}^{2n}(1)$ consisting of all $z$ in $C^n$ with $||z|| \leq 1$. We write $\mathcal{F}(Z)$ for the foliation defined by solutions of a holomorphic vector field $Z$ in some neighborhood of $M$. If the boundary of $M$ is transverse to $\mathcal{F}(Z)$, then $Z$ has only one singular point, say $p$, in $M$. Furthermore, the index of $Z$ at $p$ is equal to one.

From Theorem 1, we get an answer to the problem suggested by Figures 5 and 6.

Corollary 2. Consider a linear vector field in $C^n$: $Z = \sum_{j=1}^{n} \lambda_j z_j \partial/\partial z_j$, $\lambda_j \in C$, $\lambda_j \neq 0$. If a smooth imbedding $\varphi$ of $(2n-1)$-sphere $S^{2n-1}$ in $C^n-\{0\}$ belongs to the zero element of the homotopy group $\pi_{2n-1}(C^n-\{0\})$, then $\varphi$ is not transverse to $\mathcal{F}(Z)$.

Since the distance function for solutions of a holomorphic vector field $Z$ with respect to the origin 0 is subharmonic, each solution of $Z$ is unbounded except the singular set of $Z$. Therefore we have formulated a Poincaré-Bendixson type theorem for holomorphic vector fields.

Theorem 3. Let $M$ denote a subset of $C^n$ holomorphic and diffeomorphic to the $2n$-dimensional closed disk $\overline{D}^{2n}(1)$. Let $Z$ be a holomorphic vector field in some neighborhood of $M$. If the boundary $\partial M$ of $M$ is transverse to the foliation $\mathcal{F}(Z)$, then each solution of $Z$ which crosses $\partial M$ tends to the unique singular point $p$ of $Z$ in $M$, that is, $p$ is in the closure.
of $L$. Further, the restriction $\mathcal{F}(Z)|_{M-\{p\}}$ of $\mathcal{F}(Z)$ to $M-\{p\}$ is $C^\omega$-diffeomorphic to the foliation $\mathcal{F}(Z)|_{\partial M \times (0, 1]}$ of $M-\{p\}$, where $\mathcal{F}(Z)|_{\partial M}$ denotes the restriction of $\mathcal{F}(Z)$ to $\partial M$.

Adrien Douady proved Theorem 3 in the case $n=2$.

From Theorem 3 we get an affirmative answer to a special case of the Seifert conjecture.

Corollary 4. Let $Z$ be a holomorphic vector field in some neighborhood of $\overline{D}^4(1) \subset C^2$. If the boundary $\partial \overline{D}^4(1) = S^3(1)$ is transverse to $\mathcal{F}(Z)$, then the restriction $\mathcal{F}(Z)|_{S^3(1)}$ to $S^3$ has at least one compact leaf.

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§1. DEFINITION OF TRANSVERSALITY BETWEEN MANIFOLDS AND HOLOMORPHIC VECTOR FIELDS

Let $Z = \sum_{j=1}^{n} f_j(z) \partial/\partial z_j$ be a holomorphic vector field in the complex space $C^n$ of dimension $n$. We identify $C^n$ with the real space $R^{2n}$ of dimension $2n$ by the natural correspondence. We have a real representation of $Z$:

$$Z = \sum_{j=1}^{n} f_j(z) \partial/\partial z_j$$

$$= \sum_{j=1}^{n} (g_j(x, y) + ih_j(x, y)) \frac{1}{2} (\partial/\partial x_j - i \partial/\partial y_j)$$

$$= \frac{1}{2} \left\{ \left[ \sum_{j=1}^{n} (g_j(x, y) \partial/\partial x_j + h_j(x, y) \partial/\partial y_j) \right]$$

$$- i \left[ \sum_{j=1}^{n} (-h_j(x, y) \partial/\partial x_j + g_j(x, y) \partial/\partial y_j) \right] \right\}$$

$$= \frac{1}{2} (X - iY), \quad (1.1)$$

where we set

$$X = \sum_{j=1}^{n} (g_j(x, y) \partial/\partial x_j + h_j(x, y) \partial/\partial y_j) \quad (1.2)$$

and

$$Y = \sum_{j=1}^{n} (-h_j(x, y) \partial/\partial x_j + g_j(x, y) \partial/\partial y_j) \quad (1.3)$$
Let $J$ be the natural almost complex structure of $C^n$. The vector fields $X$ and $Y$ satisfy the following equations:

$$JX = Y, \quad JY = -X \quad \text{and} \quad [X, Y] = 0. \quad (1.4)$$

Let $N$ be a smooth manifold of dimension $2n - 1$. We define below the transversality of a smooth map $\Phi : N \to C^n$ to the foliation $\mathcal{F}(Z)$ determined by solutions of $Z$.

