<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>A generalization of close-to-convex functions</td>
</tr>
<tr>
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<td>YAGUCHI, Teruo</td>
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<td>テキストバージョン</td>
<td>publisher</td>
</tr>
</tbody>
</table>
A generalization of close-to-convex functions*

Teruo YAGUCHI

1. Introduction

Let $A$ denote the class of functions of the form:

\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \]

which are analytic in the unit disk $U = \{ z : |z| < 1 \}$. Let $P_\theta(\alpha)$ denote the class of functions of the form:

\[ f(z) = e^{-i\theta} + \sum_{k=1}^{\infty} a_k z^k \quad (-\cos^{-1} \alpha < \theta < \cos^{-1} \alpha), \]

which are analytic and $\Re f(z) > \alpha (0 \leq \alpha < 1)$ in the unit disk $U$. We set $P(\alpha) = P_0(\alpha)$.

For a function $f(z)$ in the class $A$, Salagean ([6]) defined the differential operator $D^n, n \in N_0 = \{0, 1, 2, 3, \cdots \}$, by

\[ D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z) \]

and

\[ D^{n+1} f(z) = D(D^n f(z)) \quad (n \in N = \{1, 2, 3, \cdots \}). \]

If a function $f(z) \in A$ is defined by the form (1), then

\[ D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k. \]

Salagean ([6]) also defined the subclass $S^n(\alpha)$ of the class $A$ by

\[ S^n(\alpha) = \left\{ f(z) \in A : \frac{D^{n+1} f(z)}{D^n f(z)} \in P(\alpha) \right\} \]

for some $\alpha (0 \leq \alpha < 1)$ and for some $n \in N_0$. From equalities

\[ \frac{D^1 f(z)}{D^0 f(z)} = \frac{zf''(z)}{f(z)} \quad \text{and} \quad \frac{D^2 f(z)}{D^1 f(z)} = 1 + \frac{zf'''(z)}{f'(z)}, \]

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it follows that $S^0(\alpha) = S^* (\alpha)$ and $S^1(\alpha) = K(\alpha)$, where $S^*(\alpha)$ and $K(\alpha)$ are classes consisting of all starlike and convex (univalent) functions of order $\alpha$, respectively.

Now we introduce a new class. Let $0 \leq \alpha < 1, 0 \leq \beta < 1$ and $-\cos^{-1}\beta < \theta < \cos^{-1}\beta$. Then a function $f(z) \in A$ is said to be in the class $C^\phi_{\theta}(\alpha, \beta)$ if and only if there is a function
$g(z) \in S^* (\alpha)$ and a real number $\theta$ such that

$$\frac{D^nf(z)}{e^{i\theta}D^n g(z)} \in P_{\theta}(\beta).$$

Further, we set

$$\overline{C}^n(\alpha, \beta) = \bigcup \{C_{\theta}^n(\alpha, \beta) : -\cos^{-1}\beta < \theta < \cos^{-1}\beta\}$$

and

$$\underline{C}^n(\alpha, \beta) = \bigcap \{C_{\theta}^n(\alpha, \beta) : -\cos^{-1}\beta < \theta < \cos^{-1}\beta\}.$$

Kaplan ([3]) defined the class $C^0_{\phi}(0,0)$ of close-to-convex functions, and Libera ([4]) defined the class $C^1_{\phi}(\alpha, \beta)$ of close-to-convex functions of order $\beta$ and type $\alpha$. Goodman and Saff ([2]) defined the class $C^\phi_{\phi}(0,1)$, and showed the result $C^1(0,0) = K(0)$ without its proof. The new class $C^\phi_{\theta}(\alpha, \beta)$ is a generalization of the class of close-to-convex functions of order $\alpha$ and type $\beta$. With virtue of Lemma 1, Theorems 1 and 2, a function in the class $\overline{C}^n(\alpha, \beta)$ is said to be a close-to-$S^n(\alpha)$ function of order $\beta$, or a close-to-$S^n$ function of order $\beta$ and type $\alpha$. A function $f(z)$ in the class $\overline{C}^0(\alpha,0)$ (or $\overline{C}^1(\alpha,0)$) is, respectively, known as a close-to-star function of type $\alpha$ (or a close-to-convex function of type $\alpha$).

