Quantized calculus and Teichmüller space

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1. quantized calculus.

The famous duality theorem by Fefferman states that the dual space of $\text{Re} H^1(S^1)$ is $BMO(S^1)$. On the other hand, $H^1$ can be represented as a product of two elements in $H^2(S^1)$,

$$h \in \text{Re} H^1 \iff h = g_1 H g_2 + (H g_1)g_2, \quad g_j \in L^2(S^1)$$

Here, $H$ is the Hilbert transformation. Further Fefferman showed that

$$|\int f h d\theta| \leq C \|f\|_{BMO} \|g_1\|_2 \|g_2\|_2$$

Hence for every function $f$ on $S^1$, set

$$[H, f](g) = H(f g) - f H(g) \quad g \in L^2(S^1),$$

and we have

$$\int f h d\theta = \int [H, f](g_1)g_2 d\theta.$$  

THEOREM ([1]). The operator $[H, f]$ is bounded if and only if $f \in BMO(S^1)$.

Following A. Connes, this operator is called the quantized derivative $d^Q(f)$ of $f$. In fact, considering on the real line, we have

$$[H, f](g) = \text{Const.} \int_{\mathbb{R}} \frac{f(x) - f(y)}{x-y} g(y)dy$$

and hence can considered as the "polarization" of usual differentiation. Moreover we know
Theorem ([1]). The operator $[H, f]$ is compact if and only if $f \in VMO(S^1)$.

Here $VMO(S^1)$ is the closure of $C(S^1)$ in $BMO(S^1)$. In particular, if $f \in C(S^1)$, then $[H, f]$ is compact. Recall that the dual space of $VMOA(S^1)$ is $H^1(S^1)$ and that an element of $VMO(S^1)$ is not necessarily continuous but only 'quasicontinuous'. More precisely,

$$L^\infty \cap VMO = QC \ = \ (H^\infty + C) \cap (H^\infty + C) = C + HC$$

Remark ([2]). If $f \in L^\infty$ and $|f| \in C(S^1)$, then $f \in QC$.

Now, "smoother" $f(S^1)$ in fractal sense, better $f$ as a compact operator.

Theorem ([3]). The operator $[H, f]$ belongs to the Schatten class $\mathcal{L}^p$ if and only if $f \in B^{1/p}_p(S^1)$, where $B^{1/p}_p(S^1)$ is the Besov space as below.

Here $f \in \mathcal{L}^p$ means that the sequence of eigenvalues of $|T| = (T^*T)^{1/2}$ belongs to $\ell^p$. (In particular, $\mathcal{L}^2$ is the Hilbert-Schmidt class.)

Next $f \in B^{1/p}_p(S^1)$ means that $f$ satisfies the inequality

$$\iint_{S^1 \times S^1} |f(x + t) - 2f(x) + f(x - t)|^p t^{-2} dx dt < +\infty.$$

Recall that, if $p > 1$, this inequality is equivalent to

$$\iint_{S^1 \times S^1} |f(x + t) - f(x)|^p t^{-2} dx dt < +\infty.$$

On the other hand, considering the harmonic extension on $D$, $f \in B^{1/p}_p(S^1)$ if and only if

$$\int_D \|D^2 f\|^p (1 - |z|)^{2p-2} |dz \wedge d\overline{z}| < +\infty.$$

(When $p > 1$, this is equivalent that $f$ is $p$-integrable 1-form, namely
\[ \int_{D} \| Df \|^p (1 - |z|)^{p-2} |dz \wedge d\overline{z}| < +\infty. \]

**Corollary.** \( B_{2}^{1/2}(S^{1}) \) is the Sobolev space (the harmonic Dirichlet space) \( HD(D) = W_{1}^{2}(D) \cap H(D) \), where \( D \) is the unit disk.

Boundary values form \( H^{1/2} = \{ (a_n) | \sum |n||a_n|^2 < +\infty \} \) (which S. Nag used).

