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Kyoto University
A generalization of close-to-convex functions*

Teruo YAGUCHI
(日本大学文学部数学教室 谷口泰男)

1. Introduction
Let $A$ denote the class of functions of the form:

\begin{equation}
\label{eq:1}
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\end{equation}

which are analytic in the unit disk $U = \{ z : |z| < 1 \}$. Let $P_\theta(\alpha)$ denote the class of functions of the form:

\begin{equation}
\label{eq:2}
f(z) = e^{-i\theta} + \sum_{k=1}^{\infty} a_k z^k \quad (-\cos^{-1}\alpha < \theta < \cos^{-1}\alpha),
\end{equation}

which are analytic and $\text{Re} f(z) > \alpha (0 \leq \alpha < 1)$ in the unit disk $U$. We set $P(\alpha) = P_0(\alpha)$.

For a function $f(z)$ in the class $A$, Salagean ([6]) defined the differential operator $D^n, n \in \mathbb{N}_0 = \{0, 1, 2, 3, \cdots \}$, by

\begin{align*}
D^0 f(z) &= f(z), \\
D^1 f(z) &= Df(z) = zf'(z)
\end{align*}

and

\begin{equation}
\label{eq:3}
D^{n+1} f(z) = D(D^n f(z)) \quad (n \in \mathbb{N} = \{1, 2, 3, \cdots \}).
\end{equation}

If a function $f(z) \in A$ is defined by the form (1), then

\begin{equation}
\label{eq:4}
D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.
\end{equation}

Salagean ([6]) also defined the subclass $S^n(\alpha)$ of the class $A$ by

\begin{equation}
\label{eq:5}
S^n(\alpha) = \left\{ f(z) \in A : \frac{D^{n+1} f(z)}{D^n f(z)} \in P(\alpha) \right\}
\end{equation}

for some $\alpha (0 \leq \alpha < 1)$ and for some $n \in \mathbb{N}_0$. From equalities

\begin{align*}
\frac{D^1 f(z)}{D^0 f(z)} &= \frac{zf''(z)}{f(z)} \quad \text{and} \quad \frac{D^2 f(z)}{D^1 f(z)} = 1 + \frac{zf''(z)}{f'(z)},
\end{align*}

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it follows that $S^{0}(\alpha) = S^{*}(\alpha)$ and $S^{1}(\alpha) = K(\alpha)$, where $S^{*}(\alpha)$ and $K(\alpha)$ are classes consisting of all starlike and convex (univalent) functions of order $\alpha$, respectively.

Now we introduce a new class. Let $0 \leq \alpha < 1, 0 \leq \beta < 1$ and $-\cos^{-1}\beta < \theta < \cos^{-1}\beta$. Then a function $f(z) \in A$ is said to be in the class $C^{n}_{\theta}(\alpha, \beta)$ if and only if there is a function $g(z) \in S^{n}(\alpha)$ and a real number $\theta$ such that $\frac{D^{n}f(z)}{e^{i\theta}D^{n}g(z)} \in P_{\theta}(\beta)$. Further, we set

$$\overline{C}^{n}(\alpha, \beta) = \bigcup \{C^{n}_{\theta}(\alpha, \beta) : -\cos^{-1}\beta < \theta < \cos^{-1}\beta\}$$

and

$$\underline{C}^{n}(\alpha, \beta) = \bigcap \{C^{n}_{\theta}(\alpha, \beta) : -\cos^{-1}\beta < \theta < \cos^{-1}\beta\}.$$  

Kaplan ([3]) defined the class $C^{1}_{0}(0, 0)$ of close-to-convex functions, and Libera ([4]) defined the class $C^{1}_{\theta}(\alpha, \beta)$ of close-to-convex functions of order $\beta$ and type $\alpha$. Goodman and Saff ([2]) defined the class $\overline{C}^{1}(0, 0)$, and showed the result $C^{1}(0, 0) = K(0)$ without its proof. The new class $\overline{C}^{n}(\alpha, \beta)$ is a generalization of the class of close-to-convex functions of order $\alpha$ and type $\beta$. With virtue of Lemma 1, Theorems 1 and 2, a function in the class $\overline{C}^{n}(\alpha, \beta)$ is said to be a close-to-$S^{n}(\alpha)$ function of order $\beta$, or a close-to-$S^{n}$ function of order $\beta$ and type $\alpha$. A function $f(z)$ in the class $\overline{C}^{n}(0, 0)$ (or $\overline{C}^{1}(\alpha, 0)$) is, respectively, known as a close-to-star function of type $\alpha$ (or a close-to-convex function of type $\alpha$).

