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ON A SUBCLASS OF p-VALENT FUNCTIONS OF ORDER α

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ABSTRACT. Let $T_{n+p-1}(\alpha)$ denote the class of functions

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N = \{1,2,...\})$$

which are regular in the unit disc $U = \{z: |z| < 1\}$ and satisfying

$$\mathrm{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} > \alpha, \qquad z \in U$$

where $0 \le \alpha < p$, n is any integer such that $n > -p$ and $D^{n+p-1}f(z)$ denote the $(n+p-1)$th order Ruscheweyh derivative of $f(z)$. We show that the functions in $T_{n+p-1}(\alpha)$ are p-valent. Further properties preserving integrals are considered.

KEY WORDS- Regular, p-valent, Ruscheweyh derivative.
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1. Introduction.

Let A(p) denote the class of functions

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N = \{1,2,\dots\}) \tag{1.1}$$

which are regular in the unit disc $U = \{z:|z| < 1\}$. Let f(z) be in A(p) and g(z) be in A(p). Then we denote by $f * g(z)$ the Hadamard product of f(z) and g(z), that is, if f(z) is given by (1.1) and g(z) is given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \quad (p \in N), \tag{1.2}$$

then

$$f * g(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}. \tag{1.3}$$

The (n+p-1)th order Ruscheweyh derivative $D^{n+p-1}f(z)$ of a function f(z) of A(p) is defined by

$$D^{n+p-1}f(z) = \frac{z^p(z^{n-1}f(z))^{(n+p-1)}}{(n+p-1)!} \tag{1.4}$$

where n is any integer such that $n > -p$. It is easy to see that

$$D^{n+p-1}f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z). \tag{1.5}$$

In [6] Sohi introduced the classes T_{n+p-1} of functions f(z) in A(p) satisfying the condition

$$\operatorname{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} > 0, \qquad z \in U. \tag{1.6}$$

Further Sohi [6] showed the basic property

$$T_{n+p} \subset T_{n+p-1}. \tag{1.7}$$

In this paper we consider the classes of functions $f(z) \in A(p)$ which satisfy the condition

$$\operatorname{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} > \alpha, \quad 0 \le \alpha < p, \ z \in U. \tag{1.8}$$

Using the identity

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - nD^{n+p-1}f(z) \tag{1.9}$$

condition (1.8) can be rewritten as

$$\operatorname{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right\} > \frac{n+\alpha}{n+p}, \quad n > -p, \ z \in U. \tag{1.10}$$

In this paper we prove that $T_{n+p}(\alpha) \subset T_{n+p-1}(\alpha)$. Since $T_0(\alpha)$ is the class of functions which satisfy the condition $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$, $0 \le \alpha < p$, $z \in U$, it follows that the functions in the classes $T_{n+p-1}(\alpha)$ are p-valent in U (see Umezawa [7]). Further properties preserving integrals are considered. Some known results of Bernardi [1,2], Goel [3], and Sohi [6] are extended.

2. Properties of the classes $T_{n+p-1}(\alpha)$.

In proving our main results (Theorem 1 and Theorem 2), we shall need the following lemma due to I. S. Jack [4].

Lemma. Let $w(z)$ be non-constant regular in U, $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at z_o, we have $z_o w'(z_o) = kw(z_o)$ where k is a real number, $k \geq 1$.

Theorem 1. $T_{n+p}(\alpha) \subset T_{n+p-1}(\alpha)$, $0 \leq \alpha < p$, $n > -p$.