2. Preliminaries

To get our results, we need some lemmas as follows.

**Lemma A** (MacGregor [5]). Let $0 \leq \alpha < 1$. Then $K(\alpha) \subset S^*(\phi)$, where

$$(2) \quad \begin{cases} \phi \equiv \phi(\alpha) = \frac{1 - 2\alpha}{2(2^{1-2\alpha} - 1)} \quad (\alpha \neq \frac{1}{2}) \\ \phi \equiv \phi\left(\frac{1}{2}\right) = \frac{1}{2\log 2} \quad (\alpha = \frac{1}{2}). \end{cases}$$

The value of $\phi$ satisfies that

$$\max\{\alpha, \frac{1}{2}\} < \phi(\alpha) < 1 \quad (0 \leq \alpha < 1)$$

**Lemma B** (Salagean [6]). Let $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$. Then $S^{n+1}(\alpha) \subset S^n(\phi(\alpha))$, where $\phi(\alpha)$ is given by (2).

For $0 \leq \alpha < 1$ and $\phi(\alpha)$ defined by (2), let $\{\phi_{p}\}_{0}^{\infty}$ be a sequence defined by mathematical induction as follows:

$$(3) \quad \phi_{0} = \alpha, \quad \phi_{p+1} = \phi(\phi_{p}) \quad (p \in \mathbb{N}_0).$$

The sequence $\{\phi_{p}\}$ satisfies that

$$\max\{\alpha, \frac{1}{2}\} < \phi_{1} < \cdots < \phi_{p} < \phi_{p+1} < \cdots < 1, \quad \phi_{p} \to 1 \quad (p \to \infty).$$

We get easily the following lemma with virtue of Lemma B.
**Lemma 1.** Let $n \in \mathbb{N}_0, p \in \mathbb{N}, 0 \leq \alpha < 1$ and let $\{\phi_p\}$ be defined by (3). Then

$$S^{n+p}(\alpha) \subset S^n(\phi_p) \subsetneqq S^n(\alpha).$$

**Lemma C (Bernardi [1]).** Let $0 \leq \alpha < 1, \Re c \leq \alpha$ and $f(z) \in P(\alpha)$. Then

$$\left| \frac{f'(z)}{f(z) - c} \right| \leq \frac{2(1 - \alpha)}{(1 - |z|)^2 \{1 - \Re c + (1 - 2\alpha + \Re c)|z|\}}.$$

**3. Main results**

**Theorem 1.** Let $n \in \mathbb{N}_0, 0 \leq \alpha < 1$ and $0 \leq \beta < 1$. Then $S^n(\alpha) = C^n(\alpha, \beta) \subsetneqq C^n(\alpha, \beta)$ for all real $\theta(|\theta| < \cos^{-1}\beta)$.

**Proof.** If $f(z) \in S^n(\alpha)$, then there is a function $g(z) \equiv f(z) \in S^n(\alpha)$ such that $\frac{D^n f(z)}{e^{i\theta} D^n g(z)} \equiv e^{-i\theta} \in P_\theta(\beta)$ for $0 \leq \beta < 1$ and real $0 \leq \theta(|\theta| < \cos^{-1}\beta)$, which proves $S^n(\alpha) \subset C^n(\alpha, \beta)$. Conversely, suppose $f(z) \in C^n(\alpha, \beta)$ for $0 \leq \alpha < 1$ and $0 \leq \beta < 1$. Then for all real $\theta(|\theta| < \cos^{-1}\beta)$ there is a function $g(z) \equiv g_\theta(z) \in S^n(\alpha)$ such that $\frac{D^n f(z)}{e^{i\theta} D^n g(z)} \in P_\theta(\beta)$. Applying the function $w(z)$ defined by

$$w(z) = \frac{D^n f(z)}{e^{i\theta} D^n g(z)} + 1 - e^{-i\theta} \in P(1 - \cos\theta + \beta) \quad (0 < \beta < 1)$$

to Lemma C, we have

$$\left| \frac{D^{n+1} f(z)}{D^n f(z)} - \frac{D^{n+1} g(z)}{D^n g(z)} \right| = \left| \frac{zw'(z)}{w(z) + e^{-i\theta} - 1} \right| \leq \frac{2(\cos \theta - \beta)|z|}{(1 - |z|)\{\cos \theta + (\cos \theta - 2\beta)|z|\}}$$