2. Preliminaries

To get our results, we need some lemmas as follows.

Lemma A (MacGregor [5]). Let $0 \leq \alpha < 1$. Then $K(\alpha) \subset S^{*}(\phi)$, where

$$\phi \equiv \phi(\alpha) = \frac{1 - 2\alpha}{2(2^{1 - 2\alpha} - 1)} \quad (\alpha \neq \frac{1}{2})$$

$$\phi \equiv \phi(\frac{1}{2}) = \frac{1}{2\log 2} \quad (\alpha = \frac{1}{2}).$$

The value of $\phi$ satisfies that

$$\max\{\alpha, \frac{1}{2}\} < \phi(\alpha) < 1 \quad (0 \leq \alpha < 1)$$

Lemma B (Salagean [6]). Let $0 \leq \alpha < 1$ and $n \in \mathbb{N}_{0}$. Then $S^{n+1}(\alpha) \subset S^{n}(\phi(\alpha))$, where $\phi(\alpha)$ is given by (2).

For $0 \leq \alpha < 1$ and $\phi(\alpha)$ defined by (2), let $\{\phi_{p}\}_{0}^{\infty}$ be a sequence defined by mathematical induction as follows:

$$\phi_{0} = \alpha, \quad \phi_{p+1} = \phi(\phi_{p}) \quad (p \in \mathbb{N}_{0}).$$

The sequence $\{\phi_{p}\}$ satisfies that

$$\max\{\alpha, \frac{1}{2}\} < \phi_{1} < \cdots < \phi_{p} < \phi_{p+1} < \cdots < 1, \quad \phi_{p} \to 1 \quad (p \to \infty).$$

We get easily the following lemma with virtue of Lemma B.
Lemma 1. Let \( n \in \mathbb{N}_0, p \in \mathbb{N}, 0 \leq \alpha < 1 \) and let \( \{ \phi_p \} \) be defined by (3). Then
\[
S^{n+p}(\alpha) \subset S^n(\phi_p) \subsetneqq S^n(\alpha).
\]

Lemma C (Bernardi [1]). Let \( 0 \leq \alpha < 1, \Re c \leq \alpha \) and \( f(z) \in P(\alpha) \). Then
\[
\left| \frac{f'(z)}{f(z) - c} \right| \leq \frac{2(1 - \alpha)}{(1 - |z|)(1 - \Re c + (1 - 2\alpha + \Re c)|z|)}.
\]

3. Main results

Theorem 1. Let \( n \in \mathbb{N}_0, 0 \leq \alpha < 1 \) and \( 0 \leq \beta < 1 \). Then \( S^n(\alpha) = \mathbb{C}^n(\alpha, \beta) \subsetneqq C^\theta(\alpha, \beta) \) for all real \( \theta(|\theta| < \cos^{-1}\beta) \).

Proof. If \( f(z) \in S^n(\alpha) \), then there is a function \( g(z) \equiv f(z) \in S^n(\alpha) \) such that \( \frac{D^{n+1}f(z)}{e^{i\theta}D^n g(z)} \equiv e^{-i\theta} \in P_\theta(\beta) \) for \( 0 \leq \beta < 1 \) and real \( \theta(|\theta| < \cos^{-1}\beta) \), which proves \( S^n(\alpha) \subset \mathbb{C}^n(\alpha, \beta) \). Conversely, suppose \( f(z) \in \mathbb{C}^n(\alpha, \beta) \) for \( 0 \leq \alpha < 1 \) and \( 0 \leq \beta < 1 \). Then for all real \( \theta(|\theta| < \cos^{-1}\beta) \) there is a function \( g(z) \equiv g(\alpha) \in S^n(\alpha) \) such that \( \frac{D^{n+1}f(z)}{e^{i\theta}D^n g(z)} \in P_\theta(\beta) \).

