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Biarc Spline Interpolation

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1. Introduction. Biarc splines made of circular arcs and straight line segments are of much use for approximating discrete data in a plane since they are the paths by a drafting machine that can not draw a cubic curve directly ([2], [5]). In finding an interpolating biarc spline, most authors specify the tangents at the given data points, and then determine biarcs (a biarc is a pair of circular arcs having geometric continuity of degree one – G^1 continuity) that join the points and match those tangents. That is, for the data points $\mathbf{P}_i$ ($i = 1, 2$) and the tangents m_i at $\mathbf{P}_i$, letting D_1 and D_2 be the two circular arcs, then the biarc joining $\mathbf{P}_i$ ($i = 1, 2$) satisfies :

(1) D_i ($i = 1, 2$) pass through $\mathbf{P}_i$ and are tangential to m_i at $\mathbf{P}_i$

(2) D_i ($i = 1, 2$) are tangential to each other at their joint.

The biarc has six degrees of freedom, but the above conditions (1) and (2) use only five. This shows that the biarc is not unique. Therefore, various restricting conditions have been used to make the biarc unique. For example, the difference of the curvatures of the two circular arcs could be minimized, with the result that the joint is the on the right bisector of the line segment joining the two end points.

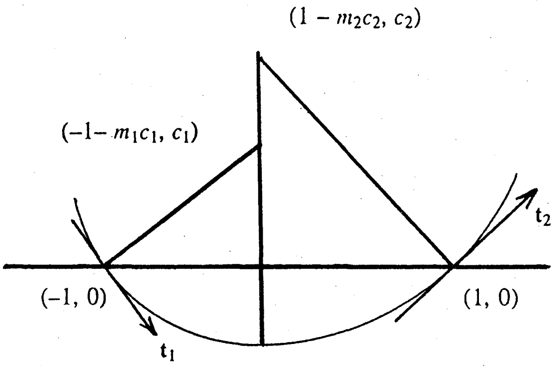
The object of Section 2 is to obtain some relations between the radii of the two circular arcs of the biarc. The object of Section 3 is to derive the locus of the joint of the biarc that have been used for the unique biarc [5]. Algorithms for finding of the biarc spline interpolation and some examples are given in Section 4.

2. Relations between radii of biarcs. It is easy to check that the following affine transformation corresponds points $(-1, 0)$ and $(1, 0)$ to ones (x_j, y_j) and (x_{j+1}, y_{j+1}), respectively:

$$(1)\qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = (1/2)\begin{bmatrix} (x_{j+1} - x_j) & -(y_{j+1} - y_j) \\ (y_{j+1} - y_j) & (x_{j+1} - x_j) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + (1/2)\begin{bmatrix} x_j + x_{j+1} \\ y_j + y_{j+1} \end{bmatrix}$$

and that an angle between any two vectors in the plane is invariant under the above affine transformation since the coefficient matrix on the right hand side of (1) means a rotation of the coordinate axes and a change of scale in any direction. Hence, it suffices to consider the case when a biarc passes through $(-1, 0)$ and $(1, 0)$ and matches two given

(or specified) unit tangents t_i ($= (\cos\theta_i, \sin\theta_i)$) ($i = 1, 2$) at those points. In what follows, $m_i = \tan\theta_i$ ($|\theta_i| \le \pi/2$) and r_i ($i = 1, 2$) are the radii of the circular arcs passing through $(-1, 0)$ and $(1, 0)$, respectively. Depending on the signs of θ_i ($i = 1, 2$), we shall derive relations between the radii r_i ($i = 1, 2$) of the biarc. First we consider the case when $\theta_1\theta_2 < 0$ for which the biarc is C-shaped.



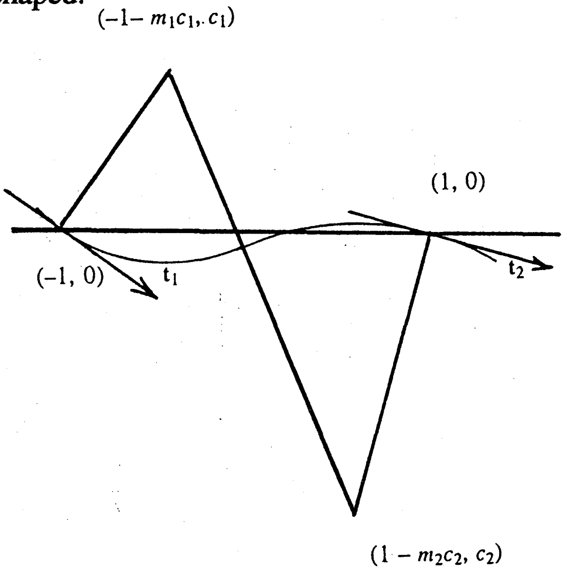


Fig. 1. C–shaped biarc ($\theta_1 < 0$, $\theta_2 > 0$). **Fig. 2.** S–shaped biarc ($\theta_1 < 0$, $\theta_2 < 0$).

