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Eigenvalue problems arising from two-component flow

By

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1 Introduction

In 1986 M. Renardy and D. D. Joseph wrote a paper " Hopf Bifurcation in Two - Component Flow " [1], where they discuss the stability of two - layer Couette flow . The physical configuration they treated is the following: Two - layers of viscous incompressible fluids are confined between two parallel plates and are separated by the interface. Two fluids are of equal density, but have different viscosities. The upper plate moves at constant speed U^* while the lower is at rest. See Figure 1.

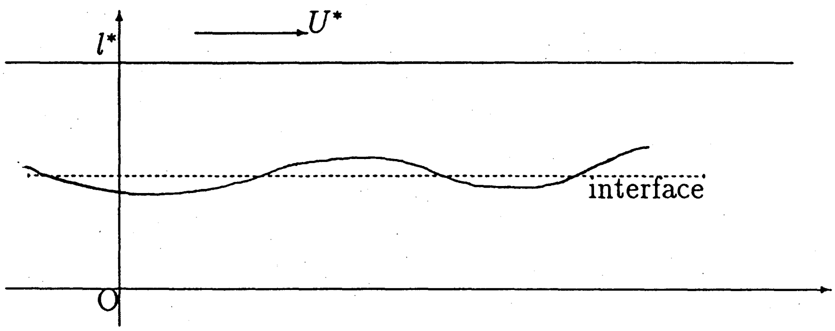

Figure 1.

In this configuration there always exists a stationary flow called "two-layer Couette flow" which has a piecewise linear velocity profile parallel to the plates.

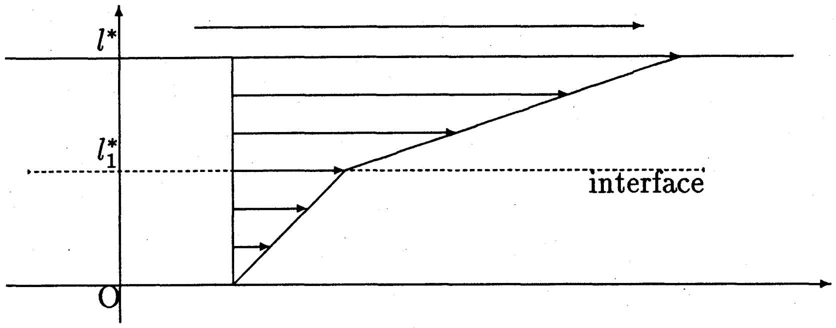

Figure 2.

In [1] they claim that , when U^* increases, the above flow becomes unstable and a bifurcation of the Hopf type is expected. After some computations for the problem derived by linearization around the above flow, they give a result (Theorem 4.1) , relying on [3]. There they assume that, at some critical speed $U^* = U_c^*$, there is a pair of complex conjugate eigenvalues which cross the imaginary axis transversally. To utilize the theory of [3], it is crucial to show the existence of such eigenvalues.

In this article we propose a method to obtain these by reducing the eigenvalue problem to the boundary value problem for the Orr – Sommerfeld equation. A number of numerical methods have been developed for dealing with this equation. (See [4] , Section 30 .) Among them we choose a shooting method to solve the boundary value problem. The eigenvalues are found by searching the zeros of the determinant of the matrix whose components are given by the fundamental solutions of the Orr – Sommerfeld equations. We finally prepare a method to study how the eigenvalue depends on parameters. Though we here outline our numerical method to obtain the desired eigenvalues, we will give an analytically rigorous result by taking " a posteori " error estimate into account.

2 Formulation of the problem

We use the same dimensionless variables as those in [2], Chap. IV. The velocity of the stationary flow in the dimensionless form is given by $(U(z), 0)$,where

$$U(z) = \begin{cases} \dfrac{1}{l_1 + m(1 - l_1)} z, & \text{for } 0 \leq z \leq l_1 \\[2ex] \dfrac{m}{l_1 + m(1 - l_1)} (z - 1) + 1, & \text{for } l_1 \leq z \leq 1. \end{cases}$$

$m = \mu_1/\mu_2$ is the viscosity ratio of two fluids. We consider $2D$ disturbances from this flow. $(u_j,\ w_j)$ and h denote a small disturbance to the velocity in fluid j, $j = I,\ II$ and to the interface position, respectively. The equations governing linear stability are

$$\begin{aligned}
&(2.1) \qquad & \frac{1}{\mathcal{R}}\Delta u_1 - \partial_x p_1 - w_1 \partial_z U - U(z)\partial_x u_1 &= \partial_t u_1, \\
&(2.2) \qquad & \frac{1}{\mathcal{R}}\Delta w_1 - \partial_z p_1 - U(z)\partial_x w_1 &= \partial_t w_1, \\
&(2.3) \qquad & \partial_x u_1 + \partial_z w_1 &= 0 \\
& & \text{in} \quad 0 < z < l_1 &
\end{aligned}$$