2. On Hausdorff dimension of quasicircles.

A Riemann map \( f \) onto a \( K \)-quasi disk has a \( (1/K) \)-Hölder continuous boundary value. Hence, for instance (also see Astata, to appear), we have

**Proposition (cf. Falconer).** The Hausdorff dimension of a \( K \)-quasicircle is at most \( 2 - \frac{1}{K} \).

On the other hand,

**Theorem (Sullivan).** Assume that there is a cocompact quasiFuchsian group \( \Gamma \) whose limit set is a quasicircle \( C \) as the limit set. Then The Hausdorff dimension of \( C \) is \( p \) if and only if a Riemman map \( f \) onto the interior of \( C \) belongs to \( B_{q}^{1/q}(S^{1}) \) for every \( q > p \).

**Corollary ([4]).** A quasicircle \( C \) as in Theorem 4 has Hausdorff dimension \( p \) if and only if

\[ p = \inf \{ q | [H, f] \in L^q \}. \]

**Problem.** Characterize such quasicircles that corresponds to finitely generated Kleinian groups.

3. Teichmüller spaces.

Here we will give new representation of the Universal Teichmüller sapce. First we recall (cf. Astala-Gehring, '86) the folllowing
THEOREM (KOEBE). \{\log f' \mid f \text{ is univalent on } D\} \text{ is bounded in the Bloch space} \mathfrak{B} = \{f \mid \sup(1-|z|^2)|f'(z)| < +\infty\}.

On the other hand, the boundary value of \log f', where \( f \in T(1) \), does not necessarily belong to \( BMO(S^1) \). (cf. Astata-Zinsmeister, '91)

Now, if \( f \) is a Riemann map onto a quasidisk, \( f \) itself has a continuous boundary value. Hence we can consider to represent Riemann maps in the above spaces.

First we set

\[ \Sigma = \{ f \mid f \text{ is univalent on } D \text{ and has a form } = \frac{1}{z} + \sum_{n=1}^{\infty} c_n z^n \text{ near } z = \infty, \} \]

and equip \( \Sigma \) with the Bers topology. Then \( \Sigma \) has the subset \( \Sigma_1 \) which we can identify with the universal Teichmüller space \( T(1) \).

**Theorem.** \( \Sigma \) can be mapped injectively in \( VMO(S^1) \).

This injection is continuous at least on \( \Sigma_1 \).

In general, \( BMO(S^1) \subset \mathfrak{B} \) and hence \( VMO(S^1) \subset \mathfrak{B}_0 \), and it is known that, for \( g \in \mathfrak{B}_0 \), \( g \) has a finite angular limit on a set of Hausdorff dimension 1 (Makarov '89). Also recall that \( AD(D) \subset VMOA(D) \) (S.Yamashita '82. Further, see Aulaskari '88), and that \( f \in \Sigma \) has a finite angular limit almost everywhere as is seen by classical Plessner's theorem.

On the other hand, Pommerenke ([6]) showed that, under the locally uniformly boundedness assumption of average multiplicity,

\[ f \in BMOA(S^1) \text{ if and only if } f \in \mathfrak{B}, \quad f \in VMOA(S^1) \text{ if and only if } f \in \mathfrak{B}_0. \]

On the other hand, since multiplication by \( z \) is an invertible \( VMO \)-multiplier, we can identify \( \Sigma \) with \( z\Sigma \subset VMOA(S^1) \). In particular, we have the following
**Corollary.** \( \Sigma \) can be mapped injectively in \( \mathfrak{B}_0 \).

This injection is continuous at least on \( \Sigma_1 \).

**Remarks.** Recall that a Riemann map has a continuous boundary value if and only if the complement is locally connected. Hence the locally connectedness conjecture of the limit set (cf. Abikoff([7])) can be restated as follows:

The image of \( \Sigma(G) \) is contained in \( C(S^1) \) for a finitely generated Kleinian group \( G \), where \( \Sigma(G) \) corresponds to \( T(G) \).

It seems interesting to characterize Riemann maps, or elements of \( \Sigma \), belonging to \( VMO(S^1) - C(S^1) \) geometrically.