Case 1: $\theta_1 < 0, \theta_2 > 0$. In this case, the centers of the two circular arcs are $(-1 - m_1c_1, c_1)$ and $(1 - m_2c_2, c_2)$, where $c_i = r_i/\sqrt{1 + m_i^2}$ ($= r_i\cos\theta_i$). The two circular arcs are joined in a G^1 manner if the distance from the one center to the other is equal to the difference of the radii, i.e.,

(2) $$(r_2 - r_1)^2 = (c_2 - c_1)^2 + (2 - m_2c_2 + m_1c_1)^2$$

or equivalently

(3) $$r_1r_2\sin^2\frac{\theta_1 - \theta_2}{2} = -r_1\sin\theta_1 + r_2\sin\theta_2 - 1 \ (= r_1|\sin\theta_1| + r_2|\sin\theta_2| - 1).$$

Here we note that $r_1 \neq r_2$ for $|\theta_1| \neq |\theta_2|$. Supposing that $r_1 = r_2$, then (3) becomes

(4) $$r_1^2\sin^2\frac{\theta_1 - \theta_2}{2} - r_1(|\sin\theta_1| + |\sin\theta_2|) + 1 = 0.$$

This quadratic equation has no real roots since the discreminant is equal to $-(\cos\theta_1 - \cos\theta_2)^2$ (< 0), proving $r_1 \neq r_2$. Hence, the joint (x, y) of the biarc is given by

(5) $$(r_1 - r_2)x = r_1 + r_2 - r_1r_2(\sin\theta_2 - \sin\theta_1), \quad (r_1 - r_2)y = r_1r_2(\cos\theta_2 - \cos\theta_1).$$

Now, in order for the biarc to be C-shaped shown in Fig. 1, it would be quite natural to demand the conditions:

(6) $$-1 < x < 1, \ y < 0.$$

We can easily check that two inequalities in (6) are valid if $r_1 > r_2$ (or $r_1 < r_2$) for

$|\theta_1| < |\theta_2|$ (or $|\theta_1| > |\theta_2|$). For example, assume that $r_1 > r_2$ for $|\theta_1| < |\theta_2|$, the case of $|\theta_1| > |\theta_2|$ being similarly checked. Since $r_1 > r_2$, $y < 0$ is immediate. Next, by (3)

(7) $$r_1 > 1/|\sin\theta_1|, \quad r_2 < |\sin\theta_1| / \sin^2\frac{\theta_1 - \theta_2}{2}.$$

This gives

(8) $$r_2 < 2/(\sin\theta_2 - \sin\theta_1) < r_1, \text{ i.e., } -1 < x < 1.$$

Thus, for the biarc to be C-shaped, we obtain relation (3) between the two radii r_i ($i = 1, 2$) where $r_1 > r_2$ for $|\theta_1| < |\theta_2|$ and $r_1 < r_2$ for $|\theta_1| > |\theta_2|$.

Case 2: $\theta_1 > 0$, $\theta_2 < 0$. The two circular arcs with centers $(-1 - m_1c_1, c_1)$ and $(1 - m_2c_2, c_2)$ ($c_i = -r_i/\sqrt{1+m_i^2}$ ($= -r_i\cos\theta_i$)) are joined in a G^1 manner if

(9) $$r_1r_2\sin^2\frac{\theta_1 - \theta_2}{2} = r_1\sin\theta_1 - r_2\sin\theta_2 - 1 \; (= r_1|\sin\theta_1| + r_2|\sin\theta_2| - 1).$$

Similarly as in case 1, for the biarc to be C-shaped, we require the condition

(10) $$r_1 > r_2 \; (\text{or } r_1 < r_2) \text{ for } |\theta_1| < |\theta_2| \; (\text{or } |\theta_1| > |\theta_2|).$$

Trivially in the case when $\theta_1\theta_2 < 0$ and $|\theta_1| = |\theta_2|$, a single circular arc with radius $r = 1/|\sin\theta_1|$ passes through $(\pm 1, 0)$ and matches ($\cos\theta_i$, $\sin\theta_i$) at those points. This case is considered to be a special one of (3) (or (9)) since then equation (3) (or (9)) becomes $(r_1|\sin\theta_1| - 1)(r_2|\sin\theta_1| - 1) = 0$ from which we get r_1 or $r_2 = r$ ($= 1/|\sin\theta_1|$). Hence we have

