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One Criterion on a Class of Certain Analytic Functions

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Let A denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

which are analytic in the unit disk $U=\{z\,;\,|z|<1\}$.

A function belonging to A is said to be a member of the class $S(\alpha)$ if it satisfies

$$(1)\qquad \frac{zf'(z)}{f(z)} \qquad 1+(1-\alpha)z$$

for some $\alpha\,(0\leqq\alpha<1)$ and for all $z\in U$. The symbol denotes the subordination. It is easily confirmed that the condition (1) is equivalent to the following

$$(2)\qquad \left|\frac{zf'(z)}{f(z)}-1\right|<1-\alpha$$

for all $z\in U$.

In [1], Fukui obtained the following result

Theorem A. If $f(z)\in A$ satisfies

$$(3)\qquad \left|\beta\left(\frac{zf'(z)}{f(z)}-1\right)+(1-\beta)\frac{zf''(z)}{f(z)}\right|<1-\alpha$$

for some $\alpha\,(0\leqq\alpha<1)$, $\beta\,(0\leqq\beta<1)$, and for all $z\in U$, then $f(z)\in S(\alpha)$.

Making a lemma, we will improve Theorem A.

In order to derive our result, we need the following lemma due to Jack[2] (or Miller and Mocanu[3]).

Lemma 1. Let $w(z)$ be analytic in U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point z_0, then we have

$$z_0w'(z_0)=kw(z_0)$$

where k is real and $k\geqq 1$.

Applying Lemma 1, we have

Main Theorem. Let $p(z)$ be analytic in U, $p(0)=1$ and suppose that

$$(4)\qquad |\beta(p(z)-1)+(1-\beta)(p^2(z)-p(z)+zp'(z))|<(1-\alpha)(1+\alpha-\alpha\beta)$$

for some $\alpha\,(0\leqq\alpha<1)$, $\beta\,(0\leqq\beta<1)$ and for all $z\in U$. Then we have

$$|p(z)-1|<1-\alpha$$

for all $z \in U$.

Proof. Let us put

$$(1-\alpha)w(z)=(p(z)-1).$$

Then we have $w(0)=0$.

By an easy calculation, we have

$$\begin{aligned}&|\beta(p(z)-1)+(1-\beta)(p^2(z)-p(z)+zp'(z))|\\&=|\beta(1-\alpha)w(z)+(1-\beta)(1-\alpha)\{(1-\alpha)w^2(z)+w(z)+zw'(z)\}|\\&=\left|(1-\alpha)w(z)\left\{1+(1-\alpha)(1-\beta)w(z)+(1-\beta)\frac{zw'(z)}{w(z)}\right\}\right|\end{aligned}$$

If there exists a point z_0 such that

$$\max_{z<z_0}|w(z)|=|w(z_0)|=1,$$

then from Lemma 1, we have

$$\begin{aligned}&\left|(1-\alpha)w(z_0)\left\{1+(1-\beta)((1-\alpha)w(z_0)+\frac{z_0w'(z_0)}{w(z_0)})\right\}\right|\\&=(1-\alpha)|1+k(1-\beta)+(1-\alpha)(1-\beta)w(z_0)|\\&\geqq(1-\alpha)(1+1-\beta-(1-\alpha)(1-\beta))\\&=(1-\alpha)(1+\alpha-\alpha\beta).\end{aligned}$$

This contradicts to (4). This shows that

$$|p(z)-1|<1-\alpha$$

for all $z \in U$. This completes our proof.

Putting

$$p(z)=\frac{zf'(z)}{f(z)}$$

then we have

$$p^2(z)-p(z)+zp'(z)=\frac{zf''(z)}{f(z)}.$$

Therefore, from the Main theorem, we have

Corollary 1. If $f(z)\in A$ satisfies

$$\left|\beta\frac{zf'(z)}{f(z)}-1+(1-\beta)\frac{z^2f''(z)}{f(z)}\right|<(1-\alpha)(1+\alpha-\alpha\beta)$$

for some $\alpha\ (0\leqq\alpha<1)$, $\beta\ (0\leqq\beta<1)$ and for all $z \in U$, then we have $f(z)\in S(\alpha)$.

This is an improvement of Theorem A.

Taking $\beta=0$ in Corollary 1, we have

Corollary 2. If $f(z)\in A$ satisfies

$$\left|\frac{z^2 f''(z)}{f(z)}\right|<1-\alpha^2$$

for some $\alpha\,(0\leqq\alpha<1)$ and for all $z\in U$, then we have $f(z)\in S(\alpha)$.
This is an improvement of [1, Corollary 1].

Taking $\beta=1/2$ in Corollary 1, we have

Corollary 3. If $f(z)\in A$ satisfies

$$\left|\frac{z f'(z)}{f(z)}-1+\frac{z^2 f''(z)}{f(z)}\right|<(2-\alpha+\alpha^2)$$

for some $\alpha\,(0\leqq\alpha<1)$ and for all $z\in U$, then we have $f(z)\in S(\alpha)$.
This is an improvement of [1, Corollary 2].

Taking $\beta=0$ in Main theorem, we have

Corollary 4. Let $p(z)$ be analytic in U, $p(0)=1$ and suppose that

$$|p^2(z)-p(z)+z p'(z)|<1-\alpha^2$$

for all $z\in U$. Then we have

$$|p(z)-1|<1-\alpha$$

for all $z\in U$.

References

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