Now to prove Theorem, we note the following fact, which follows at once from the equivalence of \( VMO(S^1) \) and \( \mathfrak{B}_0 \), and from the geometrical characterization of Bloch functions by Pommerenke ([5]).

**Proposition.** Let \( f \) be a holomorphic injection of \( D \). If \( f(D) \) is bounded, then the boundary value \( f \) belongs to \( VMO(S^1) \).

But this fact has an interesting

**Corollary.** Let \( G \) be any Kleinian group which has \( \infty \) as an ordinary point, and \( f \) be a Riemann map onto a simply connected component of \( G \). Then \( f \in VMO(S^1) \).

Here we note that \( VMO \)-ness is a local property.

**Lemma (Gotoh).** Let \( f \) be meromorphic on \( D \) and has no poles near \( \partial D \). If, for every \( \zeta \in \partial D \), there is a neighborhood \( U \) of \( \zeta \) such that \( f \circ \phi_\zeta \in VMOA(S^1) \), where \( \phi_\zeta \) is a Riemann map onto \( U \cap D \), then \( f \in VMOA(S^1) \).

**Corollary.** Let \( f \) be a meromorphic injection of \( D \). If \( \infty \in f(D) \), then the boundary value \( f \) belongs to \( VMO(S^1) \).
PROOF OF THEOREM: Since the injectivity is clear, the first assertion follows from the above Corollary.

Next suppose that $f_{n}$ converges to $f$ in $\Sigma_{1}$. Then by uniform convergence property of normalized quasiconformal maps, we can see that $f_{n}$ converges to $f$ uniformly on $\overline{D}$. In particular, $f_{n}$ converges to $f$ in $L^{\infty}(S^{1})$ and hence in $BMO(S^{1})$, which shows the second assertion, continuity of injection on $\Sigma_{1}$.

REMARK. Local character of functions in $VMO(S^{1})$ can be restated as Axler-Shapiro's theorem ([8]). On the other hand, when $C$ is the limit set of a $b$-group, every prime end of the invariant component is area 0 by Ahlfors-Thurston's $0 - 1$ theorem. Hence these facts give another proof of the above Theorem for this case.

PROBLEM. Is the above injection, say $E$, continuous on the whole $\Sigma$? If not, determine the corona, i.e. the set $E(\overline{\Sigma_{1}}) - E(\Sigma_{1})$.

Some further discussion on this problem will appear elsewhere.

Next, another representation can be obtained by considering the set

$$\tilde{S} = \{ f \mid f \text{ is univalent and holomorphic on } D \}$$

Again we write as $\tilde{S}_{1}$ the set corresponding to $T(1)$, namely, the set of Riemann maps which admits a quasiconformal extension. Then the 'VMO-ness at a point' can measure the local complexity at the point metrically. For instance, we have

PROPOSITION. Suppose that $f$ is a Riemann map onto a component $B$ of a Kleinian group $G$. If $\infty$ belongs to the boundary of $B$ and is fixed by an element of $G$ with infinite order, then $f$ does not belong to $BMO(S^{1})$.

PROOF: If $\infty$ is a parabolic fixed point, then the existence of a cusp neighborhood implies that $f \notin BMO(S^{1})$ by Pommerenke's characterization of Bloch functions
If $\infty$ is a loxodromic fixed point, then from self-similarity (invariance) of the limit set, we can conclude the assertion again by Pommerenke's characterization.

Outside of the fixed points set, the limit set of $G$ may have high complexity, at least, in the finitely generated case. Hence the Riemann map $f$ may also behave very wildly. So we may put the following

PROBLEM. If $G$ is a finitely generated Kleinian group with a component $f(D)$ and $\infty$ is not fixed by any non-trivial element of $G$, does $f$ belong to $VMO(S^{1})$?

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REFERENCES

1. R. R. Coifman, R. Rochberg and D. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math. 103 (1976), 611-635.
3. V. V. Peller, Hankel operators of class $\mathcal{S}_p$ and their applications, Math. USSR Sbornik 41 (1982), 443-479.