**Definition 1.1.** We say that the map $\Phi$ is transverse to the foliation $\mathcal{F}(Z)$ or the holomorphic vector field $Z$ if the following equation is satisfied for each point $p \in N$:

$$\Phi_*(T_p N) + \{X, Y\}_{\Phi(p)} = T_{\Phi(p)} R^{2n},$$

where $T_p N$ and $T_{\Phi(p)} R^{2n}$ are the tangent space of $N$ at $p$ and the tangent space of $R^{2n}$ at $\Phi(p)$ respectively, and $\{X, Y\}_{\Phi(p)}$ is the vector space generated by $X_{\Phi(p)}$ and $Y_{\Phi(p)}$. In particular, if $N$ is a submanifold in $C^n$, we say that $N$ is transverse to $\mathcal{F}(Z)$.

For example consider the $(2n-1)$-dimensional sphere $S^{2n-1}(r)$, consisting of all $z \in C^n$ with $||z|| = r$. $S^{2n-1}(r)$ is tangent to $\mathcal{F}(Z)$ at $p \in S^{2n-1}(r)$ if and only if the following equation is satisfied at $p$:

$$\sum_{j=1}^{n} f_j(z) \bar{z}_j = (X, N) - i(Y, N) = 0, \quad (1.6)$$

where we denote by $N = \sum_{j=1}^{n} (z_j \partial/\partial x_j + y_j \partial/\partial y_j)$ the usual normal vector field on $S^{2n-1}(r)$. We set $\Sigma = \{ z \in C^n | \sum_{j=1}^{n} f_j(z) \bar{z}_j = 0 \}$ and say that $\Sigma$ is the total contact set of spheres and $\mathcal{F}(Z)$. We denote by $R(z) = \sum_{j=1}^{n} |z_j|^2$ the distance function between $z \in C^n$ and the origin 0 in $C^n$. A critical point of the restriction $R|_L$ of $R$ to a solution $L$ of $Z$ is a contact point of $L$ and the sphere.

We will conclude this section by giving some examples of the contact set $\Sigma \cap S^{2n-1}(r)$ of $S^{2n-1}(r)$ and $\mathcal{F}(Z)$.

**Example 1.2.** Consider $Z = z_1(2+z_1+z_2) \partial/\partial z_1 + z_2(1+z_1) \partial/\partial z_2$ defined in $C^2$. The set $\text{Sing}(Z)$ of singular points of $Z$ consists of three points: $(0,0)$, $(-2,0)$, and $(-1,-1)$. Now $\text{Sing}(Z) \cap \bar{D}^4(1)$ consists of $(0,0)$ only, where $\bar{D}^4(1)$ is the four-dimensional closed disk centered at the origin in $C^2$ with radius 1. For any $r$, $0 < r < 1$, the contact set $S^3(r) \cap \Sigma$ is empty; that is, $S^3(r)$ is transverse to $\mathcal{F}(Z)$. Therefore, each solution of $Z$ which crosses $S^3(1)$ tends to the origin in $C^2$. 
Example 1.3. Let $a$ be a complex number different from zero. Define $Z$ on $\mathbb{C}^2$ by $Z = (2z_1 + az_2^2) \partial/\partial z_1 + z_2 \partial/\partial z_2$. We mention here that one can find in [3] one of the normal forms of holomorphic vector fields in $\mathbb{C}^2$:

$$\tilde{Z} = (\lambda_1 z_1 + az_2^2) \partial/\partial z_1 + \lambda_2 z_2 \partial/\partial z_2, \quad \lambda_1 = n\lambda_2.$$ 

The singular set $\text{Sing}(Z)$ consists of a single point $(0, 0)$. There exists a number $r_0 > 0$ such that

(i) if $0 < r < r_0$, $\Sigma \cap S^3(r)$ is empty;

(ii) if $r = r_0$, $\Sigma \cap S^3(r_0)$ is diffeomorphic to the circle $S^1$;

(iii) if $r_0 < r$, $\Sigma \cap S^3(r)$ is diffeomorphic to the disjoint union $S^1 \sqcup S^1$ of two copies of the circle $S^1$.

