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<td>SAITOH, HITOSHI</td>
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Kyoto University
A LINEAR OPERATOR AND SOME APPLICATIONS OF FIRST ORDER
DIFFERENTIAL SUBORDINATIONS

HITOSHI SAITOHO (斎藤斉
群馬工芸高等学校)

1. INTRODUCTION

Let $A_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}) \quad (1.1)$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$.

For functions $f_j(z) \in A_p$ ($j=1, 2$) defined by

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k ,$$

we define the convolution (or Hadamard product) $f_1 * f_2(z)$ of functions $f_1(z)$ and $f_2(z)$ by

$$f_1 * f_2(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k . \quad (1.2)$$

With the convolution above, we define

$$D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) \quad (f(z) \in A_p) \quad (1.3)$$

where $n$ is any integer greater than $-p$.

For a function $f(z) \in A_p$, we define the generalized Libera integral operator $J_{v,p}$ by

$$J_{v,p}(f(z)) = \frac{v + p}{z^v} \int_0^z t^{v-1} f(t) dt, \quad v > -p. \quad (1.4)$$
For $p = 1$ and $v \in \mathbb{N}$, the operator $J_{v,1}$ was introduced by Bernardi [1]. In particular, the operator $J_{1,1}$ was studied earlier by Libera [4] and Livingston [5]. Some interesting results for the operator $J_{v,p}$ were showed by Saitoh [12] and Saitoh et al. [13].

Now, let the function $\phi_p(a,c)$ be defined by

$$\phi_p(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p} \quad (z \in \mathbb{D}), \quad (1.5)$$

for $c \neq 0,-1,-2,\ldots$, where $(a)_n$ is the Pochhammer symbol given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0 \\ a(a+1) \ldots (a+n-1), & \text{if } n \in \mathbb{N} \end{cases} \quad (1.6)$$

Also, we define a linear operator $L_p(a,c)$ on $A_p$ by

$$L_p(a,c;z)f(z) = \phi_p(a,c;z)*f(z) \quad (1.7)$$

for $f(z) \in A_p$ and $c \neq 0,-1,-2,\ldots$.

The operator $L_1(a,c)$ was introduced by Carlson and Shaffer [2] in their systematic investigation of certain interesting classes of starlike, convex, and prestarlike hypergeometric functions.

**Remarks.**

(1) For $f(z) \in A_1 = A$,

$$L_1(n+1,1;z)f(z) = D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z)$$

is Ruscheweyh derivative of $f(z)$ ([8]).

(2) For $f(z) \in A_p$,

$$L_p(n+p,1;z)f(z) = D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z)$$

is Ruscheweyh derivative introduced by Goel and Sohi [3].
(3) For \( f(z) \in \mathcal{A}_p \),
\[
L_p(\nu+p,\nu+p+1;z)f(z) = J_{\nu,p}(f(z))
\]
is the generalized Libera integral operator ([12],[13]).

(4) \( \phi_1(a,c;z) \) is an incomplete beta function, related to the Gauss hypergeometric functions by
\[
\phi_1(a,c;z) = z^2 F_1(1,a,c;z).
\]

It follows from (1.7) that
\[
z(L_p(a,c;z)f(z))' = aL_p(a+1,c;z)f(z) - (a-p)L_p(a,c;z)f(z) \quad (1.8)
\]

Let the function \( f(z) \) and \( g(z) \) be analytic in \( U \). Then the \( f(z) \) is said to be subordinate to \( g(z) \) if there exists a function \( w(z) \) analytic in \( U \), with \( w(0)=0 \) and \( |w(z)|<1 \) (\( z \in U \)), such that \( f(z) = g(w(z)) \) (\( z \in U \)). We denote this subordination by \( f(z) \prec g(z) \).

2. MAIN RESULTS

To establish our main results, we need the following lemmas.

**Lemma 1** [6]. Let \( h(z) \) be convex and \( B(z) \) be analytic in \( U \) with \( \text{Re}B(z) \geq 0 \). If \( p(z) \) is analytic in \( U \) and \( p(0)=h(0) \), then
\[
p(z) + B(z)zp'(z) \prec h(z) \text{ implies } p(z) \prec h(z) \quad (z \in U).
\]

**Lemma 2** [9]. Let \( \gamma=0 \), \( \text{Re} \gamma \geq 0 \), and let \( h(z) \) be convex. If \( p(z) \) is analytic in \( U \) with \( p(0)=h(0) \), then
\[
p(z) + \frac{1}{\gamma}zp'(z) \prec h(z) \text{ implies } p(z) \prec \frac{\gamma}{z^\gamma} \int_0^zh(t)t^{\gamma-1}dt \quad (z \in U).
\]
**Theorem 1.** Let \( \lambda, a \) be a real number with \( \lambda \geq 0, a > 0 \). Let 
\( h(z) \) be a convex function with \( h(0) = 1, c \neq 0, -1, -2, \ldots \) and 
\( g(z) \in \mathbb{A}_p \) satisfies 
\[
\text{Re} \left\{ \frac{L_p(a+1, c; z)g(z)}{L_p(a, c; z)g(z)} \right\} > 0 \tag{2.1}
\]
If \( f(z) \in \mathbb{A}_p \) satisfies 
\[
(1-\lambda) \left\{ \frac{L_p(a, c; z)f(z)}{L_p(a, c; z)g(z)} \right\} + \lambda \left\{ \frac{L_p(a+1, c; z)f(z)}{L_p(a+1, c; z)g(z)} \right\} < h(z) \tag{2.2}
\]
then we have 
\[
\frac{L_p(a, c; z)f(z)}{L_p(a, c; z)g(z)} < h(z) \quad (z \in U). \tag{2.3}
\]

