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SOME PROPERTIES OF A LINEAR OPERATOR

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ABSTRACT

A certain linear operator defined by a Hadamard product or convolution for functions which are analytic in the open unit disk is introduced. The object of the present paper is to derive some properties of this linear operator.

I. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = \sum_{n=0}^{\infty} a_{n+1} z^{n+1} \qquad (a_1 = 1) \tag{1.1}$$

that are analytic in the open unit disk $\mathbb{U} = \{z: |z| < 1\}$.

For functions $f_j(z)$ $(j = 1,2)$ defined by

$$f_j(z) = \sum_{n=0}^{\infty} a_{n+1,j} z^{n+1} \qquad (a_{1,j} = 1), \tag{1.2}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{n+1,1} a_{n+1,2} z^{n+1}. \tag{1.3}$$

Now, we define the function $\phi(a,c;z)$ by

$$\phi(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \qquad (c \neq 0, -1, -2, \ldots), \tag{1.4}$$

so that $\phi(a,c;z)$ is an incomplete bate function with

$$\phi(a,c;z) = z\,{}_2F_1(1,a;c;z), \tag{1.5}$$

where

$$ {}_2F_1(1,a;c;z) = \sum_{n=0}^{\infty} \frac{(1)_n (a)_n}{(c)_n} \frac{z^n}{n!}, \tag{1.6}$$

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and $(x)_n$ is the Pochhammer symbol defined by

(1.7) $$(x)_n = \begin{cases} x(x+1)(x+2)\cdots(x+n-1) & (\text{if } n \in \mathbb{N} = \{1,2,3,\ldots\} \\ 1 & (\text{if } n = 0)\ . \end{cases}$$

Corresponding to the function $\phi(a,c;z)$, Carlson and Shaffer [1] defined a linear operator $L(a,c)$ on $\mathcal{A}$ by the convolution

(1.8) $$L(a,c)f(z) = \phi(a,c;z)*f(z)$$

for $f(z) \in \mathcal{A}$. Clearly, $L(a,c)$ maps $\mathcal{A}$ onto itself, and $L(c,a)$ is an inverse of $L(a,c)$, provided that $a \neq 0, -1, -2, \cdots$.

Recently, Srivastava and Owa [4] have given some properties of $L(a,c)$ concerning with univalent functions in $\mathcal{U}$. To derive our result for the linear operator, we have to recall here the following lemma due to Jack [2] (also, due to Miller and Mocanu [3]).

LEMMA 1. Let $w(z)$ be analytic in $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value in the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, then we can write

$$z_0 w'(z_0) = kw(z_0),$$

where k is real and $k \geqq 1$.

2. SOME PROPERTIES

First of all, we show the following lemma.

LEMMA 2. If $f(z) \in \mathcal{A}$, then

(2.1) $$z(L(a,c)f(z))' = aL(a+1,c)f(z) - (a-1)L(a,c)f(z),$$

where $c \neq 0, -1, -2, \cdots$.

PROOF. Note that

(2.2) $$L(a,c)f(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1}$$

and

$$\text{(2.3)} \qquad L(a+1,c)f(z) = \sum_{n=0}^{\infty} \frac{(a+1)_n}{(c)_n} a_{n+1} z^{n+1}$$

This gives that

$$\text{(2.4)} \quad aL(a+1,c)f(z) - (a-1)L(a,c)f(z)$$

$$= \sum_{n=0}^{\infty} (a+n)\frac{(a)_n}{(c)_n} a_{n+1} z^{n+1} - \sum_{n=0}^{\infty} (a-1)\frac{(a)_n}{(c)_n} a_{n+1} z^{n+1}$$

$$= \sum_{n=0}^{\infty} (n+1)\frac{(a)_n}{(c)_n} a_{n+1} z^{n+1}$$

$$= z(L(a,c)f(z))'.$$

Applying Lemma 1 and Lemma 2, we prove

THEOREM 1. If $f(z) \in \mathcal{A}$ satisfies

$$\text{(2.5)} \qquad \mathrm{Re}(L(a+1,c)f(z)) > -\alpha \qquad (z \in \mathcal{U})$$

for $\alpha > 0$, then

$$\text{(2.6)} \qquad \mathrm{Re}(L(a-j,c)f(z)) > -\alpha\beta_j \qquad (z \in \mathcal{U}),$$

where $a < 0$ and

$$\text{(2.7)} \qquad \beta_j = 2\prod_{k=0}^{j}\left(\frac{a-k}{2a-2k-1}\right).$$

PROOF. We define the function $p(z)$ by $p(z) = L(a,c)f(z)$. Then, Lemma 2 gives us that

$$\text{(2.8)} \qquad zp'(z) = aL(a+1,c)f(z) - (a-1)p(z),$$

so that,

$$\text{(2.9)} \qquad L(a+1,c)f(z) = \left(1 - \frac{1}{a}\right)p(z) + \frac{1}{a} zp'(z).$$