Theorem 1. For the case in which $\theta_1\theta_2 < 0$, there exists a C–shaped biarc (a pair of two circular arcs with radii r_i ($i = 1, 2$)) that joins $(-1, 0)$ and $(1, 0)$ and matches two given or specified unit tangents $\boldsymbol{t_i}$ ($= (\cos\theta_i, \sin\theta_i)$) ($i = 1, 2$) at those points if

$$r_1r_2\sin^2\frac{\theta_1 - \theta_2}{2} = r_1|\sin\theta_1| + r_2|\sin\theta_2| - 1$$

where $r_1 \geq r_2$ (or $r_1 < r_2$) for $|\theta_1| \leq |\theta_2|$ (or $|\theta_1| > |\theta_2|$).

Next we consider the case when $\theta_1\theta_2 \geq 0$ for which the biarc is S–shaped.

Case 3: $\theta_1 < 0$, $\theta_2 < 0$. In this case, the centers of the two circular arcs that pass through $(-1, 0)$ and $(1, 0)$ and match the given or specified unit tangents $\boldsymbol{t_i}$ ($= (\cos\theta_i, \sin\theta_i)$) ($i = 1, 2$) at those points are $(-1 - m_1c_1, c_1)$ and $(1 - m_2c_2, c_2)$ where $c_1 = r_1/\sqrt{1+m_1^2}$ ($= r_1\cos\theta_1$) and $c_2 = -r_2/\sqrt{1+m_2^2}$ ($= -r_2\cos\theta_2$). As shown in Figure 2, the two circular arcs are joined in a G^1 manner if the sum of the radii is equal to the distance between the two centers, i.e.,

(11) $$(r_2 + r_1)^2 = (c_2 - c_1)^2 + (2 - m_2c_2 + m_1c_1)^2$$

or equivalently

(12) $$r_1r_2\sin^2\frac{\theta_1 - \theta_2}{2} = r_1\sin\theta_1 + r_2\sin\theta_2 + 1 \; (= -r_1|\sin\theta_1| - r_2|\sin\theta_2| + 1).$$

Case 4: $\theta_1 > 0,\ \theta_2 > 0$. The centers of the two circular arcs are $(-1-m_1c_1, c_1)$ and $(1-m_2c_2, c_2)$ where $c_1 = -r_1/\sqrt{1+m_1^2}\ (= -r_1\cos\theta_1)$ and $c_2 = r_2/\sqrt{1+m_2^2}\ (= r_2\cos\theta_2)$. Similarly to Case 3 we obtain

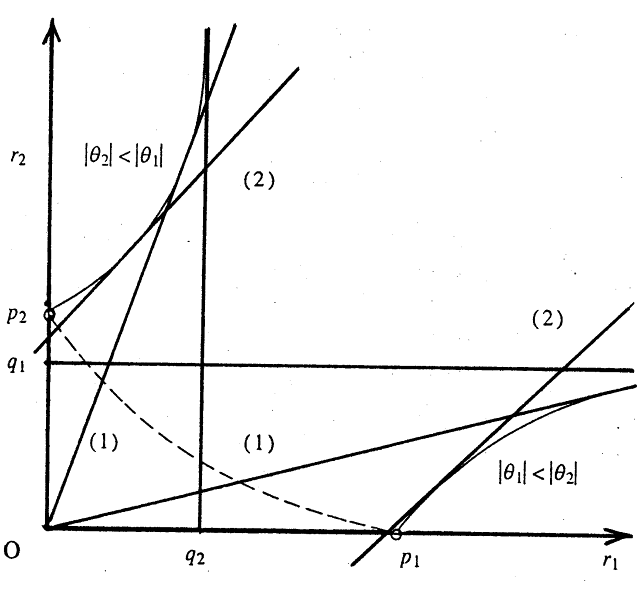
$$(13) \qquad r_1r_2\sin^2\frac{\theta_1-\theta_2}{2} = -r_1\sin\theta_1 - r_2\sin\theta_2 + 1\ (= -r_1|\sin\theta_1| - r_2|\sin\theta_2| + 1).$$

Trivially note that we can let θ_1 or θ_2 be zero in (12) and (13). Hence we have

Theorem 2. For $\theta_1\theta_2 \geq 0$, there exists an S–shaped biarc (a pair of two circular arcs with radii r_i) that joins $(-1, 0)$ and $(1, 0)$ and matches the specified unit tangents t_i ($= (\cos\theta_i, \sin\theta_i)$) ($i = 1, 2$) at those points if

$$r_1r_2\sin^2\frac{\theta_1-\theta_2}{2} = -r_1|\sin\theta_1| - r_2|\sin\theta_2| + 1$$

where for $\theta_1 = \theta_2 = 0$, the biarc reduces to the interval $(-1, 1)$.