Applying the function \( w(\alpha) \) defined by
\[
w(z) = \frac{D^n f(z)}{e^{i\theta}D^n g(z)} + 1 - e^{-i\theta} \in P(1 - \cos \theta + \beta) \quad (0 < \beta < 1)
\]
to Lemma C, we have
\[
\left| \frac{D^{n+1}f(z)}{D^n f(z)} - \frac{D^{n+1}g(z)}{D^n g(z)} \right| = \left| \frac{zw'(z)}{w(z) + e^{-i\theta} - 1} \right| \leq \frac{2(\cos \theta - \beta)|z|}{(1 - |z|)(\cos \theta + (\cos \theta - 2\beta)|z|)}
\]
and therefore
\[
\Re \frac{D^{n+1}f(z)}{D^n f(z)} \geq \Re \frac{D^{n+1}g(z)}{D^n g(z)} - \frac{2(\cos \theta - \beta)|z|}{(1 - |z|)(\cos \theta + (\cos \theta - 2\beta)|z|)}
\]
(4)
\[
\geq (1 - \alpha)\frac{1 - |z|}{1 + |z|} + \alpha - \frac{2(\cos \theta - \beta)|z|}{(1 - |z|)(\cos \theta + (\cos \theta - 2\beta)|z|)}.
\]

For fixed \( z \in U \), the value of the last formula of inequality (4) is larger than \( \alpha \) when we choose \( \theta \) such that the value of \( \cos \theta - \beta > 0 \) is sufficiently small. This proves \( f(z) \in S^n(\alpha) \) and hence \( S^n(\alpha) = \mathbb{C}^n(\alpha, \beta) \) for \( 0 < \beta < 1 \). For \( \beta = 0 \), we define the function \( p(z) \in P(0) \) by
\[
p(z) \cos \theta - i \sin \theta = \frac{D^n f(z)}{e^{i\theta}D^n g(z)} \in P_\theta(0)
\]
Then we have
\[
\Re \frac{D^{n+1}f(z)}{D^n f(z)} \geq \Re \frac{D^{n+1}g(z)}{D^n g(z)} - \left| \frac{D^{n+1}f(z)}{D^n f(z)} - \frac{D^{n+1}g(z)}{D^n g(z)} \right|
\]
(5)
\[
\geq \alpha + (1 - \alpha)\frac{1 - |z|}{1 + |z|} - \frac{zp'(z)\cos \theta}{p(z)\cos \theta - i \sin \theta}.
\]
For fixed $z \in U$, the value of the last formula of inequality (5) is larger than $\alpha$ for sufficiently small $\cos \theta > 0$. This proves $f(z) \in S^n(\alpha)$ and hence $S^n(\alpha) = C^n(\alpha, 0)$. Finally, we have to prove $S^n(\alpha) \neq C^n(\alpha, \beta)$, and hence the existence of a function in the class $C^n(\alpha, \beta) - S^n(\alpha)$ for all real $\theta(|\theta| < \cos^{-1} \beta)$. The function $f_\theta(z) \in A$ defined by $D^n f_\theta(z) = \frac{z(1 + e^{i\theta}(e^{i\theta} - 2\beta)z)}{(1 - z)^{3 - 2\alpha}}$ is in the class $C^n(\alpha, \beta)$. Because the function $g(z) \in A$ defined by $D^n g(z) = \frac{z}{(1 - z)^{2(1 - \alpha)}}$ satisfies

$$g(z) \in S^n(\alpha), \quad \frac{D^n f_\theta(z)}{e^{i\theta} D^n g(z)} = \frac{e^{-i\theta} + (e^{i\theta} - 2\beta)z}{1 - z} \in P_\theta(\beta).$$

That $f_\theta(z) \notin S^n(\alpha)$ for any $0 \leq \alpha < 1$ and $0 \leq \beta < \cos \theta$ is shown as follows. Suppose that $f_\theta(z) \in S^n(\alpha)$ for some $\alpha(0 \leq \alpha < 1)$ and some $\beta(0 \leq \beta < \cos \theta)$. Since

$$\frac{D^{n+1} f_\theta(z)}{D^n f_\theta(z)} = 2\alpha - 1 - \frac{1}{1 + e^{i\theta}(e^{i\theta} - 2\beta)z} + \frac{3 - 2\alpha}{1 - z},$$

