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<th>A Note on the Arithmeticity of the Figure-Eight Knot Orbifolds (Complex Analysis on Hyperbolic 3-Manifolds)</th>
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1. INTRODUCTION

Let $K$ be the figure-eight knot, $(K, n)$ the orbifold with underlying space $S^3$, singular set $K$ and isotropy group cyclic of order $n$.

**Proposition 1** (Thurston [5], Hilden-Lozano-Montesinos [1]). If $n > 3$, $(K, n)$ is hyperbolic. Furthermore, $(K, n)$ is arithmetic exactly for $n = 4, 5, 6, 8, 12$.

In this paper, our aim is to describe concretely the arithmeticity of $(K, n)$ for $n = 4, 5, 6, 8, 12$.

2. PRELIMINARIES

We can take a Kleinian model of $(K, n)$ as follows ([1]):

$$
\Gamma_n = \langle A, B | A^{-1}BAB^{-1}ABA^{-1}B^{-1}AB^{-1} = I, A^n = B^n = -I \rangle,
$$

$$
A = \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right), \quad B = \left( \begin{array}{cc} \mu & 1/\alpha \\ \alpha & \mu(\alpha - \mu) - 1 \end{array} \right),
$$

where

$$
\alpha = 2 \cos \frac{\pi}{n},
$$

$$
\beta = \frac{1}{2} \left( 1 + \alpha^2 + \sqrt{(\alpha^2 - 1)(\alpha^2 - 5)} \right),
$$

$$
\lambda = \frac{1}{2} (\alpha + \sqrt{\alpha^2 - 4}),
$$

$$
\mu = \frac{\lambda \beta - \alpha}{\lambda^2 - 1}.
$$

**Definition.** Let $\Gamma$ be a non-elementary Kleinian group and $\Gamma^{(2)}$ the subgroup generated by the squares of the elements of $\Gamma$. The invariant trace field of $\Gamma$ is the field $\mathbb{Q}(tr\Gamma^{(2)})$, and denoted by $k\Gamma$. The invariant quaternion algebra is given by

$$
\{ \sum a_i \gamma_i (\text{finite sum}) | a_i \in k\Gamma, \gamma_i \in \Gamma^{(2)} \},
$$

and denoted by $A\Gamma$.

In fact, we see that $A\Gamma$ is a quaternion algebra over $k\Gamma$ from:

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Lemma 2. Let $\Gamma$ be a Kleinian group of finite covolume. Then $A\Gamma$ is quaternion algebra over $k\Gamma$ if

(1) $k\Gamma$ is a number field with one complex place, and
(2) $tr\Gamma^{(2)}$ consists of algebraic integers.

Furthermore, if we define

$$R_{k\Gamma} = \{ a \in k\Gamma \mid a \text{ is an algebraic integer} \} \quad \text{and}$$

$$O\Gamma = \left\{ \sum b_i\gamma_i \text{(finite sum)} \mid b_i \in R_{k\Gamma}, \gamma_i \in \Gamma^{(2)} \right\},$$

then $O\Gamma$ is an order of $A\Gamma$.

The following lemma shows that if $\Gamma$ is arithmetic, it is sufficient to take $k\Gamma$ and $A\Gamma$ as its algebraic tools (see [2] [6]).

Lemma 3. Suppose that $\Gamma$ is an arithmetic Kleinian group. Then

$$\Gamma^{(2)} \subset P(O^1\Gamma)$$

where $P : SL(2, \mathbb{C}) \rightarrow PSL(2, \mathbb{C})$ and $O^1\Gamma = \{ x \in O\Gamma \mid \text{the norm of } x \text{ is } 1 \}$.

We shall calculate $k\Gamma_n$ for $n = 4, 5, 6, 8, 12$. Since $k\Gamma_n = \mathbb{Q}(\alpha^2, \beta)$ by [1], we see that

$$k\Gamma_4 = \mathbb{Q}(\sqrt{-3}) \quad (\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}});$$

$$k\Gamma_5 = \mathbb{Q}(\sqrt{\frac{-1-3\sqrt{5}}{2}}) \quad (\cos \frac{\pi}{5} = \frac{1+\sqrt{5}}{4});$$

$$k\Gamma_6 = \mathbb{Q}(\sqrt{-1}) \quad (\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2});$$

$$k\Gamma_8 = \mathbb{Q}(\sqrt{-1-2\sqrt{2}}) \quad (\cos \frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2});$$

$$k\Gamma_{12} = \mathbb{Q}(\sqrt{-2\sqrt{3}}) \quad (\cos \frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}).$$

All of them are the extension fields over $\mathbb{Q}$ of degree 2.