Furthre, define the function $w(z)$ by

$$(2.10) \qquad p(z) = \frac{-\gamma w(z)}{1 - w(z)} \qquad (w(z) \neq 1),$$

where $\gamma = -4a\alpha/(2a-1)$. Then we have

$$(2.11) \qquad zp'(z) = \frac{-\gamma zw'(z)}{(1 - w(z))^2}$$

It follows from (2.10) and (2.11) that

$$(2.12) \qquad L(a+1,c)f(z) = -\gamma\left(1 - \frac{1}{a}\right)\frac{w(z)}{1 - w(z)} - \frac{1}{a}\gamma\frac{zw'(z)}{(1 - w(z))^2}$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \qquad (w(z_0) \neq 1).$$

Then, using Lemma 1, we have

$$z_0 w'(z_0) = kw(z_0) \qquad (k \geq 1).$$

Therefore, letting $w(z_0) = e^{i\theta}$, we obtain

$$(2.13) \quad \mathrm{Re}(L(a+1,c)f(z_0))$$

$$= \mathrm{Re}\left\{-\gamma\left(1 - \frac{1}{a}\right)\frac{w(z_0)}{1 - w(z_0)} - \frac{1}{a}\gamma\frac{z_0 w'(z_0)}{(1 - w(z_0))^2}\right\}$$

$$= \mathrm{Re}\left\{-\gamma\left(1 - \frac{1}{a}\right)\frac{e^{i\theta}}{1 - e^{i\theta}} - \frac{1}{a}\gamma\frac{ke^{i\theta}}{(1 - e^{i\theta})^2}\right\}$$

$$= \frac{\gamma}{2}\left(1 - \frac{1}{a}\right) + \frac{1}{a}\gamma\frac{k}{2(1 - \cos\theta)}$$

$$\leq \frac{\gamma}{2}\left(1 - \frac{1}{a}\right) + \frac{k\gamma}{4a}$$

$$\leq \frac{\gamma}{2}\left(1 - \frac{1}{a}\right) + \frac{\gamma}{4a}$$

$$= -\alpha.$$

This contradicts our condition (2.5). Therefore, we conclude that $|w(z)| < 1$ for all $z \in U$, so that

$$\mathrm{Re}(p(z)) > \frac{\gamma}{2} = -\frac{2a\alpha}{2a-1} \qquad (z \in U).$$

Thus we have that

$$\mathrm{Re}(L(a,c)f(z)) > -\alpha\beta_0 \qquad (z \in U) \tag{2.14}$$

with $\beta_0 = 2a/(2a-1)$.

Further, repeating this manner, we prove that

$$\mathrm{Re}(L(a-j,c)f(z)) > -\alpha\beta_j \qquad (z \in U). \tag{2.15}$$

This completes the assertion of Theorem 1.

Similarly, we have

THEOREM 2. If $f(z) \in A$ satisfies

$$\mathrm{Re}(L(a+1,c)f(z)) < \alpha \qquad (z \in U) \tag{2.16}$$

for $\alpha > 0$, then

$$\mathrm{Re}(L(a-j,c)f(z)) < \alpha\beta_j \qquad (z \in U), \tag{2.17}$$

where $a > 0$ and

$$\beta_j = 2 \prod_{k=0}^{j} \left(\frac{a-k}{2a-ak-1} \right). \tag{2.18}$$

PROOF. Let $p(z) = L(a,c)f(z)$, and let $p(z) = -\gamma w(z)/(1-w(z))$ with $\gamma = 4a\alpha/(2a-1)$. If there exists a point $z_0 \in U$ such that

$$\max_{|z| \le |z_0|} |w(z)| = |w(z_0)| = 1 \qquad (w(z_0) \neq 1),$$

then we have

$$\mathrm{Re}(L(a+1,c)f(z_0)) = \frac{\gamma}{2}\left(1 - \frac{1}{a}\right) + \frac{1}{a}\gamma \frac{k}{2(1-\cos\theta)} \tag{2.19}$$

$$\geq \frac{\gamma}{2}\left(1 - \frac{1}{a}\right) + \frac{k\gamma}{4a}$$

$$\geq \frac{\gamma}{2}\left(1 - \frac{1}{a}\right) + \frac{\gamma}{4a}$$

$$= \alpha.$$

It follows from (2.19) that

$$\text{(2.20)} \qquad \mathrm{Re}(L(a,c)f(z)) < \alpha\beta_0 \qquad (z \in \mathbb{U}).$$

Also, repeating this step, we prove Theorem 2.

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