Quantized calculus and Teichmüller space

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1. quantized calculus.

The famous duality theorem by Fefferman states that the dual space of $ReH^1(S^1)$ is $BMO(S^1)$. On the other hand, $H^1$ can be represented as a product of two elements in $H^2(S^1)$,

$$h \in ReH^1 \iff h = g_1Hg_2 + (Hg_1)g_2, \quad g_j \in L^2(S^1)$$

Here, $H$ is the Hilbert transformation. Further Fefferman showed that

$$|\int fh d\theta| \leq C \|f\|_{BMO} \|g_1\|_2 \|g_2\|_2$$

Hence for every function $f$ on $S^1$, set

$$[H, f](g) = H(fg) - fH(g) \quad g \in L^2(S^1),$$

and we have

$$\int fh d\theta = \int [H, f](g_1)g_2 d\theta.$$

**Theorem ([1]).** The operator $[H, f]$ is bounded if and only if $f \in BMO(S^1)$.

Following A. Connes, this operator is called the quantized derivative $d^Q(f)$ of $f$. In fact, considering on the real line, we have

$$[H, f](g) = Const. \int_{\mathbb{R}} \frac{f(x) - f(y)}{x - y} g(y) dy$$

and hence can considered as the "polarization" of usual differentiation.

Moreover we know
**Theorem ([1]).** The operator $[H, f]$ is compact if and only if $f \in VMO(S^1)$.

Here $VMO(S^1)$ is the closure of $C(S^1)$ in $BMO(S^1)$. In particular, if $f \in C(S^1)$, then $[H, f]$ is compact. Recall that the dual space of $VMOA(S^1)$ is $H^1(S^1)$ and that an element of $VMO(S^1)$ is not necessarily continuous but only 'quasicontinuous'. More precisely,

$$L^\infty \cap VMO = QC \left(= (H^\infty + C) \cap (H^\infty + C) = C + HC\right)$$

**Remark ([2]).** If $f \in L^\infty$ and $|f| \in C(S^1)$, then $f \in QC$.

Now, "smoother" $f(S^1)$ in fractal sense, better $f$ as a compact operator.

**Theorem ([3]).** The operator $[H, f]$ belongs to the Schatten class $\mathcal{L}^p$ if and only if $f \in B_p^{1/p}(S^1)$, where $B_p^{1/p}(S^1)$ is the Besov space as below.

Here $f \in \mathcal{L}^p$ means that the sequence of eigenvalues of $|T| = (T^*T)^{1/2}$ belongs to $\ell^p$. (In particular, $\mathcal{L}^2$ is the Hilbert-Schmidt class.)

Next $f \in B_p^{1/p}(S^1)$ means that $f$ satisfies the inequality

$$\iint_{S^1 \times S^1} |f(x + t) - 2f(x) + f(x - t)|^p t^{-2} dx dt < +\infty.$$ 

Recall that, if $p > 1$, this inequality is equivalent to

$$\iint_{S^1 \times S^1} |f(x + t) - f(x)|^p t^{-2} dx dt < +\infty.$$ 

On the other hand, considering the harmonic extension on $D$, $f \in B_p^{1/p}(S^1)$ if and only if

$$\int_D \|D^2 f\| |p(1 - |z|)^{2p-2}|dz \wedge d\overline{z}| < +\infty.$$ 

(When $p > 1$, this is equivalent that $f$ is $p$-integrable 1-form, namely
\[ \int_D \|Df\|^p (1 - |z|)^{p-2} |dz \wedge d\overline{z}| < +\infty. \]

COROLLARY. \( B_{2}^{1/2}(S^{1}) \) is the Sobolev space (the harmonic Dirichlet space) \( HD(D) = W_{1}^{2}(D) \cap H(D) \), where \( D \) is the unit disk.

Boundary values form \( H^{1/2} = \{ (a_n) | \sum |n||a_n|^2 < +\infty \} \) (which S. Nag used).