On the other hand, by [1],

$$A\Gamma_n = \frac{\frac{1}{4}(\lambda^2 - \lambda^{-2})^2, \alpha^2(\mu(\alpha - \mu) - 1)}{k\Gamma_n},$$
where
\[ 1 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} \frac{1}{2}(\lambda^2 - \lambda^{-2}) & 0 \\ 0 & -\frac{1}{2}(\lambda^2 - \lambda^{-2}) \end{pmatrix} \]
and
\[ j = \begin{pmatrix} 0 & 1 \\ \alpha^2\{\mu(\alpha - \mu) - 1\} & 0 \end{pmatrix}. \]
In fact,
\[ i^2 = \frac{1}{4}(\lambda^2 - \lambda^{-1})^2, \quad j^2 = \alpha^2\{\mu(\alpha - \mu) - 1\}, \quad ij = -ji. \]

3. MAIN THEOREM AND ITS PROOF

In the beginning of this section, we mention our main theorem.

**Theorem 4.** For the Kleinian model $\Gamma_n$ of $(K, n)$, the following are satisfied:

1. If $n = 4, 6, 8, 12$, then
   \[ \Gamma_n \cap P(O^1\Gamma_n) = \Gamma_n^{(2)} \quad \text{and} \quad [\Gamma_n : \Gamma_n \cap P(O^1\Gamma_n)] = 2. \]

2. For $n = 5$,
   \[ \Gamma_5 \cap P(O^1\Gamma_5) = \Gamma_5, \quad \text{that is,} \quad [\Gamma_5 : \Gamma_5 \cap P(O^1\Gamma_5)] = 1. \]

To prove this theorem, we need the next lemma.

**Lemma 5.** Let $\Gamma$ be a finitely generated group, $m$ the number of the generators of $\Gamma$. Then
\[ [\Gamma : \Gamma^{(2)}] \leq 2^m. \]

**Proof.** See [6].

**Proof of Theorem 4.** Lemma 3 and Lemma 5 show that
\[ [\Gamma_n : \Gamma_n \cap P(O^1\Gamma_n)] \leq [\Gamma_n : \Gamma_n^{(2)}] \leq 4. \]
Furthermore by the relation
\[ A^{-1}BAB^{-1}ABA^{-1}B^{-1}AB^{-1} = I, \]
we see that $AB \in \Gamma_n^{(2)}$. Hence $[\Gamma_n : \Gamma_n \cap P(O^1\Gamma_n)] \leq 2$, and it is sufficient to consider $A$ (or $B$). We set $A = a_0 \cdot 1 + a_1 i + a_2 j + a_3 ij$. In this case, solving linear equations, we see that
\[ a_0 = \alpha/2, \quad a_1 = 1/\alpha, \quad a_2 = a_3 = 0. \]
And since $\lambda^2 - \lambda^{-2} = \alpha\sqrt{\alpha^2 - 4}$, the norm of $A$ equals to 1.

Now, we shall classify into two cases. In case $n = 4, 6, 8, 12$: Since $\alpha \not\in k\Gamma_n$, we see $A \not\in A\Gamma_n$. Hence $A \not\in O\Gamma_n$. In case $n = 5$: By $A^5 = -I$, we have $-A = A^{-4} \in \Gamma_5^{(2)}$. On the other hand, since $\Gamma_5^{(2)} \subset \Gamma_5 \cap P(O^1\Gamma_5)$,

$$A = \sum (-b_i) \gamma_i$$

where $-b_i \in R_{k\Gamma_5}$, $\gamma_i \in \Gamma_5^{(2)}$ and the norm of $A$ equals to 1. Therefore $A \in O^1\Gamma_5$.

The proof of Theorem 4 is now completed.

4. THE DIFFICULTY ABOUT THE COMPLEMENT

For $S^3 - K$, there is Riley's model $\Gamma$ as its Kleinian model ([4]), so we know it is arithmetic. But it is difficult to calculate its arithmeticity same as the case of $(K, n)$. The difficulty comes from the lack of definite information about the order $O\Gamma$, but by relations in the fundamental group of $S^3 - K$ and experimental calculation in [6], we shall except the next problem.

**Problem 6.** For the Kleinian model $\Gamma$ of $S^3 - K$, is it satisfied that $\Gamma \cap P(O^1\Gamma) = \Gamma$? In other words,

$$[\Gamma : \Gamma \cap P(O^1\Gamma)] = 1?$$

In future, our subject is to investigate geometrical properties of arithmetic hyperbolic 3-manifolds.

REFERENCES


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