In the case (ii), the circle $\Sigma \cap S^3(r_0)$ consists of degenerate critical points. If $L_p$ is the solution of $Z$ passing through $p \in \Sigma \cap S^3(r_0)$, then $L_p \cap \Sigma$ is a singleton set $\{p\}$.

In the case (iii), one circle of $\Sigma \cap S^3(r)$ consists of minimal points and the other consists of saddle points. In particular, for $p \in \Sigma \cap S^3(r)$ the set $L_p \cap \Sigma$ consists of two points $p$ and $q$, $p \neq q$. More precisely, one of these two points is a saddle point of $R|_{L_p}$ and the other a minimal point of $R|_{L_p}$.

Example 1.4. One finds in [4] the following example of a one-form $\omega$ on $\mathbb{C}^2$: $\omega = z_2(1 - i - z_1z_2)dz_1 - z_1(1 + i - z_1z_2)dz_2$. We consider here $Z = z_1(1 + i - z_1z_2) \partial/\partial z_1 + z_2(1 - i - z_1z_2) \partial/\partial z_2$ on $\mathbb{C}^2$. The singular set $\text{Sing}(Z)$ consists of a single point, namely $(0, 0)$. If $0 < r < \sqrt{2}$, $\Sigma \cap S^3(r)$ is empty. If $r = \sqrt{2}$, $\Sigma \cap S^3(\sqrt{2})$ is diffeomorphic to the circle $S^1$. Indeed $\Sigma \cap S^3(\sqrt{2})$ belongs to the solution $z_1z_2 = 1$ of $Z$. If $r > \sqrt{2}$, $\Sigma \cap S^3(r)$ is diffeomorphic to the disjoint union $S^1 \sqcup S^1$ of two copies of the circle $S^1$, and consists of saddle points.

§2. PROOF OF THEOREM 1

In this section we shall use the same notation as in the previous sections.

First, we note that the following property of analytic sets in $\mathbb{C}^n$: the set of singular points of $Z$ in $M$ consists of isolated finite points. Since the boundary $\partial M$ of $M$ is transverse to $\mathcal{F}(Z)$, there exists a smooth vector field $\xi$ in some neighborhood of $\partial M$ such that

(i) $\xi$ is represented by $aX + bY \neq 0$, where $a$ and $b$ are smooth functions defined in some neighborhood of $\partial M$;

(ii) $\xi$ is required to point outward at each point of $\partial M$.

We obtain a smooth map $(a, b)$ of some neighborhood of $\partial M$ to $\mathbb{R}^2 - \{0\}$. When $n \geq 2$ using obstruction theory (see [9]), we can extend the map $(a, b)$ to a smooth map $(\alpha, \beta)$ of some neighborhood of $M$ to $\mathbb{R}^2 - \{0\}$ such that the restriction of $(\alpha, \beta)$ to some neighborhood of $\partial M$ is the map $(a, b)$.
There should be no confusion if we use $\xi$ for the extended smooth vector field $\xi = \alpha X + \beta Y$. By the definition of $\xi$ on a neighborhood of $M$, the set $\text{Sing}(Z)$ of the singular points of $Z$ coincides with that of $\xi$.

In order to calculate the index of $\xi$ at $p \in \text{Sing}(Z)$, we may think of the vector field $\xi$ as a map $\xi : M \to \mathbb{R}^n$. Similarly we may think of the holomorphic vector field $Z$ as a map $Z : M \subset \mathbb{C}^n \to \mathbb{C}^n$ or as a map $Z : M \subset \mathbb{R}^n \to \mathbb{R}^n$. We say that the vector field $Z$ is non-degenerate at $p \in \text{Sing}(Z)$ if the Jacobian $\det(D(Z)(p))$ of $Z$ at $p$ is different from zero. By a direct calculation we obtain the following:

$$\det(D(\xi)(p)) = \det \left( \begin{array}{cc} \alpha(p)I_n & -\beta(p)I_n \\ \beta(p)I_n & \alpha(p)I_n \end{array} \right) \det(D(Z)(p))$$

$$= |\det((\alpha(p) + i\beta(p))I_n)|^2 \left| \det \left( \frac{\partial g_j}{\partial x_k}(p) + i \frac{\partial g_j}{\partial y_k}(p) \right) \right|^2,$$

where $\det A$ denotes the determinant of a matrix $A$ and $I_n$ is the identity matrix of $\text{GL}(n, \mathbb{R})$. In particular, since $\det(D(Z)(p))$ is positive at a non-degenerate singular point $p \in \text{Sing}(Z)$, the index of $\xi$ at $p$ is one (see [6]).