Proof. Let $f(z) \in T_{n+p}(\alpha)$. Then

$$\mathrm{Re}\left\{\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)}\right\} > \alpha, \qquad z \in U. \tag{2.1}$$

We have to show that (2.1) implies the inequality

$$\mathrm{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} > \alpha, \qquad z \in U. \tag{2.2}$$

Define a regular function $w(z)$ in U such that $w(0) = 0$, $w(z) \neq -1$ by

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = \frac{p+(2\alpha-p)w(z)}{1+w(z)}. \tag{2.3}$$

Using (1.9), (2.3) may be written as

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{(p+n) + (2\alpha+n-p)w(z)}{1 + w(z)}. \tag{2.4}$$

Logarithmic differentiation of (2.4) yields

$$\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} = \left[(p-\alpha)\frac{1-w(z)}{1+w(z)} + \alpha\right] - \frac{2(p-\alpha)zw'(z)}{[(p+n)+(2\alpha+n-p)w(z)](1+w(z))}. \tag{2.5}$$

We claim that $|w(z)| < 1$ in U. For otherwise (by Jack's lemma) there exists z_o in U such that

$$z_o w'(z_o) = k\, w(z_o) \tag{2.6}$$

where $|w(z_o)| = 1$ and $k \geq 1$. From (2.5) and (2.6), we obtain

$$\frac{z_o(D^{n+p}f(z_o))'}{D^{n+p}f(z_o)} = \left[(p-\alpha)\frac{1-w(z_o)}{1+w(z_o)} + \alpha\right] - \frac{2(p-\alpha)kw(z_o)}{[(p+n)+(2\alpha+n-p)w(z_o)](1+w(z_o))}. \tag{2.7}$$

Since $\mathrm{Re}\left\{\frac{1-w(z_o)}{1+w(z_o)}\right\} = 0, \quad k \geq 1,$

$$2(p-\alpha)k\ \mathrm{Re}\left\{\frac{w(z_o)}{[(p+n)+(2\alpha+n-p)w(z_o)](1+w(z_o))}\right\} \geq \frac{(p-\alpha)}{2(n+\alpha)},$$

we see that $\mathrm{Re}\left\{\frac{z_o(D^{n+p}f(z_o))'}{D^{n+p}f(z_o)}\right\} \leq \alpha - \frac{(p-\alpha)}{2(n+\alpha)} < \alpha$, which contradicts (2.1). Hence $|w(z)| < 1$, $z \in U$ and from (2.3) it follows that $f(z) \in T_{n+p-1}(\alpha)$.

Theorem 2. Let $f(z) \in A(p)$ satisfying the condition

$$\mathrm{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} > \alpha - \frac{(p-\alpha)}{2(c+\alpha)},\quad c > 0,\ 0 \leq \alpha < p, \tag{2.8}$$

then

$$F(z) = \frac{p+c}{z^c}\int_0^z t^{c-1}f(t)\,dt \tag{2.9}$$

belongs to $T_{n+p-1}(\alpha)$.

Proof. Let $w(z)$ be a regular function in U, $w(0) = 0$, $w(z) \neq -1$ defined by

$$\frac{z(D^{n+p-1}F(z))'}{D^{n+p-1}F(z)} = \frac{p-(p-2\alpha)w(z)}{1+w(z)}. \tag{2.10}$$

Using the identity

$$z(D^{n+p-1}F(z))' = (c+p)D^{n+p-1}f(z) - c\, D^{n+p-1}F(z) \qquad (2.11)$$

(2.10) can be rewritten as

$$\frac{D^{n+p-1}f(z)}{D^{n+p-1}F(z)} = \frac{(c+p)+(c+2\alpha-p)w(z)}{(1+w(z))(c+p)}. \qquad (2.12)$$

Taking logarithmic differentiation of (2.12), we get after a simple computation

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = \left[(p-\alpha)\frac{1-w(z)}{1+w(z)} + \alpha\right]$$

$$- \frac{2(p-\alpha)zw'(z)}{[(c+p)+(c+2\alpha-p)w(z)](1+w(z))}. \qquad (2.13)$$

Now we claim that $|w(z)| < 1$ for $z \in U$, for otherwise (by Jack's lemma) there exists a z_o, $z_o \in U$, such that

$$z_o w'(z_o) = k\, w(z_o) \qquad (2.14)$$

with $|w(z_o)| = 1$ and $k \geq 1$.