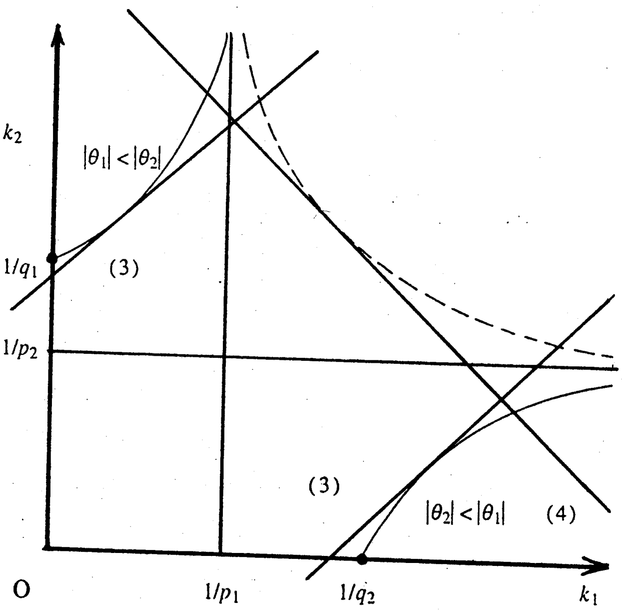


Fig. 3. Relations between r_1 and r_2. **Fig. 4.** Relations between k_1 and k_2.

In Figs 3 and 4, solid and dashed lines mean the relations between the radii r_i (or the curvatures k_i) ($i = 1, 2$) for C–shaped and S–shaped biarcs, respectively where $(p_i, q_i) = (1/|\sin\theta_i|, |\sin\theta_i|/\sin^2\{(\theta_1-\theta_2)/2\})$. Therein, the points of tangent of straight line segments (1) – (4) and the relations in Theorems 1 and 2 correspond to choices for the minimization of $|r_1/r_2 - 1|$, $|r_1 - r_2|$, $|k_1 - k_2|$ and $|k_1 + k_2|$.

3. Locii of joints. In this section, we shall show that a locus of the joint of the biarc joining $(\pm 1, 0)$ is a circular arc passing through $(\pm 1, 0)$.

Theorem 3. In cases 1 and 2 (i.e., $\theta_1\theta_2 < 0$), the joint of the biarc is on the circle C with center $\mathbf{C}\ (0, -1/\tan\{(\theta_1-\theta_2)/2\})$ and radius $r = 1/|\sin\{(\theta_1-\theta_2)/2\}|$, i.e.,

$$x^2 + (y + 1/\tan\{(\theta_1 - \theta_2)/2\})^2 = 1/\sin^2\{(\theta_1 - \theta_2)/2\}, \quad (x, y) \neq (\pm 1, 0)$$

where for case 1,

(14) $y \leq (x-1)\tan(\frac{\theta_1+\theta_2}{2})$ $(|\theta_1| \leq |\theta_2|)$ or $y \leq (x+1)\tan(\frac{\theta_1+\theta_2}{2})$ $(|\theta_1| > |\theta_2|)$

and for case 2,

(15) $y \geq (x-1)\tan(\frac{\theta_1+\theta_2}{2})$ $(|\theta_1| \leq |\theta_2|)$ or $y \geq (x+1)\tan(\frac{\theta_1+\theta_2}{2})$ $(|\theta_1| > |\theta_2|)$.

PROOF of Theorem 3. In case 1, solving for k_i ($= 1/r_i$) ($i = 1, 2$) from (5), and manipulating to obtain

(16) $k_1 (= 1/r_1) = (p - q)\sin\{(\theta_1 - \theta_2)/2\}$, $k_2 (= 1/r_2) = (p + q)\sin\{(\theta_1 - \theta_2)/2\}$

where $py = x\sin\{(\theta_1 + \theta_2)/2\} - y\cos\{(\theta_1 + \theta_2)/2\}$, $qy = \sin\{(\theta_1 + \theta_2)/2\}$. Substituting these quantities of k_i into relation(4) and rearranging, we get