2. On Hausdorff dimension of quasicircles.

A Riemann map \( f \) onto a K-quasi disk has a \((1/K)\)-Hölder continuous boundary value. Hence, for instance (, also see Astata, to appear), we have

**Proposition (cf. Falconer).** The Hausdorff dimension of a K-quasicircle is at most \( 2 - \frac{1}{K} \).

On the other hand,

**Theorem (Sullivan).** Assume that there is a cocompact quasiFuchsian group \( \Gamma \) whose limit set is a quasicircle \( C \) as the limit set. Then The Hausdorff dimension of \( C \) is \( p \) if and only if a Riemman map \( f \) onto the interior of \( C \) belongs to \( B_{q}^{1/q}(S^{1}) \) for every \( q > p \).

**Corollary ([4]).** A quasicircle \( C \) as in Theorem 4 has Hausdorff dimension \( p \) if and only if

\[ p = \inf\{ q \mid [H, f] \in L^q \}. \]

**Problem.** Characterize such quasicircles that corresponds to finitely generated Kleinian groups.

3. Teichmüller spaces.

Here we will give new representation of the Universal Teichmüller sapce. First we recall (cf. Astala-Gehring, '86) the folllowing
THEOREM (KOEBe). \{\log f' \mid f \text{ is univalent on } D\} \text{ is bounded in the Bloch space}

\[ \mathfrak{B} = \{f \mid \sup(1 - |z|^2)|f'(z)| < +\infty\}. \]

On the other hand, the boundary value of \(\log f'\), where \(f \in T(1)\), does not necessarily belong to \(BMO(S^1)\). (cf. Astata-Zinsmeister, '91)

Now, if \(f\) is a Riemann map onto a quasidisk, \(f\) itself has a continuous boundary value. Hence we can consider to represent Riemann maps in the above spaces.

First we set

\[ \Sigma = \{f \mid f \text{ is univalent on } D \text{ and has a form } \frac{1}{z} + \sum_{n=1}^{\infty} c_n z^n \text{ near } z = \infty, \} \]

and equip \(\Sigma\) with the Bers topology. Then \(\Sigma\) has the subset \(\Sigma_1\) which we can identify with the universal Teichmüller space \(T(1)\).

THEOREM. \(\Sigma\) can be mapped injectively in \(VMO(S^1)\).

This injection is continuous at least on \(\Sigma_1\).

In general, \(BMO(S^1) \subset \mathfrak{B}\) and hence \(VMO(S^1) \subset \mathfrak{B}_0\), and it is known that, for \(g \in \mathfrak{B}_0\), \(g\) has a finite angular limit on a set of Hausdorff dimension 1 (Makarov '89).

Also recall that \(AD(D) \subset VMOA(D)\) (S.Yamashita '82. Further, see Aulaskari '88), and that \(f \in \Sigma\) has a finite angular limit almost everywhere as is seen by classical Plessner's theorem.

On the other hand, Pommerenke ([6]) showed that, under the locally uniformly boundedness assumption of average multiplicity,

\[ f \in BMOA(S^1) \text{ if and only if } f \in \mathfrak{B}, \quad f \in VMOA(S^1) \text{ if and only if } f \in \mathfrak{B}_0. \]

On the other hand, since multiplication by \(z\) is an invertible \(VMO\)-multiplier, we can identify \(\Sigma\) with \(z\Sigma \subset VMOA(S^1)\). In particular, we have the following
**Corollary.** $\Sigma$ can be mapped injectively in $\mathfrak{B}_0$.

This injection is continuous at least on $\Sigma_1$.

**Remark.** Recall that a Riemann map has a continuous boundary value if and only if the complement is locally connected. Hence the locally connectedness conjecture of the limit set (cf. Abikoff[7]) can be restated as follows;

The image of $\Sigma(G)$ is contained in $C(S^1)$ for a finitely generated Kleinian group $G$, where $\Sigma(G)$ corresponds to $T(G)$.