From (2.13) and (2.14) we have

$$\frac{z_o(D^{n+p-1}f(z_o))'}{D^{n+p-1}f(z_o)} = \left[(p-\alpha)\frac{1-w(z_o)}{1+w(z_o)} + \alpha\right]$$

$$- \frac{2(p-\alpha)kw(z_o)}{[(c+p)+(c+2\alpha-p)w(z_o)](1+w(z_o))}. \qquad (2.15)$$

Since $\text{Re}\left\{\frac{1-w(z_o)}{1+w(z_o)}\right\} = 0$, $k \geq 1$,

$$2(p-\alpha)k\ \text{Re}\left\{\frac{w(z_o)}{[(c+p)+(c+2\alpha-p)w(z_o)](1+w(z_o))}\right\} \geq \frac{(p-\alpha)}{2(c+\alpha)},$$

it follows from (2.15) that

$$\text{Re}\left\{\frac{z_o(D^{n+p-1}f(z_o))'}{D^{n+p-1}f(z_o)}\right\} \leq \alpha - \frac{(p-\alpha)}{2(c+\alpha)},$$

which contradicts (2.8). Hence $|w(z)| < 1$ in U and from (2.10) it follows that $F(z) \in T_{n+p-1}(\alpha)$. This completes the proof of Theorem 2.

Putting p = 1 and taking n = 0 and n = 1 in Theorem 2, we obtain the following extensions of the earlier results of Bernardi [1] and Sohi [6].

<u>Corollary 1</u>. If f(z) is starlike of order $\alpha - \frac{1-\alpha}{2(c+\alpha)}$, then the function

$$F(z) = \frac{1+c}{z^c}\int_0^z t^{c-1}f(t)\,dt$$

is starlike of order α, $0 \leq \alpha < 1$.

<u>Corollary 2</u>. If f(z) is convex of order $\alpha - \frac{1-\alpha}{2(c+\alpha)}$, then the function

$$F(z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t)\, dt$$

is convex of order α, $0 \le \alpha < 1$.

Theorem 3. If $F(z) \in T_{n+p-1}(\alpha)$, $0 \le \alpha < p$, and $f(z)$ is defined by (2.9), then $f(z) \in T_{n+p-1}(\alpha)$ for

$$|z| < \frac{c+p}{\sqrt{2p+1+c^2+\alpha(\alpha+2c-2)} + (p+1-\alpha)}$$

the result is sharp.

Proof. Since $F(z) \in T_{n+p-1}(\alpha)$, $0 \le \alpha < p$, therefore we can write

$$\frac{\dfrac{z(D^{n+p-1}F(z))'}{D^{n+p-1}F(z)} - \alpha}{p - \alpha} = u(z) \qquad (2.16)$$

where $u(z)$ is regular in U and satisfies the conditions $\mathrm{Re}\{u(z)\} > 0$ and $u(0) = 1$.

From (2.11) and (2.16) we have

$$\frac{D^{n+p-1}f(z)}{D^{n+p-1}F(z)} = \frac{(p-\alpha)u(z) + (\alpha+c)}{(c+p)}. \qquad (2.17)$$

Taking logarithmic differentiation of (2.17) and using (2.16), we get after a simple computation

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} - \alpha = (p-\alpha)\left\{u(z) + \frac{z\,u'(z)}{(p-\alpha)u(z)+(\alpha+c)}\right\}. \quad (2.18)$$

It is well known [5] that for $|z| = r < 1$,

$$|z\,u'(z)| \le \frac{2r}{1-r^2}\,\mathrm{Re}\{u(z)\}. \quad (2.19)$$

Thus from (2.18) and (2.19) we have for $|z| = r < 1$,

$$\mathrm{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} - \alpha\right\} \ge (p-\alpha)\left\{1 - \frac{2r}{(1-r)[c+p+(c+2\alpha-p)r]}\right\}\mathrm{Re}\{u(z)\}. \quad (2.20)$$