**Proof.** Put 
\[
H(z) = (1-\lambda) \left\{ \frac{L_p(a, c; z)f(z)}{L_p(a, c; z)g(z)} \right\} + \lambda \left\{ \frac{L_p(a+1, c; z)f(z)}{L_p(a+1, c; z)g(z)} \right\}
\]
From assumption, \( h(z) \) is convex with \( h(0) = 1 \) and \( f(z) \in \mathbb{A}_p \) satisfies 
\( H(z) < h(z) \) \( (z \in U) \), where \( \lambda \geq 0 \) and \( g(z) \in \mathbb{A}_p \) satisfies \( (2.1) \).

Set 
\[
B(z) = \frac{L_p(a, c; z)g(z)}{L_p(a+1, c; z)g(z)}
\]
According to \( (2.1) \), we have \( \text{Re}B(z) > 0 \). Define \( p(z) \) by 
\[
p(z) = \frac{L_p(a, c; z)f(z)}{L_p(a, c; z)g(z)} \tag{2.4}
\]
We can see that \( p(z) \) is analytic in \( U \) and \( p(0) = 1 \). Logarithmic differentiating \( (2.4) \) and using \( (1.8) \), we can get 
\[
\frac{L_p(a+1, c; z)f(z)}{L_p(a+1, c; z)g(z)} = p(z) + \frac{1}{\lambda} B(z)zp'(z) \quad (z \in U). \tag{2.5}
\]
Then \( (2.2) \) can be written
\[ p(z) + B(z)zp'(z) \prec h(z). \]

By Lemma 1, we have
\[ p(z) \prec h(z). \]

Thus, we have
\[ \frac{L_p(a,c;z)f(z)}{L_p(a,c;z)g(z)} \prec h(z). \]

This completes the proof of Theorem 1.

Putting \( a = n+p, \ c=1 \) in Theorem 1, we have

**COROLLARY 1.** Let \( \lambda \) be a real number with \( \lambda \geq 0 \). Let \( h(z) \) be a convex function with \( h(0) = 1 \) and \( g(z) \in A_p \) satisfies
\[ \text{Re} \left\{ \frac{D^{n+p}g(z)}{D^{n+p-1}g(z)} \right\} > 0. \]

If \( f(z) \in A_p \) satisfies
\[ (1-\lambda)\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)} + \lambda \frac{D^{n+p}f(z)}{D^{n+p}g(z)} \prec h(z), \]

then we have
\[ \frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)} \prec h(z) \quad (z \in U). \]

Making \( a = \nu+p, \ c=\nu+p+1 \) in Theorem 1, we can show

**COROLLARY 2.** Let \( \lambda \) be a real number with \( \lambda \geq 0 \). Let \( h(z) \) be a convex function with \( h(0) = 1 \) and \( g(z) \in A_p \) satisfies
\[ \text{Re} \left\{ \frac{g(z)}{J_{\nu,p}(g(z))} \right\} > 0. \]

If \( f(z) \in A_p \) satisfies
\[ (1-\lambda)\frac{J_{\nu,p}(f(z))}{J_{\nu,p}(g(z))} + \lambda \frac{f(z)}{g(z)} \prec h(z), \]

then we have
\[ \frac{J_{\nu,p}(f(z))}{J_{\nu,p}(g(z))} \prec h(z). \]
Next, we prove

**Theorem 2.** Let \( f(z) \in A_p \) and \( h(z) \) be a convex function with \( h(0) = 1 \). Then for any complex number \( \lambda \) with \( \Re \lambda \geq 0 \) \((\lambda > 0), \alpha > 0\),

\[
(1-\lambda)\left\{ \frac{L_p(a,c;z)f(z)}{z^p} \right\} + \lambda\left\{ \frac{L_p(a+1,c;z)f(z)}{z^p} \right\} < h(z) \quad (z \in U) \tag{2.6}
\]

implies

\[
\frac{L_p(a,c;z)f(z)}{z^p} < \frac{a}{\lambda z^{a/\lambda}} \int_0^z h(t)t^{a/\lambda-1}dt < h(z) \quad (z \in U) \tag{2.7}
\]

The result is sharp.

**Proof.** Choosing \( g(z) = z^p \) and \( B(z) = \frac{\lambda}{a} \), and use Lemma 2,

Theorem 2 follows from Theorem 1.

**References**