and

$$\begin{aligned}
&(2.4) \qquad & \frac{1}{m\mathcal{R}}\Delta u_2 - \partial_x p_2 - w_2 \partial_z U - U(z)\partial_x u_2 &= \partial_t u_2, \\
&(2.5) \qquad & \frac{1}{m\mathcal{R}}\Delta w_2 - \partial_z p_2 - U(z)\partial_x w_2 &= \partial_t w_2, \\
&(2.6) \qquad & \partial_x u_2 + \partial_z w_2 &= 0 \\
& & \text{in} \quad l_1 < z < 1. &
\end{aligned}$$

Here $\mathcal{R} = U^* l^* \rho/\mu_1$ is the Reynolds number based on the fluid I. See Fig. 1 for l^* . ρ is the density common to both fluids. The conditions at the plates are

$$(2.7) \qquad u_1 = w_1 = 0 \qquad \text{at } z = 0$$

and

$$(2.8) \qquad u_2 = w_2 = 0 \qquad \text{at } z = 1.$$

The kinematic boundary condition is written as

$$(2.9) \qquad w_1 - U(l_1)\frac{\partial h}{\partial x} = \partial_t h.$$

As in [1] we assume the periodicity in the streamwise direction x. We now introduce the stream functions ψ_j, $j = I, II$ for each fluid and rewrite the problem (2.1) - (2.6) in terms of ψ_j, where

$$u_j = \frac{\partial \psi_j}{\partial z}, \quad w_j = -\frac{\partial \psi_j}{\partial x}, \qquad j = I, II\,.$$

From the periodicity in $x-axis$, we assume that ψ_j is of the following form:

$$\psi_j(x,z,t) = \sum \psi_{j,\alpha}(z)\exp(i\alpha x + \sigma t), \quad j = I,\ II\,.$$

The interface position $h(x,t)$ is also expanded as above. Since we are concerned about only searching the pure imaginary eigenvalues, we stick to some mode $\alpha > 0$, which is fixed from now on. (2.1) - (2.6) yield the problem for $\psi_{j,\alpha},\ j = I, II$:

$$L_I\,\psi_I = 0, \qquad 0 < z < l_1, \tag{2.10}$$

$$\psi_I(0) = \frac{d\psi_I}{dz}(0) = 0, \tag{2.11}$$

$$L_{II}\,\psi_{II} = 0, \qquad l_1 < z < 1, \tag{2.12}$$

$$\psi_{II}(1) = \frac{d\psi_{II}}{dz}(1) = 0, \tag{2.13}$$

where

$$L_I = \left(\left(\frac{d}{dz}\right)^2 - \alpha^2\right)^2 - i\alpha\mathcal{R}(U(z)-c)\left(\left(\frac{d}{dz}\right)^2 - \alpha^2\right), \tag{2.14}$$

$$L_{II} = \left(\left(\frac{d}{dz}\right)^2 - \alpha^2\right)^2 - i\alpha m\mathcal{R}(U(z)-c)\left(\left(\frac{d}{dz}\right)^2 - \alpha^2\right). \tag{2.15}$$

Here and hereafter we set $\sigma = -i\alpha c$ and omit the subscript α . The interface conditions at $z = l_1$ are the following

$$\psi_I = \psi_{II}, \tag{2.16}$$

$$\frac{d\psi_I}{dz} + \frac{1-m}{l_1+ml_2}h = \frac{d\psi_{II}}{dz} \qquad (\, l_2 = 1 - l_1 \,), \tag{2.17}$$

$$\frac{d^2\psi_I}{dz^2} + \alpha^2\psi_I = \frac{1}{m}\left(\frac{d^2\psi_{II}}{dz^2} + \alpha^2\psi_{II}\right), \tag{2.18}$$

$$\begin{aligned} -\frac{d^3\psi_I}{dz^3} &+ \left(i\alpha\mathcal{R}\frac{l_1}{l_1+ml_2} - i\alpha c\mathcal{R} + 3\alpha^2\right)\frac{d\psi_I}{dz} \\ &-i\alpha\mathcal{R}\frac{l_1}{l_1+ml_2}\psi_I + i\alpha^3 Sh \\ = -\frac{1}{m}\frac{d^3\psi_{II}}{dz^3} &+ \left(i\alpha\mathcal{R}\frac{l_1}{l_1+ml_2} - i\alpha c\mathcal{R} + \frac{3\alpha^2}{m}\right)\frac{d\psi_{II}}{dz} \\ &-i\alpha\mathcal{R}\frac{m}{l_1+ml_2}\psi_{II} \end{aligned} \tag{2.19}$$

S in (2.19) is a surface tension number. The interface position h can be recovered from (2.9) so that we can substitute

$$h = \frac{1}{c - U(l_1)}\psi_1$$

in (2.17) and (2.19). For (2.16) - (2.19) see [2].