In order to calculate the index of $\xi$ at a degenerate singular point $p \in \text{Sing}(Z)$, we recall the following

**Proper mapping theorem ([5]).** Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map such that $F(0)$ is equal to $0$. Assume that $0$ is an isolated point in $F^{-1}(0)$ and $\det(D(F)(0))$ is $0$. Then there exists a number $\epsilon > 0$ together with a neighborhood $W$ of $0$ such that $F|_W : W \to \Delta(0 : \epsilon) = \{ z \in \mathbb{C}^n | ||z|| < \epsilon \}$ is surjective.

Using the proper mapping theorem we find a sufficiently small number $\epsilon > 0$ and a neighborhood $W$ of $p \in \text{Sing}(Z)$ such that $W \cap \text{Sing}(Z)$ is a singleton set. Since there exist regular values $y$ of $Z$ in $\Delta(0 : \epsilon)$, by (2.1), we may select a regular value $y$ of $\xi$ in $\Delta(0 : \epsilon_1) = \{ y \in \mathbb{R}^{2n} ||y|| < \epsilon_1 \}$, $0 < \epsilon_1 < \epsilon$. The set $N_1 = \xi^{-1}(\Delta(0 : \epsilon_1)) \cap W$ is compact. We then choose a compact set $N$ with $W \supset N \supset N_1$ and a smooth function $\lambda$ which takes on the value one at $x \in N_1$ and zero at $x \notin N$. Define $\tilde{\xi}$ by $\tilde{\xi}(x) = \xi(x) - \lambda(x)y$. Then $\tilde{\xi}$ is different from zero at each point $x \in N - N_1$; hence $\tilde{\xi}^{-1}(0) \cap W$ is compact and each point $\tilde{p} \in \tilde{\xi}^{-1}(0) \cap W$ is non-degenerate. Now we are ready to calculate the index of the vector field $\xi$ at a degenerate point $p \in \text{Sing}(Z)$:

$$\text{index}_p \xi = \sum_{\tilde{p} \in \tilde{\xi}^{-1}(0) \cap W} \text{index}_p \tilde{\xi}$$

$$= \text{the number of elements of } \tilde{\xi}^{-1}(0) \cap W \geq 1,$$  \hspace{1cm} (2.2)

where $\text{index}_p \xi$ denotes the index of $\xi$ at $p$. 
On the other hand, by the Poincaré–Hopf theorem we have the following:

\[ 1 = \chi(M) = \sum_{p \in \text{Sing}(Z) \cap M} \text{index}_p \xi, \quad (2.3) \]

where \( \chi(M) \) denotes the Euler number of \( M \). From (2.2) and (2.3) we conclude that the number of elements of \( \text{Sing}(Z) \) in \( M \) is one. This completes the proof of Theorem 1.

§3. PROOF OF THEOREM 3

We continue to use the same notation.

Since \( M \) is holomorphic, diffeomorphic to the \( 2n \)-dimensional closed disk \( \bar{D}^{2n}(1) \), we give a proof of Theorem 3 for \( \bar{D}^{2n}(1) \). Using a Möbius transformation, we can assume that the sole singular point of \( Z \) in \( \bar{D}^{2n}(1) \) is the origin \( 0 \). We define a function \( F \) in some neighborhood of \( \bar{D}^{2n}(1) \) minus the origin \( 0 \) by

\[ F(z) = \frac{\sum_{j=1}^{n} f_j(z) \bar{z}_j}{\sum_{j=1}^{n} |z_j|^2}. \]

Since the boundary \( S^{2n-1}(1) \) of \( \bar{D}^{2n}(1) \) is transverse to \( \mathcal{F}(Z) \), the restriction \( F|_{S^{2n-1}(1)} \) of \( F \) to \( S^{2n-1}(1) \) takes on the values in \( C - \{0\} \). Consider a complex line \( l_z \) through a point \( z \in S^{2n-1}(1) \): \( l_z = \{tz \in C^n | t \in C \} \).