(17)
$$\left\{p + \cos(\frac{\theta_1 + \theta_2}{2})\right\}^2 - \left\{q - \sin(\frac{\theta_1 + \theta_2}{2})/\tan(\frac{\theta_1 - \theta_2}{2})\right\}^2 = -\left\{\sin(\frac{\theta_1 + \theta_2}{2})/\sin(\frac{\theta_1 - \theta_2}{2})\right\}^2.$$

From this, it is easy to obtain the desired circle C. To examine the necessity of condition (14), we assume that $|\theta_1| \leq |\theta_2|$, the other case of $|\theta_1| > |\theta_2|$ being similarly done. Since $0 \leq k_1 \leq k_2$ for $|\theta_1| \leq |\theta_2|$, we get $p + q \leq p - q \leq 0$, i.e., $p \leq q \leq 0$ which are equivalent to (18). In case 2, k_i ($i = 1, 2$) are also given by the same equations (16) where $py = -x\sin\{(\theta_1 + \theta_2)/2\} + y\cos\{(\theta_1 + \theta_2)/2\}$ and $qy = -\sin\{(\theta_1 + \theta_2)/2\}$. Similarly as in case 1, the joint is on the same circle C. For condition (15), we assume that $|\theta_1| \leq |\theta_2|$ since the other case of $|\theta_1| > |\theta_2|$ can be treated in a similar manner. Note that $\sin\{(\theta_1 - \theta_2)/2\} > 0$ for $|\theta_1| \leq |\theta_2|$ in case 2, then $k_2 \geq k_1 \geq 0$ imply $p + q \geq p - q \geq 0$, i.e., $p \geq q \geq 0$ which are equivalent to (15) for $|\theta_1| \leq |\theta_2|$, completing the proof of Theorem 3.

For the geometric meaning of the circle C, we denote $(-1, 0)$ and $(1, 0)$ by **P** and **Q**, respectively. In addition, **R** means the intersection of the two tangents of the circular arcs at **P** and **Q** whose slopes are m_i ($i = 1, 2$). Then we obtain the following corollary that is of use for geometrically determining the locus of the joint.

COROLLARY 1. The center **C** of the circle C is on the circum-circle of Δ**PQR**, and C passes through the incenter **I** of the triangle which has been used as a joint of the C-shaped biarc ([5]).

PROOF. We consider case 1 where in addition $|\theta_1| \leq |\theta_2|$, the other cases being similarly treated. By Theorem 3, we get $\angle$**PCQ** $= \theta_2 - \theta_1$. Since $\angle$**PRQ** $= \pi - (\theta_2 - \theta_1)$, **C** is on the circum-circle of Δ**PQR**. Since $\angle$**PIQ** $= \pi - (\theta_2 - \theta_1)/2$, the circle C passes through the incenter **I**. For geometric proof of this Corollary, see [5].

The following corollary 2 can be checked by an application of Lagrange's method of indeterminate coefficient to the relation between the two radii of the biarc in Theorem 1.

COROLLARY 2. (i) $|1/r_1 - 1/r_2| = \min$ if $1/r_1 + 1/r_2 = |\sin\theta_1| + |\sin\theta_2|$,

(ii) $|r_1 - r_2| = \min$ if $r_1 + r_2 = (|\sin\theta_1| + |\sin\theta_2|)/\sin^2\dfrac{\theta_1-\theta_2}{2}$,

(iii) $|r_1/r_2 - 1| = \min$ if $|\sin\theta_1| + r_2|\sin\theta_2| = 2$,

where case (i) has a unique solution (r_1, r_2) if and only if $1/3 \le |\theta_1|/|\theta_2| \le 3$.

The restriction on the magnitude of $|\theta_1|/|\theta_2|$ in corollary 2 will be seen from the following geometric meaning of the joints **D**, **H** and **I** on the circle C in assertion (i) - (iii). We consider case 1 where in addition $|\theta_1| \le |\theta_2|$, case 2 being similarly done. In view of (i), the x-coordinate of the joint **D** is given by

(18) $$\{r_1 + r_2 - r_1r_2(\sin\theta_2 - \sin\theta_1)\}/(r_1 - r_2) = 0.$$

That is, the joint **D** is on the y-axis, i.e., the bisector of the line segment joining the two end points $(\pm 1, 0)$, and in addition $\angle$**QPD** $= (\theta_2 - \theta_1)/4$. Next in (ii), the vector joining the two centers is equal to

(19) $$(1 - m_2c_2, c_2) - (-1 - m_1c_1, c_1) = (2 + r_1\sin\theta_1 - r_2\sin\theta_2,\ r_2\cos\theta_1 - r_1\cos\theta_2).$$