For convenience we define $\mathbf{C}^4$- valued function

$$\widehat{\varphi} = \left[\varphi, \frac{d\varphi}{dz}, \frac{d^2\varphi}{dz^2}, \frac{d^3\varphi}{dz^3} \right]^T$$

for scalar function φ. By use of this notation we can express (2.16) - (2.19) as

$$Z_I \, \widehat{\psi_I} = Z_{II} \, \widehat{\psi_{II}}, \tag{2.20}$$

where Z_I and Z_{II} are 4×4 matrices.

3 Method of analysis

We are now in a position to characterize the eigenvalue $\sigma = -i\alpha c$. If we can find a nontrivial solution $(\psi_I, \ \psi_{II})$ to (2.10), (2.11), (2.12), (2.13) and (2.20) for some σ, we call this value an eigenvalue of our linearized problem. Since the equation (2.10) is of the fourth order, The fundamental solutions of this ODE consist of four linearly independent solutions. As two of these we can take $\exp(-\alpha z)$ and $\exp(\alpha z)$ by the form of L_I. As other two we can take the solutions $f_{I,1}$ and $f_{I,2}$ with the initial conditions $\widehat{f_{I,1}}(0) = [0,0,1,0]^T$ and $\widehat{f_{I,2}}(0) = [0,0,0,1]^T$, respectively. Since the eigenfunction (ψ_I, ψ_{II}) must satisfy (2.11), the first component must be represented as a linear combination $C_1 f_{I,1} + C_2 f_{I,2}$. By same reasoning the second must be represented as $\psi_{II} = C_3 f_{II,1} + C_4 f_{II,2}$, where $f_{II,1}$ and $f_{II,2}$ are the solutions of (2.12) with the initial conditions $\widehat{f_{II,1}}(1) = [0,0,1,0]$ and $\widehat{f_{I,2}}(1) = [0,0,0,1]$, respectively. Therefore (2.20) takes the form

$$C_1 Z_I \widehat{f_{I,1}}(l_1) + C_2 Z_I \widehat{f_{I,2}}(l_1) = C_3 Z_{II} \widehat{f_{II,1}}(l_1) + C_4 Z_{II} \widehat{f_{II,2}}(l_1).$$

Hence, in order that σ becomes an eigenvalue, it is necessary and sufficient that the 4×4 matrix

$$\left[Z_I \widehat{f_{I,1}}(l_1), \ Z_I \widehat{f_{I,2}}(l_1), \ Z_{II} \widehat{f_{II,1}}(l_1), \ Z_{II} \widehat{f_{II,2}}(l_1) \right] \tag{3.1}$$

becomes singular. Set $\mathcal{F} = \det$ of (3.1). Since we set $\sigma = -i\alpha c$ and are interested in only pure imaginary eigenvalues, we restrict c to be real. So we regard $\mathcal{F}$ as a $\mathbf{C}$-valued function of $(c, \mathcal{R}) \in \mathbf{R}^2$. We can now reduce our eigenvalue problem to find zero of $\mathcal{F}(c, \mathcal{R})$. Since $\mathcal{F}$ is $\mathbf{C}$-valued, we can regard $\mathcal{F}(c, \mathcal{R}) = \text{real}(c, \mathcal{R}) + i\text{imag}(c, \mathcal{R})$ as an $\mathbf{R}^2$-valued function. Thus, regarding $\mathcal{F}$ as

$$\begin{bmatrix} \text{real}(c, \mathcal{R}) \\ \text{imag}(c, \mathcal{R}) \end{bmatrix} : \mathbf{R}^2 \longmapsto \mathbf{R}^2 ,$$

we can apply the Newton-Raphson method :

$$\begin{bmatrix} c_{n+1} \\ \\ \mathcal{R}_{n+1} \end{bmatrix} = \begin{bmatrix} c_n \\ \\ \mathcal{R}_n \end{bmatrix} - \begin{bmatrix} \frac{\partial}{\partial c}(\text{real}) & \frac{\partial}{\partial \mathcal{R}}(\text{real}) \\ \\ \frac{\partial}{\partial c}(\text{imag}) & \frac{\partial}{\partial \mathcal{R}}(\text{imag}) \end{bmatrix}^{-1} \begin{bmatrix} \text{real}(c_n, \mathcal{R}_n) \\ \\ \text{imag}(c_n, \mathcal{R}_n) \end{bmatrix}$$

to solve $\mathcal{F}(c, \mathcal{R}) = 0$. The values $\widehat{f_I}(l_1) \sim \widehat{f_{II}}(l_1)$ are obtained by numerical integration. In order to find the derivatives of $\mathcal{F}(c, \mathcal{R}) = \text{real}(c, \mathcal{R}) + i\text{imag}(c, \mathcal{R})$ we differentiate the equations and the boundary conditions with respect to c and $\mathcal{R}$ and solve these numerically.

