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Kyoto University
Certain subclasses of starlike functions of order $\alpha$

1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. Let $S$ denote the subclass of $A$ consisting of functions which are univalent in the unit disk $U$.

A function $f(z)$ in $A$ is said to be starlike of order $\alpha$ if it satisfies

$$\Re \frac{zf'(z)}{f(z)} > \alpha$$

for some $\alpha(0 \leq \alpha < 1)$ and for all $z$ in $U$.

We denote by $S^*(\alpha)$ the subclass of $S$ consisting of all starlike functions of order $\alpha$ in the unit disk $U$.

Let $S_1(\alpha, a)$, $S_2(\alpha, a)$ and $S_3(\alpha, a)$ denote the subclasses of $S$ consisting of functions which satisfy

$$\left| \frac{zf'(z)}{f(z)} - a \right| < a - \alpha \quad (0 \leq \alpha < 1),$$

for $1 \leq a \leq 2$, $a > 2$ and $\frac{1+\alpha}{2} < a < 1$, respectively.

It is clear that $S_i(\alpha, a) \subset S^*(\alpha)$ and $S_i(\alpha, a) \subset S_i(\alpha, b)(a \leq b)$ for $i = 1, 2, 3$.

In [1. Theorem 1], Silverman have showed the sufficient condition for a function in $S$ belongs to $S^*(\alpha)$. In this paper, we shall reconsider the sufficient condition by using the subclasses $S_i(\alpha, a)(i = 1, 2, 3)$ of $S^*(\alpha)$ defined above. Further, we determine the distortion theorems for certain subclasses $\tilde{S}_i(\alpha, a)$ of $S_i(\alpha, a)(i = 1, 2, 3)$.

2. Coefficient inequality

We shall now prove the following theorems in a same way of Theorem 1 of Silverman [1].

**Theorem 1** Let $f(z) \in S$, $0 \leq \alpha < 1$ and $1 \leq a \leq 2$. If

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq 1 - \alpha,$$

then $f(z) \in S_1(\alpha, a)$.

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Proof. We have

\[
\left| \frac{zf'(z)}{f(z)} - a \right| = \frac{1 - a + \sum_{n=2}^{\infty} (n - a)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \leq a - 1 + \sum_{n=2}^{\infty} (n - a)|a_n||z|^{n-1}
\]

This last expression is bounded above by \(a - \alpha\) if

\[
a - 1 + \sum_{n=2}^{\infty} (n - a)|a_n||z|^{n-1} \leq (a - \alpha) \left(1 - \sum_{n=2}^{\infty} |a_n| \right).
\]

Since the hypothesis of Theorem 1 is equivalent to the coefficient inequality (1), Theorem 1 is proved.

In the proof of Theorem 1 of Silverman [1], only the case of \(a=1\) of the Theorem 1 above was considered.

**Theorem 2** Let \(f(z) \in S\), \(0 \leq \alpha < 1\) and \(a > 2\). If

\[
\sum_{n=2}^{j} (2a - n - \alpha)|a_n| + \sum_{n=j+1}^{\infty} (n - \alpha)|a_n| \leq 1 - \alpha
\]

for \(j < a \leq j + 1\), then \(f(z) \in S_2(\alpha, a)\).

Proof. We have

\[
\left| \frac{zf'(z)}{f(z)} - a \right| = \frac{a - 1 + \sum_{n=2}^{j} (a - n)a_n z^{n-1} - \sum_{n=j+1}^{\infty} (n - a)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \leq a - 1 + \sum_{n=2}^{\infty} (n - a)|a_n|.
\]
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\[ a - 1 + \sum_{n=2}^{j} (a-n)|a_n||z|^{n-1} + \sum_{n=j+1}^{\infty} (n-a)|a_n||z|^{n-1} \leq \frac{1 - \sum_{n=2}^{j} |a_n||z|^{n-1} - \sum_{n=j+1}^{\infty} |a_n||z|^{n-1}}{1 - \sum_{n=2}^{j} |a_n| - \sum_{n=j+1}^{\infty} |a_{\overline{n}}|} \]

This last expression is bounded above by \( a - \alpha \) if

\[ a - 1 + \sum_{n=2}^{j} (a-\alpha)|a_n| + \sum_{n=j+1}^{\infty} (n-\alpha)|a_n| \leq (a - \alpha) \left( 1 - \sum_{n=2}^{j} |a_n| - \sum_{n=j+1}^{\infty} |a_n| \right). \]  

(2)

(2) is equivalent to the hypothesis of Theorem 2. This completes the proof.

We can prove the following theorem by using the same way of Theorem 1 and Theorem 2.

**Theorem 3** Let \( f(z) \in S \), \( 0 \leq \alpha < 1 \) and \( \frac{1+\alpha}{2} < a < 1 \). If

\[ \sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq 2a - 1 - \alpha, \]

then \( f(z) \in S_3(a, \alpha) \).

3. Distortion theorems

We denote by \( \tilde{S}_1(\alpha, a) \) the subclass of \( S(\alpha, a) \) \( (0 \leq \alpha < 1, 1 \leq a \leq 2) \) consisting of functions which satisfy

\[ \sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq 1 - \alpha. \]  

(3)

Since a function

\[ f(z) = z + \frac{1 - \alpha}{n - \alpha}z^n \]  

(4)

in \( S \) belongs to \( S(\alpha, a) \) and satisfies the coefficient inequality (3), this function (4) belongs to \( \tilde{S}_1(\alpha, a) \). Therefore, the subclass \( \tilde{S}_1(\alpha, a) \) is not empty.