Since the magnitude of this vector is $|r_1 - r_2|$, letting the angles subtended by the two circular arcs be α and β, then

(20) $$r_1|r_1 - r_2|\cos\alpha = r_1^2 + 2r_1\sin\theta_1 - r_1r_2\cos(\theta_1 - \theta_2)$$

and

(21) $$r_2|r_1 - r_2|\cos\beta = -r_2^2 + 2r_2\sin\theta_2 + r_1r_2\cos(\theta_1 - \theta_2).$$

From these, the angles of the two circular arcs are equal if

(22) $$r_1 + 2\sin\theta_1 - r_2\cos(\theta_1 - \theta_2) = -r_2 + 2\sin\theta_2 + r_1\cos(\theta_1 - \theta_2)$$

or equivalently

(23) $$r_1 + r_2 = (|\sin\theta_1| + |\sin\theta_2|)/\sin^2\frac{\theta_1 - \theta_2}{2}$$

as is the given relation in (ii). Then, a simple calculation gives $\angle$**QPH** $= (-3\theta_1 - \theta_2)/4$.

Finally, in view of (iii), by (19) the x-component of the vector joining the two centers is equal to $2 + r_1\sin\theta_1 - r_2\sin\theta_2 = 0$ from which the common tangent vector at the joint is parallel to the x-axis, i.e., the vector joining the points of the tangent of the two circular arcs. Since then the angles subtended by the two arcs of the biarc are equal to $-\theta_1$ and θ_2 respectively, Sabin' choice is the one that makes the ratio of the radii as close to one as possible ([2]). A simple calculation gives $\angle$**QPI** $= -\theta_1/2$ and in addition, the joint **I** is the incenter of the triangle Δ**PQR**. Here note that $\angle$**QPD** $>$ $\angle$**QPI**$>$ $\angle$**QPH**. In Fig. 5, **L** is the joint of C and the straight line segment : $y = (x-1)\tan\{(\theta_1+\theta_2)/2\}$ and is also the joint of C and the straight line segment : $y = (x+1)\tan\theta_1$, i.e., $\angle$**QPL** $= -\theta_1$. Therefore, assertion (i) has its solution if and only if $\angle$**QPL** $\geq \angle$**QPD**, i.e., $-\theta_1 \geq (-\theta_1 - \theta_2)/4$ or $|\theta_2| \leq 3|\theta_1|$.

THEOREM 4. In cases 3 and 4 (i.e., $\theta_1\theta_2 > 0$), the joint (x, y) is on the same circle C in Theorem 3 where $y < 0$ ($\theta_1 < \theta_2$) or $0 < y$ ($\theta_2 < \theta_1$). For $\theta_1 = \theta_2$ ($>$ or < 0), the circle C reduces to the interval $(-1, 1)$.

PROOF of Theorem 4. In case 3, note that the joint (x, y) is given by $(r_1 + r_2)x = r_1 - r_2 - r_1r_2(\sin\theta_1 - \sin\theta_2)$ and $(r_1 + r_2)y = r_1r_2(\cos\theta_1 - \cos\theta_2)$ since the centers of the two circular arcs are $(-1 - r_1\sin\theta_1,\ r_1\cos\theta_1)$ and $(1 + r_2\sin\theta_2,\ -r_2\cos\theta_2)$ (see Fig. 2). Hence, k_i ($= 1/r_i$) ($i = 1, 2$) are given by the same (16) in the proof of Theorem 3 where $py = -\sin\{(\theta_1 + \theta_2)/2\}$ and $qy = -x\sin\{(\theta_1 + \theta_2)/2\} + y\cos\{(\theta_1 + \theta_2)/2\}$. Substituting k_i ($i = 1, 2$) into (12) and manipulating to obtain,

$$(24)\qquad \left\{p + \sin\left(\frac{\theta_1+\theta_2}{2}\right)/\tan\left(\frac{\theta_1-\theta_2}{2}\right)\right\}^2 - \left\{q - \cos\left(\frac{\theta_1+\theta_2}{2}\right)\right\}^2 = \left\{\sin\left(\frac{\theta_1+\theta_2}{2}\right)/\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right\}^2$$