An example: We obtain det $= 9.42964\ E - 08 + i(-8.70205\ E - 09)$ at $\alpha = 1.0$, $m = 0.5$, $l_1 = 0.5$, $S = 0.0031598565$ and $(\sigma, \mathcal{R}) = (0.593171 \times i, 9.9996984943046)$.

We finally propose a method to calculate $\frac{\partial \sigma}{\partial \mathcal{R}}$. Let $L_j^*\ j = I, II$ be the formal adjoint of (2.14) and (2.15) respectively. Set

$$\widetilde{Q_I} = \widetilde{Q_{II}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then the boundary conditions (2.11) - (2.13) are rewritten as

$$\widetilde{Q_I}\widehat{\psi_I}(0) = \widetilde{Q_{II}}\widehat{\psi_{II}}(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let Q_j be the 4×4 nonsingular matrix obtained from $\widetilde{Q_j}$ by adding two row vectors ($j = I, II$). Since Z_I and Z_{II} are of *rank* 4, we can find 4×4 nonsingular matrices J_j and K_j ($j = I, II$) so that , for smooth functions f_j, g_j ($j = I, II$), it holds that

$$\begin{aligned}(3.2)\qquad & (L_I f_I\,,\, g_I)_{L^2(0,l_1)} - (f_I\,,\, L_I^* g_I)_{L^2(0,l_1)} + \\ & \qquad (L_{II} f_{II}\,,\, g_{II})_{L^2(l_1,1)} - (f_{II}\,,\, L_{II}^* g_{II})_{L^2(l_1,1)} \\ &= \left(Q_I\widehat{f_I}(0)\,,\, J_I\widehat{g_I}(0)\right)_{\mathbf{C}^4} + \left(Z_I\widehat{f_I}(l_1)\,,\, K_I\widehat{g_I}(l_1)\right)_{\mathbf{C}^4} \\ &+ \left(-Z_{II}\widehat{f_{II}}(l_1)\,,\, K_{II}\widehat{g_{II}}(l_1)\right)_{\mathbf{C}^4} + \left(Q_{II}\widehat{f_{II}}(1)\,,\, J_{II}\widehat{g_{II}}(1)\right)_{\mathbf{C}^4}.\end{aligned}$$

We can show that, if the boundary value problem (2.10), (2.11), (2.12), (2.13), and (2.20) has a nontrivial solution, then the "adjoint" problem

$$(3.3)\qquad L_I^*\psi_I^* = 0, \qquad 0 < z < l_1,$$

$$(3.4)\qquad L_{II}^*\psi_{II}^* = 0; \qquad l_1 < z < 1,$$

$$(3.5)\qquad \widetilde{J_I}\widehat{\psi_I}^*(0) = \widetilde{J_{II}}\widehat{\psi_{II}}^*(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$(3.6)\qquad K_I\widehat{\psi_I}^*(l_1) = K_{II}\widehat{\psi_{II}}^*(l_1).$$

also has a nontrivial solution. Here $\widetilde{J_j}$ is the 2×4 matrix which consists of the lower two row vectors of J_j (j = I, II). After differentiating (2.10) and (2.12) with respect to $\mathcal{R}$, take L^2 inner product of the resulting equations with ψ_I^* and ψ_{II}^* respectively. By using (3.2), we can derive

$$
\begin{aligned}
(3.7) \qquad & \Big(i\alpha(U-c)(\psi_I'' - \alpha^2\psi_I)\ ,\ \psi_I^*\Big)_{L^2} + \\
& \qquad \Big(i\alpha m(U-c)(\psi_{II}'' - \alpha^2\psi_{II})\ ,\ \psi_{II}^*\Big)_{L^2} \\
& - \frac{\partial c}{\partial \mathcal{R}}\left\{i\alpha\mathcal{R}\left(\psi_I'' - \alpha^2\psi_I, \psi_I^*\right)_{L^2} + i\alpha m\mathcal{R}\left(\psi_{II}'' - \alpha^2\psi_{II}, \psi_{II}^*\right)_{L^2}\right\} \\
& \qquad = \left(Z_I \widehat{\frac{\partial \psi_I}{\partial \mathcal{R}}}(l_1) \ - \ Z_{II}\widehat{\frac{\partial \psi_{II}}{\partial \mathcal{R}}}(l_1)\ ,\ K_I \widehat{\psi_I^*}(l_1)\right)_{\mathbf{C}^4} .
\end{aligned}
$$

From this equality we can calculate $\dfrac{\partial \sigma}{\partial \mathcal{R}}$ numerically.

References

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