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A LINEAR OPERATOR AND SOME APPLICATIONS OF FIRST ORDER DIFFERENTIAL SUBORDINATIONS

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1. INTRODUCTION

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$.

For functions $f_j(z) \in \mathcal{A}_p$ $(j=1,2)$ defined by

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k ,$$

we define the convolution (or Hadamard product) $f_1 * f_2(z)$ of functions $f_1(z)$ and $f_2(z)$ by

$$f_1 * f_2(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k . \tag{1.2}$$

With the convolution above, we define

$$D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) \qquad (f(z) \in \mathcal{A}_p)$$

$$= \frac{z^p (z^{n-1} f(z))^{(n+p-1)}}{(n+p-1)!} , \tag{1.3}$$

where n is any integer greater than -p.

For a function $f(z) \in \mathcal{A}_p$, we define the generalized Libera integral operator $J_{\nu,p}$ by

$$J_{\nu,p}(f(z)) = \frac{\nu + p}{z^\nu} \int_0^z t^{\nu-1} f(t)\,dt, \qquad \nu > -p. \tag{1.4}$$

For $p = 1$ and $\nu \in \mathbb{N}$, the operator $J_{\nu,1}$ was introduced by Bernardi [1]. Inparticular, the operator $J_{1,1}$ was studied earlier by Libera [4] and Livingston [5]. Some interesting results for the operator $J_{\nu,p}$ were showed by Saitoh [12] and Saitoh et al. [13].

Now, let the function $\phi_p(a,c)$ be defined by

$$\phi_p(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p} \qquad (z \in \mathbb{U}), \tag{1.5}$$

for $c \neq 0,-1,-2,\cdots$, where $(a)_n$ is the Pochhammer symbol given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 , & \text{if } n = 0 \\ a(a+1) \quad (a+n-1) , & \text{if } n \in \mathbb{N} \end{cases} \tag{1.6}$$

Also, we define a linear operator $L_p(a,c)$ on $\mathcal{A}_p$ by

$$L_p(a,c;z)f(z) = \phi_p(a,c;z)*f(z) \tag{1.7}$$

for $f(z) \in \mathcal{A}_p$ and $c \neq 0,-1,-2,\cdots$.

The operator $L_1(a,c)$ was introduced by Carlson and Shaffer [2] in their systematic investigation of certain interesting classes of starlike, convex, and prestarlike hypergeometric functions.

REMARKS. (1) For $f(z) \in \mathcal{A}_1 = \mathcal{A}$,

$$L_1(n+1,1;z)f(z) = D^n f(z) = \frac{z}{(1-z)^{n+1}} *f(z)$$

is Ruscheweyh derivative of $f(z)$ ([8]).

(2) For $f(z) \in \mathcal{A}_p$,

$$L_p(n+p,1;z)f(z) = D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} *f(z)$$

is Ruscheweyh derivative introduced by Goel and Sohi [3].

(3) For $f(z) \in \mathcal{A}_p$,

$$L_p(\nu+p,\nu+p+1;z)f(z) = J_{\nu,p}(f(z))$$

is the generalized Libera integral operator ([12],[13]).

(4) $\phi_1(a,c;z)$ is an incomplete beta function, related to the Gauss hypergeometric functions by

$$\phi_1(a,c;z) = z\cdot{}_2F_1(1,a;c;z) .$$

It follows from (1.7) that

$$z(L_p(a,c;z)f(z))' = aL_p(a+1,c;z)f(z)-(a-p)L_p(a,c;z)f(z) \quad (1.8)$$

Let the function $f(z)$ and $g(z)$ be analytic in U. Then the $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $w(z)$ analytic in U, with $w(0)=0$ and $|w(z)|<1$ $(z\varepsilon U)$, such that $f(z) = g(w(z))$ $(z\varepsilon U)$. We denote this subordination by $f(z)\prec g(z)$.

2. Main results

To establish our main results, we need the following lemmas.