which becomes the desired circle C. For the restriction on y, we assume $\theta_1 < \theta_2$ (< 0), the other case of $\theta_2 < \theta_1$ (< 0) being similarly treated. Then, since $\sin\{(\theta_1 - \theta_2)/2\} < 0$, $k_i \geq 0$ ($i = 1, 2$) require $p - q,\ p + q \leq 0$, i.e., $(x+1)\tan\{(\theta_1+\theta_2)/2\} \leq y < 0$. Since the slope of the tangent of the circle C at $(-1, 0)$ is greater than the that of the straight line on the right hand side of the first inequality, i.e., $\tan\{(\theta_1-\theta_2)/2\} > \tan\{(\theta_1+\theta_2)/2\}$, the first inequality is always valid. Hence we have y < 0 for $\theta_1 < \theta_2$ (< 0). In case 4, k_i ($i = 1, 2$) are also given by (16) where $py = \sin\{(\theta_1+\theta_2)/2\}$ and $qy = x\sin\{(\theta_1+\theta_2)/2\} - y\cos\{(\theta_1+\theta_2)/2\}$. Substitute k_i into (13), and then rearrange the terms appropriately to get

$$(25)\qquad \left\{p - \sin\left(\frac{\theta_1+\theta_2}{2}\right)/\tan\left(\frac{\theta_1-\theta_2}{2}\right)\right\}^2 - \left\{q + \cos\left(\frac{\theta_1+\theta_2}{2}\right)\right\}^2 = \left\{\sin\left(\frac{\theta_1+\theta_2}{2}\right)/\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right\}^2.$$

From this, it is straightforward to get the desired circle C. For the restriction on y, we assume that ($0 <$)$\theta_1 < \theta_2$, the other case ($0 <$)$\theta_2 < \theta_1$ being similarly treated. From $k_i \geq 0$ ($i = 1, 2$), we get $p - q,\ p + q \leq 0$, i.e., $(x-1)\tan\{(\theta_1+\theta_2)/2\} \leq y < 0$ where the first inequality is always valid since the slope of the tangent line of the circle C at $(1, 0)$ and the straight line segment are given by $-\tan\{(\theta_1 - \theta_2)/2\}$ and $\tan\{(\theta_1 + \theta_2)/2\}$, respectively. Hence $y < 0$. For $\theta_1 = \theta_2$, trivially $x = (r_1 - r_2)/((r_1 + r_2)$ and $y = 0$ with $(r_1 + r_2)|\sin\theta_1| = 1$ from which the locus reduces to the interval $(-1, 1)$, Theorem 4 being proved.

Using the same notation in Corollary 1, we have the following corollaries that are of use for a geometric determination of the joint for the *S*-shaped biarc.

COROLLARY 3. The center **C** of the circle *C* is on the circum-circle of Δ**PQR**.

Proof of Corollary 3. We consider case 3 where in addition $\theta_1 < \theta_2 < 0$, the other cases being similarly treated. Since then $\angle \mathbf{PCQ} = \theta_2 - \theta_1$ ($= \angle \mathbf{PQR}$) by Theorem 4, **C** is on the circum-circle of Δ**PQR**.

Here we remark that the circle *C* passes through the excenter of Δ**PQR** lying on its opposite side with respect to **QR** (or **PR**) for $|\theta_2| < |\theta_1|$ (or $|\theta_1| < |\theta_2|$) where the excenter itself can not be used as the joint but it is of use for determining the locus of the joint. Lagrange's method of indeterminate coefficient to the relation between the radii of the biarc in Theorem 2 or an elementary calculation gives (i) and (ii), respectively.

COROLLARY 4. (i) The difference of the signed curvatures $|k_1 - (-k_2)|$ ($= 1/r_1 + 1/r_2$) $= \min$ if $1/r_1 - 1/r_2$ ($= k_1 - k_2$) $= |\sin\theta_1| - |\sin\theta_2|$.

(ii) $r_1 = r_2$ if r ($= r_1 = r_2$) is the positive root of the quadratic equation:

$$r^2 \sin^2 \frac{\theta_1 - \theta_2}{2} + r(|\sin\theta_1| + |\sin\theta_2|) - 1 = 0.$$

Since then the x-coordinate of the joint **D** in (i) is equa l to $r_1(1 - m_2 c_2) + r_2(-1 - m_1 c_1) = r_1 r_2 (k_2 - k_1 + |\sin\theta_1| - |\sin\theta_2|) = 0$, the joint **D** is on the bisector of the line segment joining the end points (± 1, 0), i.e., y-axis. For $r_1 = r_2$ in case 3, the proof of Theorem 4 easily shows that the coordinate (x, y) of the joint **K** in (ii) satisfies $y/x = \tan\{(\theta_1 + \theta_2)/2\}$, i.e., $\angle \mathbf{QOK} = -(\theta_1 + \theta_2)/2$. In case 4, $\angle \mathbf{POK} = (\theta_1 + \theta_2)/2$.