We define a holomorphic function \( \tilde{F}(t : z) \) in some neighborhood of \( \bar{D}^2(1 : 0) = \{ t \in C | |t| \leq 1 \} \) by

\[ \tilde{F}(t : z) = \begin{cases} \frac{\sum_{j=1}^{n} f_j(tz) \bar{z}_j}{tt} , & \text{if } t \neq 0 \\ \sum_{j,k=1}^{n} \frac{\partial f_j}{\partial z_k}(0) z_k \bar{z}_j , & \text{if } t = 0. \end{cases} \]

Then the degree of \( \tilde{F}|_{|t|=1} \) is zero, because \( F|_{S^{2n-1}(1)} \) is homotopic to a constant map. Hence, for any \( z \in S^{2n-1}(1) \), \( \tilde{F}(t : z) \) is not zero; that is, the only element of \( \Sigma \cap \bar{D}^{2n}(1) \) is the origin \( 0 \) in \( C^n \). In other words, \( S^{2n-1}(r), \ 0 < r \leq 1 \), are transverse to \( \mathcal{F}(Z) \). Let \( \bar{N} \in T\mathcal{F}(Z) \) be the vector field of the projection of \( N \) to \( T\mathcal{F}(Z) \). The set of singular points of \( \bar{N} \) in \( \bar{D}^{2n}(1) \) is the singleton set \( \{0\} \) in \( C^n \). Then each solution of \( Z \) which crosses \( S^{2n-1}(1) \) tends to \( 0 \) along the orbit of \( \bar{N} \). Furthermore, the restricted foliation \( \mathcal{F}(Z)|_{S^{2n-1}(r)} \) of \( S^{2n-1}(r) \) is \( C^\infty \)-diffeomorphic to the foliation \( \mathcal{F}(Z)|_{S^{2n-1}(1)} \) of \( S^{2n-1}(1) \) by the correspondence along orbits of \( \bar{N} \). This completes the proof of Theorem 3.
§4. A SPECIAL CASE OF SEIFERT CONJECTURE

The notation used in the Introduction, §1 and §3 carries over in the present section.

We first recall the Seifert conjecture. Consider the vector field $\mathbf{e} = z_1 \partial/\partial z_1 + z_2 \partial/\partial z_2$ on $\mathbb{C}^2$. All leaves of the restricted foliation $\mathcal{F}(\mathbf{e})|_{S^3(1)}$ of $S^3(1)$ are fibres of the Hopf fibration $S^3 \to S^2$. On the other hand, consider the vector field $\mathbf{e}_\epsilon = (z_1 + \epsilon z_2) \partial/\partial z_1 + z_2 \partial/\partial z_2$, where the number $\epsilon$ is sufficiently small. Then the restricted foliation $\mathcal{F}(\mathbf{e}_\epsilon)|_{S^3(1)}$ of $S^3(1)$ has one closed orbit $|z_1| = 1$ but all other leaves are diffeomorphic to $\mathbb{R}^1$. In [8] H. Seifert proved the following

Theorem (H. Seifert). A continuous vector field on the three-sphere which differs sufficiently little from $\mathcal{F}(\mathbf{e})|_{S^3(1)}$ and which sends through every point exactly one integral curve, has at least one closed integral curve.

The Seifert conjecture says "every non-singular vector field on the three-dimensional sphere $S^3$ has a closed integral curve".

In [7] Paul Schweitzer constructed a counterexample to the Seifert conjecture: There exists a non-singular $C^1$ vector field on $S^3$ which has no closed integral curves.

In this section we investigate a certain property of a non-singular vector field on $S^3$ induced by a holomorphic vector field in some neighborhood of $D^4(1)$ which is transverse to $S^3(1)$. This will prove Corollary 4.

Proof of Corollary 4. Using a Möbius transformation, we can assume that the only singular point of $Z$ in $D^4(1)$ is the origin. First, we note that the existence of a separatrix of $Z$ at 0 was proved by C. Camacho and P. Sad [2]. Let $L$ be a separatrix of $Z$ at 0. There is a sufficiently small number $\epsilon > 0$ together with a holomorphic function $f$ defined in $D^4(\epsilon)$ such that $D^4(\epsilon) \cap \overline{L} = \{ f = 0 \}$. Then for each $\epsilon_1$, $0 < \epsilon_1 < \epsilon$, $S^3(\epsilon_1) \cap L$ is a circle. Since $\mathcal{F}(F)|_{S^3(\epsilon_1)}$ is $C^\omega$-diffeomorphic to $\mathcal{F}(F)|_{S^3(1)}$, the latter has at least one compact leaf. This completes the proof of Corollary 4.

References