It seems interesting to characterize Riemann maps, or elements of $\Sigma$, belonging to $VMO(S^1) - C(S^1)$ geometrically.

Now to prove Theorem, we note the following fact, which follows at once from the equivalence of $VMO(S^1)$ and $\mathfrak{B}_0$, and from the geometrical characterization of Bloch functions by Pommerenke ([5]).

**Proposition.** Let $f$ be a holomorphic injection of $D$. If $f(D)$ is bounded, then the boundary value $f$ belongs to $VMO(S^1)$.

But this fact has an interesting

**Corollary.** Let $G$ be any Kleinian group which has $\infty$ as an ordinary point, and $f$ be a Riemann map onto a simply connected component of $G$. Then $f \in VMO(S^1)$.

Here we note that $VMO$-ness is a local property.

**Lemma (Gotoh).** Let $f$ be meromorphic on $D$ and has no poles near $\partial D$. If, for every $\zeta \in \partial D$, there is a neighborhood $U$ of $\zeta$ such that $f \circ \phi_\zeta \in VMOA(S^1)$, where $\phi_\zeta$ is a Riemann map onto $U \cap D$, then $f \in VMOA(S^1)$.

**Corollary.** Let $f$ be a meromorphic injection of $D$. If $\infty \in f(D)$, then the boundary value $f$ belongs to $VMO(S^1)$. 
Proof of Theorem: Since the injectivity is clear, the first assertion follows from the above Corollary.

Next suppose that \( f_n \) converges to \( f \) in \( \Sigma_1 \). Then by uniform convergence property of normalized quasiconformal maps, we can see that \( f_n \) converges to \( f \) uniformly on \( \overline{D} \). In particular, \( f_n \) converges to \( f \) in \( L^\infty(S^1) \) and hence in \( BMO(S^1) \), which shows the second assertion, continuity of injection on \( \Sigma_1 \).

Remark. Local character of functions in \( VMO(S^1) \) can be restated as Axler-Shapiro's theorem ([8]). On the other hand, when \( C \) is the limit set of a \( b \)-group, every prime end of the invariant component is area 0 by Ahlfors-Thurston's 0 – 1 theorem. Hence these facts give another proof of the above Theorem for this case.

Problem. Is the above injection, say \( E \), continuous on the whole \( \Sigma \)? If not, determine the corona, i.e. the set \( E(\Sigma_1) - E(\Sigma_1) \).

Some further discussion on this problem will appear elsewhere.

Next, another representation can be obtained by considering the set

\[ \tilde{S} = \{ f \mid f \text{ is univalent and holomorphic on } D \} \]

Again we write as \( \tilde{S}_1 \) the set corresponding to \( T(1) \), namely, the set of Riemann maps which admits a quasiconformal extension. Then the 'VMO-ness at a point' can measure the local complexity at the point metrically. For instance, we have

Proposition. Suppose that \( f \) is a Riemann map onto a component \( B \) of a Kleinian group \( G \). If \( \infty \) belongs to the boundary of \( B \) and is fixed by an element of \( G \) with infinite order, then \( f \) does not belongs to \( BMO(S^1) \).

Proof: If \( \infty \) is a parabolic fixed point, then the existence of a cusp neighborhood implies that \( f \not\in BMO(S^1) \) by Pommerenke's characterization of Bloch functions
If $\infty$ is a loxodromic fixed point, then from self-similarity (invariance) of the limit set, we can conclude the assertion again by Pommerenke's characterization.

Outside of the fixed points set, the limit set of $G$ may have high complexity, at least, in the finitely generated case. Hence the Riemann map $f$ may also behave very wildly. So we may put the following

**Problem.** *If $G$ is a finitely generated Kleinian group with a component $f(D)$ and $\infty$ is not fixed by any non-trivial element of $G$, does $f$ belong to $VMO(S^1)$?*

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**References**