LEMMA 1 [6]. Let $h(z)$ be convex and $B(z)$ be analytic in U with $\mathrm{Re}B(z)\geqq 0$. If $p(z)$ is analytic in U and $p(0)=h(0)$, then

$$p(z) + B(z)zp'(z)\prec h(z) \text{ implies } p(z)\prec h(z) \quad (z\varepsilon U).$$

LEMMA 2 [9]. Let $\gamma=0$, $\mathrm{Re}\gamma\geqq 0$, and let $h(z)$ be convex. If $p(z)$ is analytic in U with $p(0)=h(0)$, then

$$p(z)+\frac{1}{\gamma}zp'(z)\prec h(z) \text{ implies } p(z)\prec\frac{\gamma}{z^\gamma}\int_0^z h(t)t^{\gamma-1}dt \quad (z\varepsilon U).$$

THEOREM 1. Let λ, a be a real number with $\lambda \geqq 0$, $a > 0$. Let $h(z)$ be a convex function with $h(0)=1$, $c \neq 0,-1,-2,\cdots$ and $g(z) \in A_p$ satisfies

$$\mathrm{Re}\left\{\frac{L_p(a+1,c;z)g(z)}{L_p(a,c;z)g(z)}\right\} > 0 \tag{2.1}$$

If $f(z) \in A_p$ satisfies

$$(1-\lambda)\left\{\frac{L_p(a,c;z)f(z)}{L_p(a,c;z)g(z)}\right\} + \lambda\left\{\frac{L_p(a+1,c;z)f(z)}{L_p(a+1,c;z)g(z)}\right\} \prec h(z) \tag{2.2}$$

then we have

$$\frac{L_p(a,c;z)f(z)}{L_p(a,c;z)g(z)} \prec h(z) \qquad (z \in U). \tag{2.3}$$

PROOF. Put

$$H(z)=(1-\lambda)\left\{\frac{L_p(a,c;z)f(z)}{L_p(a,c;z)g(z)}\right\} + \lambda\left\{\frac{L_p(a+1,c;z)f(z)}{L_p(a+1,c;z)g(z)}\right\}$$

From assumption, $h(z)$ is convex with $h(0)=1$ and $f(z) \in A_p$ satisfies $H(z) \prec h(z)$ $(z \in U)$, where $\lambda \geqq 0$ and $g(z) \in A_p$ satisfies (2.1).

Set $B(z)= \dfrac{\lambda}{a} \cdot \dfrac{L_p(a,c;z)g(z)}{L_p(a+1,c;z)g(z)}$

According to (2.1), we have $\mathrm{Re}B(z) \geqq 0$. Define $p(z)$ by

$$p(z) = \frac{L_p(a,c;z)f(z)}{L_p(a,c;z)g(z)} \tag{2.4}$$

We can see that $p(z)$ is analytic in U and $p(0)=1$. Logarithmic differentiating (2.4) and using (1.8), we can get

$$\frac{L_p(a+1,c;z)f(z)}{L_p(a+1,c;z)g(z)} = p(z) + \frac{1}{\lambda} B(z) \cdot zp'(z) \qquad (z \in U). \tag{2.5}$$

Then (2.2) can be written

$$p(z) + B(z)zp'(z) \prec h(z).$$

By Lemma 1, we have

$$p(z) \prec h(z) .$$

Thus, we have

$$\frac{L_p(a,c;z)f(z)}{L_p(a,c;z)g(z)} \prec h(z) .$$

This completes the proof of Theorem 1.