hence the inequality

$$\text{Re} \frac{D^{n+1} f_\theta(-re^{-i\theta})}{D^n f_\theta(-re^{-i\theta})} = 2\alpha - 1 - \frac{1 + 2\beta r - r \cos \theta}{(1 + 2\beta r - r \cos \theta)^2 + r^2 \sin^2 \theta} + \frac{(3 - 2\alpha)(1 + r \cos \theta)}{(1 + r \cos \theta)^2 + r^2 \sin^2 \theta} > \alpha$$

has to hold true for some $\alpha(0 \leq \alpha < 1)$, some $\beta(0 < \beta < 1)$, all $r(0 \leq r < 1)$ and all $\theta(|\theta| < \cos^{-1} \beta)$, and the inequality

$$\text{Re} \frac{D^{n+1} f_\theta(-re^{-2i\theta})}{D^n f_\theta(-re^{-2i\theta})} = 2\alpha - 1 - \frac{1}{1 - r} + \frac{(3 - 2\alpha)(1 + r \cos 2\theta)}{1 + 2r \cos 2\theta + r^2} > \alpha$$

has to hold true for some $\alpha(0 \leq \alpha < 1), \beta = 0$, all $r(0 \leq r < 1)$ and all $\theta(|\theta| < \frac{\pi}{2})$. When $0 \leq \alpha < 1$ and $0 < \beta < 1$, we have

$$\lim_{r \to 1^-} \text{Re} \frac{D^{n+1} f_\theta(-re^{-i\theta})}{D^n f_\theta(-re^{-i\theta})} = \alpha - \frac{2\beta(\cos \theta - \beta)}{(1 + 2\beta - \cos \theta)^2 + \sin^2 \theta} < \alpha$$

for fixed $\theta$ and $\alpha$, which contradicts the inequality (6). When $0 \leq \alpha < 1$ and $\beta = 0$, we have

$$\lim_{r \to 1^-} \text{Re} \frac{D^{n+1} f_\theta(-re^{-2i\theta})}{D^n f_\theta(-re^{-2i\theta})} = -\infty < \alpha$$

for fixed $\theta$ and $\alpha$, which contradicts the inequality (7). This proves $f_\theta(z) \notin S^n(\alpha)$. $\square$
Theorem 2. Let \( n \in \mathbb{N}, 0 \leq l \leq n - 1 \) and \( 0 \leq \alpha < 1 \). Then

\[
S^{n-1}(\alpha) \subset C^n_0(\alpha, \beta) \quad (0 \leq \beta \leq \alpha)
\]

and

\[
S^l(\alpha) \not\subset \overline{C}^n(\alpha, \beta) \quad (\alpha < \beta < 1).
\]

Proof. Let \( f(z) \in S^{n-1}(\alpha) \), and \( g(z) = \int_0^z \frac{f(z)}{z}dz \). Then we have

\[
zg'(z) = f(z), \quad D^n g(z) = D^{n-1} f(z) \in S^*(\alpha)
\]

Therefore there is the function \( g(z) \in S^n(\alpha) \) such that \( \frac{D^n f(z)}{D^n g(z)} = \frac{D^{n-1} f(z)}{D^{n-1} g(z)} \in P(\alpha) \). This proves \( S^{n-1}(\alpha) \subset C^n_0(\alpha, \alpha) \) and (8). We define the function \( f_\alpha(z) \in A \) by

\[
D^{n-1} f_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \in S^*(\alpha)
\]

Since \( f_\alpha(z) \in S^{n-1}(\alpha) \), we have only to prove \( f_\alpha(z) \notin C^n_\theta(\alpha, \beta) \) for all \( \alpha, \beta \) and \( \theta(0 \leq \alpha < \beta < \cos \theta \leq 1) \) to prove (9). If \( f_\alpha(z) \in C^n_\theta(\alpha, \beta) \) for some \( \alpha, \beta \) and \( \theta(0 \leq \alpha < \beta < \cos \theta \leq 1) \), then there is a function \( g(z) \in S^n(\alpha) \) such that \( \frac{D^n f_\alpha(z)}{e^{i\theta} D^n g(z)} \in P_\theta(\beta) \). We define the function \( w(z) \) by

\[
w(z) = \frac{\{D^{n-1} f_\alpha(z)\}'}{e^{i\theta} \{D^{n-1} g(z)\}'} = \frac{D^n f_\alpha(z)}{e^{i\theta} D^n g(z)} \in P_\theta(\beta).
\]