and therefore

$$\Re \frac{D^{n+1} f(z)}{D^n f(z)} \geq \Re \frac{D^{n+1} g(z)}{D^n g(z)} - \frac{2(\cos \theta - \beta)|z|}{(1 - |z|)\{\cos \theta + (\cos \theta - 2\beta)|z|\}}$$

$$\geq (1 - \alpha) \frac{1 - |z|}{1 + |z|} + \alpha - \frac{2(\cos \theta - \beta)|z|}{(1 - |z|)\{\cos \theta + (\cos \theta - 2\beta)|z|\}}.$$

For fixed $z \in U$, the value of the last formula of inequality (4) is larger than $\alpha$ when we choose $\theta$ such that the value of $\cos \theta - \beta > 0$ is sufficiently small. This proves $f(z) \in S^n(\alpha)$ and hence $S^n(\alpha) = C^n(\alpha, \beta)$ for $0 < \beta < 1$. For $\beta = 0$, we define the function $p(z) \in P(0)$ by

$$p(z) \cos \theta - i \sin \theta = \frac{D^n f(z)}{e^{i\theta} D^n g(z)} \in P_\theta(0)$$

Then we have

$$\Re \frac{D^{n+1} f(z)}{D^n f(z)} \geq \Re \frac{D^{n+1} g(z)}{D^n g(z)} - \left| \frac{D^{n+1} f(z)}{D^n f(z)} - \frac{D^{n+1} g(z)}{D^n g(z)} \right|$$

$$\geq \alpha + (1 - \alpha) \frac{1 - |z|}{1 + |z|} - \frac{zp'(z) \cos \theta}{p(z) \cos \theta - i \sin \theta}.$$
For fixed $z \in U$, the value of the last formula of inequality (5) is larger than $\alpha$ for sufficiently small $\cos \theta > 0$. This proves $f(z) \in S^n(\alpha)$ and hence $S^n(\alpha) = C^n(\alpha, 0)$. Finally, we want to prove $S^n(\alpha) \neq C^n(\alpha, \beta)$, and hence the existence of a function in the class $C^n(\alpha, \beta) - S^n(\alpha)$ for all real $\theta(\theta < \cos^{-1} \beta)$. The function $f_\theta(z) \in A$ defined by

$$D^n f_\theta(z) = \frac{z \{1 + e^{i\theta} (e^{i\theta} - 2\beta) z\}}{(1 - z)^{3 - 2\alpha}}$$

is in the class $C^n(\alpha, \beta)$. Because the function $g(z) \in A$ defined by $D^n g(z) = \frac{z}{(1 - z)^{2(1 - \alpha)}}$ satisfies

$$g(z) \in S^n(\alpha), \quad \frac{D^n f_\theta(z)}{e^{i\theta} D^n g(z)} = \frac{e^{-i\theta} + (e^{i\theta} - 2\beta) z}{1 - z} \in P_\theta(\beta).$$

That $f_\theta(z) \notin S^n(\alpha)$ for any $0 \leq \alpha < 1$ and $0 \leq \beta < \cos \theta$ is shown as follows. Suppose that $f_\theta(z) \in S^n(\alpha)$ for some $\alpha(0 \leq \alpha < 1)$ and some $\beta(0 \leq \beta < \cos \theta)$. Since

$$\frac{D^{n+1} f_\theta(z)}{D^n f_\theta(z)} = 2\alpha - 1 - \frac{1}{1 + e^{i\theta} (e^{i\theta} - 2\beta) z} + \frac{3 - 2\alpha}{1 - z},$$

hence the inequality

$$\text{Re} \frac{D^{n+1} f_\theta(-re^{-i\theta})}{D^n f_\theta(-re^{-i\theta})} = 2\alpha - 1 - \frac{1 + 2\beta r - r \cos \theta}{(1 + 2\beta r - r \cos \theta)^2 + r^2 \sin^2 \theta} + \frac{(3 - 2\alpha)(1 + r \cos \theta)}{(1 + r \cos \theta)^2 + r^2 \sin^2 \theta} > \alpha$$

