Quantized calculus and Teichmüller space

MASAHIKO TANIGUCHI 谷口 雅彦（京大）
Department of Mathematics, Faculty of Science, Kyoto University

1. quantized calculus.

The famous duality theorem by Fefferman states that the dual space of \( \text{Re} H^1(S^1) \) is \( BMO(S^1) \). On the other hand, \( H^1 \) can be represented as a product of two elements in \( H^2(S^1) \),

\[
h \in \text{Re} H^1 \longleftrightarrow h = g_1 H g_2 + (H g_1) g_2, \quad g_j \in L^2(S^1)
\]

Here, \( H \) is the Hilbert transformation. Further Fefferman showed that

\[
|\int f hd\theta| \leq C \|f\|_{BMO} \|g_1\|_2 \|g_2\|_2
\]

Hence for every function \( f \) on \( S^1 \), set

\[
[H, f](g) = H(f g) - f H(g) \quad g \in L^2(S^1),
\]

and we have

\[
\int f hd\theta = \int [H, f](g_1) g_2 d\theta.
\]

**Theorem (1).** The operator \([H, f] \) is bounded if and only if \( f \in BMO(S^1) \).

Following A. Connes, this operator is called the quantized derivative \( d^Q(f) \) of \( f \). In fact, considering on the real line, we have

\[
[H, f](g) = \text{Const.} \int_{\mathbb{R}} \frac{f(x) - f(y)}{x - y} g(y) dy
\]

and hence can considered as the "polarization" of usual differentiation.

Moreover we know
Theorem ([1]). The operator $[H, f]$ is compact if and only if $f \in VMO(S^1)$.

Here $VMO(S^1)$ is the closure of $C(S^1)$ in $BMO(S^1)$. In particular, if $f \in C(S^1)$, then $[H, f]$ is compact. Recall that the dual space of $VMOA(S^1)$ is $H^1(S^1)$ and that an element of $VMO(S^1)$ is not necessarily continuous but only 'quasicontinuous'. More precisely,

$$L^\infty \cap VMO = QC \ (= (H^\infty + C) \cap (\overline{H^\infty} + C) = C + HC)$$

Remark ([2]). If $f \in L^\infty$ and $|f| \in C(S^1)$, then $f \in QC$.

Now, "smoother" $f(S^1)$ in fractal sense, better $f$ as a compact operator.

Theorem ([3]). The operator $[H, f]$ belongs to the Schatten class $\mathcal{L}^p$ if and only if $f \in B_p^{1/p}(S^1)$, where $B_p^{1/p}(S^1)$ is the Besov space as below.

Here $f \in \mathcal{L}^p$ means that the sequence of eigenvalues of $|T| = (T^*T)^{1/2}$ belongs to $\ell^p$. (In particular, $\mathcal{L}^2$ is the Hilbert-Schmidt class.)

Next $f \in B_p^{1/p}(S^1)$ means that $f$ satisfies the inequality

$$\iint_{S^1 \times S^1} |f(x + t) - 2f(x) + f(x - t)|^p t^{-2} dx dt < +\infty.$$ 

Recall that, if $p > 1$, this inequality is equivalent to

$$\iint_{S^1 \times S^1} |f(x + t) - f(x)|^p t^{-2} dx dt < +\infty.$$ 

On the other hand, considering the harmonic extension on $D$, $f \in B_p^{1/p}(S^1)$ if and only if

$$\int_D \|D^2 f\|^p (1 - |z|)^{2p-2} |dz \wedge d\overline{z}| < +\infty.$$ 

(When $p > 1$, this is equivalent that $f$ is $p$-integrable 1-form, namely
\[
\int_D \|Df\|^p (1-|z|)^{p-2} |dz \wedge d\overline{z}| < +\infty.
\]

**Corollary.** \(B_{2}^{1/2}(S^{1})\) is the Sobolev space (the harmonic Dirichlet space) \(HD(D)\) 
\(= W_{1}^{2}(D) \cap H(D)\), where \(D\) is the unit disk.