Since \( D^{n-1} g(z) \in K(\alpha) \),

\[
\frac{zw'(z)}{w(z)} = \frac{z \{D^{n-1} f_\alpha(z)\}''}{\{D^{n-1} f_\alpha(z)\}'} - \frac{z \{D^{n-1} g(z)\}''}{\{D^{n-1} g(z)\}'}
\]

hence we have

\[
\text{Re} \frac{zw'(z)}{w(z)} = \text{Re} \left(1 + \frac{z \{D^{n-1} f_\alpha(z)\}''}{\{D^{n-1} f_\alpha(z)\}'} - \frac{z \{D^{n-1} g(z)\}''}{\{D^{n-1} g(z)\}'}\right)
\]

\[
\leq \text{Re} \left(1 + \frac{(1-2\alpha)z}{1(1+1-2\alpha)z} + \frac{(3-2\alpha)z}{1-z} - (1-\alpha)\frac{1-|z|}{1+|z|} - \alpha\right)
\]

\[
= 2(1-\alpha)\text{Re} \left(\frac{2z + (1-2\alpha)z^2}{(1-z)(1+1-2\alpha)z} + \frac{|z|}{1+|z|}\right), \quad (|z| < 1)
\]

and

\[
\text{Re} \frac{-rw'(-r)}{w(-r)} \leq -\frac{2(1-\alpha)r}{(1+r)(1+(1-2\alpha)r)} \quad (0 \leq r < 1).
\]

Otherwise, from the relation \( \frac{w(z) + \imath \sin \theta}{\cos \theta} \in P(\frac{\beta}{\cos \theta}) \) and Lemma C, we also have

\[
\left| \text{Re} \frac{zw'(z)}{w(z)} \right| \leq \frac{2(\cos \theta - \beta)|z|}{(1-|z|)(\cos \theta + (\cos \theta - 2\beta)|z|)} \quad (|z| < 1),
\]
and
\[
(11) \quad \frac{-r w'(-r)}{w(-r)} \geq -\frac{2(\cos \theta - \beta)r}{(1-r)\{\cos \theta + (\cos \theta - 2\beta)r\}} \quad (0 \leq r < 1).
\]
Therefore, with virtue of inequalities (10) and (11), we have
\[
(12) \quad \frac{1-\alpha}{(1+r)\{1-(1-2\alpha)r\}} < \frac{\cos \theta - \beta}{(1-r)\{\cos \theta + (\cos \theta - 2\beta)r\}}
\]
for some $\alpha, \beta$ and $\theta(0 \leq \alpha < \beta < \cos \theta \leq 1)$, and all $r(0 < r < 1)$. Letting $r \to 0$ in the both sides of the inequality (12), we get $\beta \leq \alpha \cos \theta \leq \alpha$, which contradicts $\alpha < \beta$. This proves (9) for $l = n - 1$.

Many mathematicians have given the class of close-to-convex functions geometrical meanings. One of the meanings is that the boundary curve of the image $f(U)$ of the unit disk $U$ by a close-to-convex function $f(z)$ has no “hair pin” bend that exceeds $\pi$. Another is that the complex plane minus the image $f(U)$ is the union of closed half-lines such that the corresponding open half-lines are disjoint.

We give the class $C^n(\alpha, \beta)$ of close-to-$S^n(\alpha)$ functions of order $\beta$ set-theorecal meanings as follows:

\[
(13) \quad \begin{cases} 
S^n(\alpha) \subset S^n(\alpha, \beta) = C^n(\alpha, \beta) \subset C^n(\alpha, \beta) & (0 \leq \alpha < 1, 0 \leq \beta < 1, n < m), \\
S^l(\alpha) \subset C^l(\alpha, \beta) \subset C^n(\alpha, \beta) & (0 \leq \alpha < \beta < 1, 0 \leq l \leq n - 1), \\
S^{n-1}(\alpha) \subset C^n(\alpha, \beta) \subset C^n(\alpha, \beta) & (0 \leq \beta \leq \alpha < 1).
\end{cases}
\]

Putting $n = 1$ and $\beta = 0$ in the last inclusion relation of (13), we have the following Corollary which is well-known.

**Corollary.** A starlike function of order $\alpha$ is a close-to-convex of order $\alpha$.

**References**


Teruo YAGUCHI
Department of Mathematics
College of Humanities & Sciences
Nihon University
Sakurajousui, Setagaya, Tokyo 156
JAPAN