In Fig. 5, is given the graph of the locus of the joint in case 1 where $(1 \le) |\theta_2|/|\theta_1| \le 3$; otherwise the solid circle (the limiting case when one of the circular arcs of the biarc is a straight line segment) is in the fourth quadrant instead of the third one, i.e., the locus does not pass through the point **D** $(0, -\tan\{(\theta_1 - \theta_2)/4\})$ corresponding to the choice of the minimization of the difference of the two curvatures.

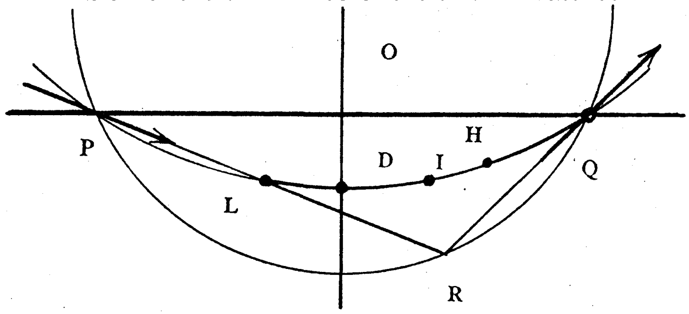


Fig. 5. Locus of joint in case 1 ($|\theta_1| < |\theta_2|$).

4. Finding of biarc spline and Examples. In numerical examples, for a unique determination of biarcs is used the conditions that minimizes the difference of the

(signed) curvatures. By Corollaries 3 and 4, the point **D** is chosen as the joint of the biarc, i.e., we can use the following algorithm for a unique determination of a biarc joining **P** and **Q**:

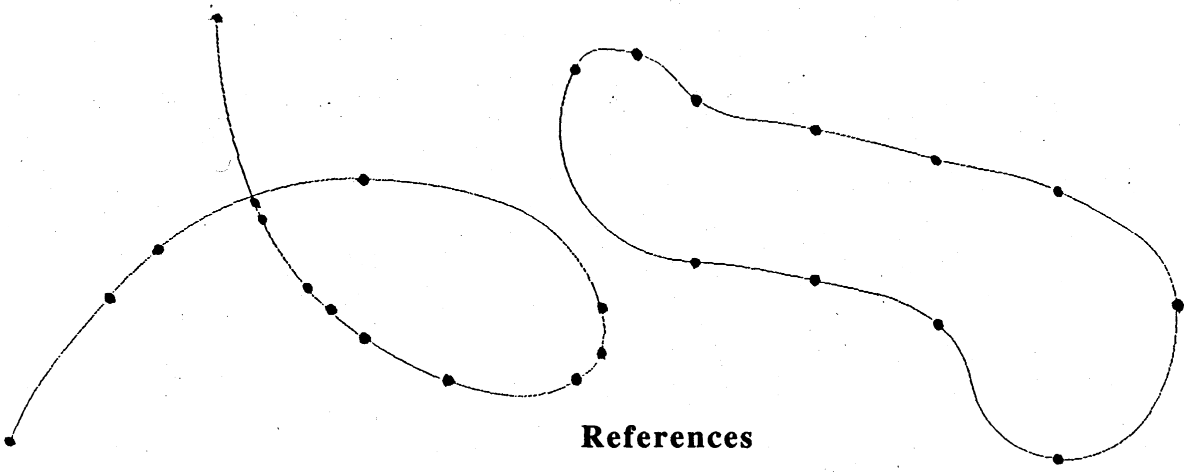
(i) draw Δ**PQR** where **R** is the intersection of the tangents at **P** and **Q** specified by means of an appropriate method without solving a large system of equations

(ii) determine the point **C** (the center of the circle C) as the intersection the circum-circle Δ**PQR** and the bisector of the line segment **PQ** where **C** and **R** are on the opposite (or same) sides of **PQ** for $\theta_1\theta_2 < 0$ (or $\theta_1\theta_2 \geq 0$)

(iii) determine the point **D** as the intersection of the circle C passing through **P** (**Q**) and the bisector of **PQ**

(iv) determine the center of each circular arc as the joint of the straight line passing through **P** (or **Q**) which is perpendicular to the tangent line at **P** (or **Q**) and the bisector of the line segment **PD** (or **QD**) and draw each circular arc.

Some numerical examples are given where the data points are shown as small circles. Arc splines have G^1 continuity, but the loss of curvature continuity does not appear to have a significantly adverse affect on the visual quality of the interpolatory curves.



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