Putting a = n+p, c=1 in Theorem 1, we have

COROLLARY 1. Let λ be a real number with $\lambda \geqq 0$. Let $h(z)$ be a convex function with $h(0)=1$ and $g(z) \in A_p$ satisfies

$$\mathrm{Re}\left\{\frac{D^{n+p}g(z)}{D^{n+p-1}g(z)}\right\} > 0 .$$

If $f(z) \in A_p$ satisfies

$$(1-\lambda)\left\{\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)}\right\} + \lambda\left\{\frac{D^{n+p}f(z)}{D^{n+p}g(z)}\right\} \prec h(z) ,$$

then we have

$$\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)} \prec h(z) \qquad (z \in U).$$

Making a = ν+p, c = ν+p+1 in Theorem 1, we can show

COROLLARY 2. Let λ be a real number with $\lambda \geqq 0$. Let $h(z)$ be a convex function with $h(0)=1$ and $g(z) \in A_p$ satisfies

$$\mathrm{Re}\left\{\frac{g(z)}{J_{\nu,p}(g(z))}\right\} > 0 .$$

If $f(z) \in A_p$ satisfies

$$(1-\lambda)\left\{\frac{J_{\nu,p}(f(z))}{J_{\nu,p}(g(z))}\right\} + \lambda\frac{f(z)}{g(z)} \prec h(z) ,$$

then we have

$$\frac{J_{\nu,p}(f(z))}{J_{\nu,p}(g(z))} \prec h(z) .$$

Next, we prove

THEOREM 2. Let $f(z)\varepsilon A_p$ and $h(z)$ be a convex function with $h(0)=1$. Then for any complex number λ with $\mathrm{Re}\lambda \geqq 0$ $(\lambda=0)$, $a>0$,

$$(1-\lambda)\left\{\frac{L_p(a,c;z)f(z)}{z^p}\right\}+\lambda\left\{\frac{L_p(a+1,c;z)f(z)}{z^p}\right\}\prec h(z) \quad (z\varepsilon U) \qquad (2.6)$$

implies

$$\frac{L_p(a,c;z)f(z)}{z^p}\prec\frac{a}{\lambda z^{a/\lambda}}\int_0^z h(t)t^{a/\lambda-1}dt\prec h(z) \quad (z\varepsilon U) \qquad (2.7)$$

The result is sharp.

PROOF. Choosing $g(z)=z^p$ and $B(z)=\frac{\lambda}{a}$, and use Lemma 2, Theorem 2 follows from Theorem 1.

REFERENCES

[1] S.D.Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc. 135(1969), 429-446.

[2] B.C.Carlson and D.B.Shaffer, Starlike and prestarlike hypergeometric functions, SIAM J. Math. Anal. 15(1984), 737-745.

[3] R.M.Goel and N.S.Sohi, A new criterion for p-valent functions, Proc. Amer. Math. Soc. 78(1980), 353-357.

[4] R.J.Libera, Some classes of regular univalent functions, ibid. 16(1965), 755-758.

[5] A.E.Livingston, On the radius of univalence of certain analytic functions, ibid. 17(1966), 352-357.

[6] S.S.Miller and P.T.Mocanu, Differential subordination and univalent functions, Michigan Math. J. 28(1981), 157-171.

[7] S.Owa, H.M.Srivastava and C.Y.Shen, Applications of a certain linear operator defined by a Hadamard product or convolution, Utilitas Math. 33(1988), 173-181.

[8] S.Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49(1975), 109-115.

[9] D.J.Hallenbeck and S.Ruscheweyh, Subordination by convex functions, Proc. Amer. Math. Soc. 52(1975), 191-195.

[10] S.Ponnusamy and O.P.Juneja, Some applications of first order differential subordinations, Glasnik Math. 25(45) (1990), 287-296.

[11] H.Saitoh, Some properties of certain multivalent functions, Tsukuba J. Math. 15(1991), 105-111.

[12] H.Saitoh, On certain class of multivalent functions, Math. Japon. 37(1992), 871-875.

[13] H.Saitoh, S.Owa, T.Sekine, M.Nunokawa and R.Yamakawa, An application of a certain integral operator, 5(1992), 21-24.

[14] H.M.Srivastava, C.Y.Shen and S.Owa, A linear fractional calculus operator and its applications to certain subclasses of analytic functions, J. Math. Anal. Appl. 143(1989), 138-147.