has to hold true for some $\alpha(0 \leq \alpha < 1)$, some $\beta(0 \leq \beta < 1)$, all $r(0 \leq r < 1)$ and all $\theta(\theta < \cos^{-1} \beta)$, and the inequality

$$\text{Re} \frac{D^{n+1} f_\theta(-re^{-2i\theta})}{D^n f_\theta(-re^{-2i\theta})} = 2\alpha - 1 - \frac{1}{1 - r} + \frac{(3 - 2\alpha)(1 + r \cos 2\theta)}{1 + 2r \cos 2\theta + r^2} > \alpha$$

has to hold true for some $\alpha(0 \leq \alpha < 1)$, $\beta = 0$, all $r(0 \leq r < 1)$ and all $\theta(\theta < \frac{\pi}{2})$. When $0 \leq \alpha < 1$ and $0 \leq \beta < 1$, we have

$$\lim_{r \to 1-0} \text{Re} \frac{D^{n+1} f_\theta(-re^{-i\theta})}{D^n f_\theta(-re^{-i\theta})} = \alpha - \frac{2\beta (\cos \theta - \beta)}{(1 + 2\beta - \cos \theta)^2 + \sin^2 \theta} < \alpha$$

for fixed $\theta$ and $\alpha$, which contradicts the inequality (6). When $0 \leq \alpha < 1$ and $\beta = 0$, we have

$$\lim_{r \to 1-0} \text{Re} \frac{D^{n+1} f_\theta(-re^{-2i\theta})}{D^n f_\theta(-re^{-2i\theta})} = -\infty < \alpha$$

for fixed $\theta$ and $\alpha$, which contradicts the inequality (7). This proves $f_\theta(z) \notin S^n(\alpha)$. □
Theorem 2. Let $n \in \mathbb{N}, 0 \leq l \leq n - 1$ and $0 \leq \alpha < 1$. Then

$$S^{n-1}(\alpha) \subset C_0^n(\alpha, \beta) \quad (0 \leq \beta \leq \alpha)$$

and

$$S^l(\alpha) \not\subset \overline{C}^n(\alpha, \beta) \quad (\alpha < \beta < 1).$$

Proof. Let $f(z) \in S^{n-1}(\alpha)$, and $g(z) = \int_0^z \frac{f(z)}{z} dz$. Then we have

$$zg'(z) = f(z), \quad D^n g(z) = D^{n-1} f(z) \in S^*(\alpha)$$

Therefore there is the function $g(z) \in S^n(\alpha)$ such that $\frac{D^n f(z)}{D^n g(z)} = \frac{D^{n-1} f(z)}{D^{n-1} g(z)} \in P(\alpha)$. This proves $S^{n-1}(\alpha) \subset C_0^n(\alpha, \alpha)$ and (8). We define the function $f_\alpha(z) \in A$ by

$$D^{n-1} f_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \in S^*(\alpha) \quad (0 \leq \alpha < \beta < 1).$$

Since $f_\alpha(z) \in S^{n-1}(\alpha)$, we have only to prove $f_\alpha(z) \not\in C_\theta^n(\alpha, \beta)$ for all $\alpha, \beta$ and $\theta(0 \leq \alpha < \beta < \cos \theta \leq 1)$ to prove (9). If $f_\alpha(z) \in C_\theta^n(\alpha, \beta)$ for some $\alpha, \beta$ and $\theta(0 \leq \alpha < \beta < \cos \theta \leq 1)$, then there is a function $g(z) \in S^n(\alpha)$ such that $\frac{D^n f_\alpha(z)}{e^{i\theta} D^n g(z)} \in P_\theta(\beta)$. We define the function $w(z)$ by

$$w(z) = \frac{\{D^{n-1} f_\alpha(z)\}'}{e^{i\theta} \{D^{n-1} g(z)\}'} = \frac{D^n f_\alpha(z)}{e^{i\theta} D^n g(z)} \in P_\theta(\beta).$$