**Theorem 4** If \( f(z) \in \tilde{S}_1(\alpha, a) \) \( (0 \leq \alpha < 1, 1 \leq a \leq 2) \), then

\[ |z| - \frac{1 - \alpha}{2 - \alpha}|z|^2 \leq |f(z)| \leq |z| + \frac{1 - \alpha}{2 - \alpha}|z|^2. \]

Equality holds for the function

\[ f(z) = z + \frac{1 - \alpha}{2 - \alpha}z^2. \]
Proof. By the assumption, note that

$$(2 - \alpha) \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} (n - \alpha)|a_n| \leq 1 - \alpha,$$

that is, that

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{1 - \alpha}{2 - \alpha}.$$

Thus, we have

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^n$$
$$\leq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n|$$
$$\leq |z| + |z|^2 \frac{1 - \alpha}{2 - \alpha},$$

and

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^n$$
$$\geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n|$$
$$\geq |z| - |z|^2 \frac{1 - \alpha}{2 - \alpha}.$$

We denote by $\tilde{S}_2(\alpha, a)$ the subclass of $S(\alpha, a) \ (0 \leq \alpha < 1, a > 2)$ consisting of functions which satisfy

$$\sum_{n=2}^{j}(2a - n - \alpha)|a_n| + \sum_{n=j+1}^{\infty} (n - \alpha)|a_n| \leq 1 - \alpha,$$

where $j < a \leq j + 1$. Just as in the case of $\tilde{S}_1(\alpha, a)$, by taking a function

$$f(z) = z + \frac{1 - \alpha}{2(2a - k - \alpha)} z^k + \frac{1 - \alpha}{2(l - \alpha)} z^l,$$

where $2 \leq k \leq j$ and $j + 1 \leq l < \infty$, we see that $\tilde{S}_2(\alpha, a)$ is not empty.

**Theorem 5** If $f(z) \in \tilde{S}_2(\alpha, a) \ (0 \leq \alpha < 1, a > 2)$, then

$$|z| - \frac{1 - \alpha}{2a - j - \alpha} |z|^2 \leq |f(z)| \leq |z| + \frac{1 - \alpha}{2a - j - \alpha} |z|^2,$$

(5)

for $a - \frac{1}{2} \leq j < a$
and

\[ |z| - \frac{1 - \alpha}{j + 1 - \alpha}|z|^2 \leq |f(z)| \leq |z| + \frac{1 - \alpha}{j + 1 - \alpha}|z|^2, \tag{6} \]

for \( a - 1 \leq j < a - \frac{1}{2} \).

These results are sharp.

Proof. Since \( 2a - n - \alpha \) and \( n - \alpha \) are decreasing and increasing for \( n \), respectively, by the assumption we note that

\[ (2a - j - \alpha) \sum_{n=2}^{j} |a_n| + (j + 1 - \alpha) \sum_{n=j+1}^{\infty} |a_n| \leq 1 - \alpha. \tag{7} \]

Then, we have

\[ \sum_{n=2}^{\infty} |a_n| \leq \frac{1 - \alpha}{2a - j - \alpha} \quad (a - \frac{1}{2} \leq j < a), \]

and

\[ \sum_{n=2}^{\infty} |a_n| \leq \frac{1 - \alpha}{j + 1 - \alpha} \quad (a - 1 \leq j < a - \frac{1}{2}). \tag{8} \]

By using (7), we have

\[ |f(z)| \leq |z| + \sum_{n=2}^{\infty} |a_n||z|^n \leq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \leq |z| + |z|^2 \frac{1 - \alpha}{2a - j - \alpha}, \]

and

\[ |f(z)| \geq |z| - \sum_{n=2}^{\infty} |a_n||z|^n \geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \geq |z| - |z|^2 \frac{1 - \alpha}{2a - j - \alpha}. \]

Equality of (5) holds for the function

\[ f(z) = z + \frac{1 - \alpha}{2a - j - \alpha}z^2. \]
Using coefficient inequality (8) we can obtain the latter of Theorem 1 similarly. Equality of (6) holds for the function

\[ f(z) = z + \frac{1 - \alpha}{j + 1 - \alpha} z^2. \]

Let \( \tilde{S}_3(\alpha, a) \) be the subclass of \( S(\alpha, a) \) \((0 \leq \alpha < 1, \frac{1 + \alpha}{2} < a < 1) \) consisting of functions which satisfy

\[ \sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 2a - 1 - \alpha. \]

Then, the subclass \( \tilde{S}_3(\alpha, a) \) is not empty, because there exists a function

\[ f(z) = z + \frac{2a - 1 - \alpha}{n - \alpha} z^n \]

in \( \tilde{S}_3(\alpha, a) \).

We can prove the following theorem in the same way of Theorem 4 and 5.

**Theorem 6** If \( f(z) \in \tilde{S}_3(\alpha, a) \) \((0 \leq \alpha < 1, \frac{1 + \alpha}{2} < a < 1) \), then

\[ |z| - \frac{2a - 1 - \alpha}{2 - \alpha} |z|^2 \leq |f(z)| \leq |z| + \frac{2a - 1 - \alpha}{2 - \alpha} |z|^2. \]

Equality holds for the function

\[ f(z) = z + \frac{2a - 1 - \alpha}{2 - \alpha} z^2. \]

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