Boundary values form \(H^{1/2} = \{(a_{n}) | \sum |n||a_{n}|^2 < +\infty \}\) (which S. Nag used).

**2. On Hausdorff dimension of quasicircles.**

A Riemann map \(f\) onto a K-quasi disk has a \((1/K)\)-Hölder continuous boundary value. Hence, for instance \((,\) also see Astata, to appear\), we have

**Proposition (cf. Falconer).** The Hausdorff dimension of a K-quasicircle is at most \(2 - \frac{1}{K}\).

On the other hand,

**Theorem (Sullivan).** Assume that there is a cocompact quasiFuchsian group \(\Gamma\) whose limit set is a quasicircle \(C\) as the limit set. Then The Hausdorff dimension of \(C\) is \(p\) if and only if a Riemman map \(f\) onto the interior of \(C\) belongs to \(B_{q}^{1/q}(S^{1})\) for every \(q > p\).

**Corollary ([4]).** A quasicircle \(C\) as in Theorem 4 has Hausdorff dimension \(p\) if and only if

\[p = \inf\{q \mid [H, f] \in \mathcal{L}^q\}.
\]

**Problem.** Characterize such quasicircles that corresponds to finitely generated Kleinian groups.

**3. Teichmüller spaces.**

Here we will give new representation of the Universal Teichmüller sapce. First we recall (cf. Astala-Gehring, '86) the folllowing
THEOREM (KOEBE). \{\log f' \mid f \text{ is univalent on } D\} \text{ is bounded in the Bloch space}

\[ \mathfrak{B} = \{f \mid \sup(1 - |z|^2)|f'(z)| < +\infty\}. \]

On the other hand, the boundary value of \(\log f'\), where \(f \in T(1)\), does not necessarily belong to \(BMO(S^1)\). (cf. Astata-Zinsmeister, '91)

Now, if \(f\) is a Riemann map onto a quasidisk, \(f\) itself has a continuous boundary value. Hence we can consider to represent Riemann maps in the above spaces.

First we set

\[ \Sigma = \{f \mid f \text{ is univalent on } D \text{ and has a form } \frac{1}{z} + \sum_{n=1}^{\infty} c_n z^n \text{ near } z = \infty, \} \]

and equip \(\Sigma\) with the Bers topology. Then \(\Sigma\) has the subset \(\Sigma_1\) which we can identify with the universal Teichmüller space \(T(1)\).

**THEOREM.** \(\Sigma\) can be mapped injectively in \(VMO(S^1)\).

This injection is continuous at least on \(\Sigma_1\).

In general, \(BMO(S^1) \subset \mathfrak{B}\) and hence \(VMO(S^1) \subset \mathfrak{B}_0\), and it is known that, for \(g \in \mathfrak{B}_0\), \(g\) has a finite angular limit on a set of Hausdorff dimension 1 (Makarov '89). Also recall that \(AD(D) \subset VMOA(D)\) (S.Yamashita '82. Further, see Aulaskari '88), and that \(f \in \Sigma\) has a finite angular limit almost everywhere as is seen by classical Plessner's theorem.

On the other hand, Pommerenke ([6]) showed that, under the locally uniformly boundedness assumption of average multiplicity,

\[ f \in BMOA(S^1) \text{ if and only if } f \in \mathfrak{B}, \quad f \in VMOA(S^1) \text{ if and only if } f \in \mathfrak{B}_0. \]

On the other hand, since multiplication by \(z\) is an invertible \(VMO\)-multiplier, we can identify \(\Sigma\) with \(z\Sigma \subset VMOA(S^1)\). In particular, we have the following
Corollary. $\Sigma$ can be mapped injectively in $\mathfrak{B}_0$.

This injection is continuous at least on $\Sigma_1$.

Remark. Recall that a Riemann map has a continuous boundary value if and only if the complement is locally connected. Hence the locally connectedness conjecture of the limit set (cf. Abikoff[7]) can be restated as follows;

The image of $\Sigma(G)$ is contained in $C(S^1)$ for a finitely generated Kleinian group $G$, where $\Sigma(G)$ corresponds to $T(G)$?