Since

$$D^{n-1} g(z) \in K(\alpha), \quad \frac{zw'(z)}{w(z)} = z \frac{\{D^{n-1} f_\alpha(z)\}''}{\{D^{n-1} f_\alpha(z)\}'} - z \frac{\{D^{n-1} g(z)\}''}{\{D^{n-1} g(z)\}''}$$

hence we have

$$\Re \frac{zw'(z)}{w(z)} = \Re \left( 1 + z \frac{\{D^{n-1} f_\alpha(z)\}''}{\{D^{n-1} f_\alpha(z)\}'} \right) - \Re \left( 1 + z \frac{\{D^{n-1} g(z)\}''}{\{D^{n-1} g(z)\}'} \right)$$

$$\leq \Re \left( 1 + \frac{(1-2\alpha)z}{1+(1-2\alpha)z} + \frac{(3-2\alpha)z}{1-z} \right) - (1-\alpha) \frac{1-|z|}{1+|z|} - \alpha$$

$$= 2(1-\alpha) \Re \left( \frac{2z+(1-2\alpha)z^2}{(1-z)(1+1-2\alpha)z^2} \right) \left( \frac{|z|}{1+|z|} \right), \quad (|z| < 1)$$

and

$$\Re \frac{-rw'(-r)}{w(-r)} \leq -\frac{2(1-\alpha)r}{(1+r)(1-(1-2\alpha)r)} \quad (0 \leq r < 1).$$

Otherwise, from the relation $\frac{w(z)+i\sin \theta}{\cos \theta} \in P(\frac{\beta}{\cos \theta})$ and Lemma C, we also have

$$\left| \Re \frac{zw'(z)}{w(z)} \right| \leq \frac{2(\cos \theta - \beta)|z|}{(1-|z|)(\cos \theta + (\cos \theta - 2\beta)|z|)} \quad (|z| < 1),$$

by
and
\[(11) \quad \text{Re} \frac{-r w'(-r)}{w(-r)} \geq -\frac{2(\cos \theta - \beta)r}{(1-r)(\cos \theta + (\cos \theta - 2\beta)r)} \quad (0 \leq r < 1).\]

Therefore, with virtue of inequalities (10) and (11), we have
\[(12) \quad \frac{1 - \alpha}{(1+r)(1-(1-2\alpha)r)} < \frac{\cos \theta - \beta}{(1-r)(\cos \theta + (\cos \theta - 2\beta)r)}\]
for some $\alpha, \beta$ and $\theta(0 \leq \alpha < \beta < \cos \theta \leq 1)$, and all $r(0 < r < 1)$. Letting $r \to 0$ in the both sides of the inequality (12), we get $\beta \leq \alpha \cos \theta \leq \alpha$, which contradicts $\alpha < \beta$. This proves (9) for $l = n - 1$. By Lemma 1, we prove the assertion (9) for $0 \leq l \leq n - 1$. □

Many mathematicians have given the class of close-to-convex functions geometrical meanings. One of the meanings is that the boundary curve of the image $f(U)$ of the unit disk $U$ by a close-to-convex function $f(z)$ has no "hair pin" bend that exceeds $\pi$. Another is that the complex plane minus the image $f(U)$ is the union of closed half-lines such that the corresponding open half-lines are disjoint.

We give the class $\overline{C'}^n(\alpha, \beta)$ of close-to-$S^n(\alpha)$ functions of order $\beta$ set-theorecal meanings as follows:
\[(13) \quad \begin{cases} 
S^n(\alpha) \subset \mathcal{C}^{n}(\alpha, \beta) \subset \overline{\mathcal{C}}^{n}(\alpha, \beta) & (0 \leq \alpha < 1, 0 \leq \beta < 1, n < m), \\
S^{n-l}(\alpha) \subset \mathcal{C}^{n-l}(\alpha, \beta) \subset \overline{\mathcal{C}}^{n-l}(\alpha, \beta) & (0 \leq \alpha < \beta < 1, 0 \leq l \leq n - 1), \\
S^{n-1}(\alpha) \subset C^{n}(\alpha, \beta) \subset \overline{\mathcal{C}}^{n}(\alpha, \beta) & (0 \leq \beta \leq \alpha < 1).
\end{cases}\]

Putting $n = 1$ and $\beta = 0$ in the last inclusion relation of (13), we have the following Corollary which is well-known.

**Corollary.** A starlike function of order $\alpha$ is a close-to-convex of order $\alpha$.

**References**


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