Thus $\mathrm{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} - \alpha\right\} > 0$

if

$$|z| < \frac{c+p}{\sqrt{2p+1+c^2+\alpha(\alpha+2c-2)} + (p+1-\alpha)}.$$

The result is sharp for the function

$$f(z) = \frac{z^{1-c}}{p+c}\,(z^c F(z))',$$

where F(z) is given by

$$\frac{z(D^{n+p-1}F(z))'}{D^{n+p-1}F(z)} = (p-\alpha)\frac{1-z}{1+z} + \alpha, \qquad 0 \le \alpha < p.$$

Putting n + p = 1 = c in Theorem 3, we obtain

Corollary 3. If $F(z) \in S^*(p,\alpha)$ (the class of p-valent starlike functions of order α $(0 \le \alpha < p)$) then $f(z) = \frac{1}{p+1}(zF(z))' \in S^*(p,\alpha)$ for $|z| < \frac{p+1}{\sqrt{2+2p+\alpha^2} + (p+1-\alpha)}$. The result is sharp.

Remark 1. Putting $\alpha = 0$ in Corollary 3, we get the result obtained by Goel [3].

Putting p = 1 and n = 0 in Theorem 3, we obtain

Corollary 4. If $F(z) \in S^*(\alpha)$ (the class of starlike functions of order α $(0 \le \alpha < 1)$) then $f(z) = \frac{1}{2}(zF(z))' \in S^*(\alpha)$ for $|z| < \frac{c+1}{\sqrt{3+c^2+\alpha(\alpha+2c-2)}+(2-\alpha)}$. The result is sharp.

Remark 2. Putting $\alpha = 0$ in Corollary 4, we get the result obtained by Bernardi [2].

Theorem 4. If $f(z) \in T_{n+p-1}(\alpha)$, and $F(z)$ is defined by

$$F(z) = \frac{n+p}{z^n} \int_0^z t^{n-1} f(t)\, dt, \tag{2.21}$$

then $F(z) \in T_{n+p}(\alpha)$.

<u>Proof</u>. From (2.21) we have

$$n\, F(z) + z\, F'(z) = (n+p) f(z).$$

Therefore

$$n\, D^{n+p-1}F(z) + D^{n+p-1}(zF'(z)) = (n+p)D^{n+p-1}f(z)$$

or

$$n\, D^{n+p-1}F(z) + z(D^{n+p-1}F(z))' = (n+p)D^{n+p-1}f(z). \tag{2.22}$$

Using (1.10), we conclude from (2.22) that

$$D^{n+p}F(z) = D^{n+p-1}f(z). \tag{2.23}$$

Taking logarithmic differentiation of (2.23) and using the fact that $f(z) \in T_{n+p-1}(\alpha)$ we have

$$\mathrm{Re}\left\{\frac{z(D^{n+p}F(z))'}{D^{n+p}F(z)}\right\} = \mathrm{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} > \alpha, \ 0 \le \alpha < p, \ z \in U.$$

Hence $F(z) \in T_{n+p}(\alpha)$.

<u>Remark 3</u>. Taking $\alpha = 0$ in the above theorems, we get the results obtained by Sohi [6].

References

1. S.D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc. 135 (1969), 429–446.

2. S.D. Bernardi. The radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 24 (1970), 312–318.

3. R.M. Goel, On radii of starlikeness, convexity, close-to-convexity for p-valent functions, Arch. ration. Mech. Anal. 44 (1972), 320–328.

4. I.S. Jack, Functions starlike and convex of order α, J. London Math. Soc. 2 (1971), no. 3, 469–474.

5. T.H. MacGregor, Functions whose derivative has a positive real part, Trans. Amer. Math. Soc. 104 (1962), 532–537.

6. N. S. Sohi, On a subclass of p-valent functions, Indian J. Pure Appl. Math. 11 (1980), no. 11, 1504–1508.

7. T. Umezawa, Multivalently close-to-convex functions, Proc. Amer. Math. Soc. 8 (1957), 869–874.

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