It seems interesting to characterize Riemann maps, or elements of $\Sigma$, belonging to $VMO(S^1) - C(S^1)$ geometrically.

Now to prove Theorem, we note the following fact, which follows at once from the equivalence of $VMO(S^1)$ and $\mathfrak{B}_0$, and from the geometrical characterization of Bloch functions by Pommerenke ([5]).

Proposition. Let $f$ be a holomorphic injection of $D$. If $f(D)$ is bounded, then the boundary value $f$ belongs to $VMO(S^1)$.

But this fact has an interesting

Corollary. Let $G$ be any Kleinian group which has $\infty$ as an ordinary point, and $f$ be a Riemann map onto a simply connected component of $G$. Then $f \in VMO(S^1)$.

Here we note that $VMO$-ness is a local property.

Lemma (Gotoh). Let $f$ be meromorphic on $D$ and has no poles near $\partial D$. If, for every $\zeta \in \partial D$, there is a neighborhood $U$ of $\zeta$ such that $f \circ \phi_\zeta \in VMOA(S^1)$, where $\phi_\zeta$ is a Riemann map onto $U \cap D$, then $f \in VMOA(S^1)$.

Corollary. Let $f$ be a meromorphic injection of $D$. If $\infty \in f(D)$, then the boundary value $f$ belongs to $VMO(S^1)$. 
PROOF OF THEOREM: Since the injectivity is clear, the first assertion follows from the above Corollary.

Next suppose that $f_n$ converges to $f$ in $\Sigma_1$. Then by uniform convergence property of normalized quasiconformal maps, we can see that $f_n$ converges to $f$ uniformly on $\overline{D}$. In particular, $f_n$ converges to $f$ in $L^\infty(S^1)$ and hence in $BMO(S^1)$, which shows the second assertion, continuity of injection on $\Sigma_1$.

REMARK. Local character of functions in $VMO(S^1)$ can be restated as Axler-Shapiro’s theorem ([8]). On the other hand, when $C$ is the limit set of a $b$-group, every prime end of the invariant component is area 0 by Ahlfors-Thurston’s 0 – 1 theorem. Hence these facts give another proof of the above Theorem for this case.

PROBLEM. Is the above injection, say $E$, continuous on the whole $\Sigma$? If not, determine the corona, i.e. the set $\overline{E(\Sigma_1)} - E(\Sigma_1)$.

Some further discussion on this problem will appear elsewhere.

Next, another representation can be obtained by considering the set

$$\tilde{S} = \{ f \mid f \text{ is univalent and holomorphic on } D \}$$

Again we write as $\tilde{S}_1$ the set corresponding to $T(1)$, namely, the set of Riemann maps which admits a quasiconformal extension. Then the 'VMO-ness at a point' can measure the local complexity at the point metrically. For instance, we have

PROPOSITION. Suppose that $f$ is a Riemann map onto a component $B$ of a Kleinian group $G$. If $\infty$ belongs to the boundary of $B$ and is fixed by an element of $G$ with infinite order, then $f$ does not belongs to $BMO(S^1)$.

PROOF: If $\infty$ is a parabolic fixed point, then the existence of a cusp neighborhood implies that $f \not\in BMO(S^1)$ by Pommerenke’s characterization of Bloch functions.
If $\infty$ is a loxodromic fixed point, then from self-similarity (invariance) of the limit set, we can conclude the assertion again by Pommerenke's characterization.

Outside of the fixed points set, the limit set of $G$ may have high complexity, at least, in the finitely generated case. Hence the Riemann map $f$ may also behave very wildly. So we may put the following

**Problem.** If $G$ is a finitely generated Kleinian group with a component $f(D)$ and $\infty$ is not fixed by any non-trivial element of $G$, does $f$ belong to $VMO(S^1)$?

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**References**