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<td>TANABE, MASAHARU</td>
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Kyoto University
A REMARK ON THEOREMS OF DE FRANCHIS AND SEVERI

MASAHARU TANABE
田辺 正晴（東工大）

1. INTRODUCTION

The purpose of this paper is to study holomorphic maps between compact Riemann surfaces. There are two famous finiteness theorems related to this problem. Let \( \tilde{X} \) be a compact Riemann surface of genus \( \gg 1 \). Then one is that, for fixed compact Riemann surface \( X \) of genus \( > 1 \), the number of nonconstant holomorphic maps \( \tilde{X} \rightarrow X \) is finite, and another is that there are only finitely many compact Riemann surfaces \( \{X_i\} \) of genus \( > 1 \) such that, for each \( X_i \), there exists a nonconstant holomorphic map \( \tilde{X} \rightarrow X_i \). The first assertion is due to de Franchis, and second one is due to Severi.

Let \( S(\tilde{X}) = \{X_i\} \), where \( \{X_i\} \) is as in Severi’s theorem. Let
\[
\sum_{X \in S(\tilde{X})} \# \{h : \tilde{X} \rightarrow X\text{ nonconstant holomorphic}\}.
\]

Then, by the theorems above, we see \( n < \infty \) at once. Howard and Sommese [2] showed that there is a bound on \( n \) which depends only on the genus of \( \tilde{X} \), by giving an explicit estimate.

Here we will give some theorems related to rigidity of holomorphic maps between compact Riemann surfaces, and show that we may take an explicit bound on \( n \) depending only on the genus of \( \tilde{X} \) smaller than one in [2].

2. PRELIMINARIES

Let \( \tilde{X}, X \) be compact Riemann surfaces of genera \( \tilde{g}, g(> 1) \). We denote by \( H_1(X) \) the first homology group (with integer coefficients) of \( X \). Any basis of \( H_1(X) \) with intersection matrix
\[
J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}
\]
will be called a canonical homology basis, where \( E \) is the identity matrix of \( g \times g \) sized. Similarly for \( \tilde{X} \). Let \( \{\tilde{\chi}_1, \ldots, \tilde{\chi}_{2\tilde{g}}\} \) \((\{\chi_1, \ldots, \chi_{2g}\})\) be a canonical homology basis for \( H_1(\tilde{X}) \) \((H_1(X)\) respectively). Let \( \{\tilde{w}_1, \ldots, \tilde{w}_{\tilde{g}}\} \) and \( \{w_1, \ldots, w_g\} \) be dual bases for holomorphic differentials on \( \tilde{X}, X \) (i.e. \( \int_{X_j} w^k = \delta_{jk} \) where \( \delta_{jk} \) is Kronecker’s delta), and \( \tilde{\Pi} = (\tilde{E}, \tilde{Z}), \Pi = (E, Z) \) be the associated period matrices. Let \( h : \tilde{X} \rightarrow X \) be a nonconstant holomorphic map. Then \( h \) induces a homomorphism \( h_* : H_1(\tilde{X}) \rightarrow H_1(X) \). Let \( M = (m_{kj}) \in M(2g, 2\tilde{g}; \mathbb{Z}) \), where \( h_*(\tilde{\chi}_j) = \sum_{k=1}^{2g} m_{kj} \chi_k \) (We denote by \( M(m, n; K) \) the set of \( m \times n \) matrices with \( K \)-coefficients.) We will call \( M \) the matrix representation of \( h \) with respect to \( \{\tilde{\chi}_1, \ldots, \tilde{\chi}_{2\tilde{g}}\} \) and \( \{\chi_1, \ldots, \chi_{2g}\} \). The integral \( \int_{h_*(\tilde{\chi}_j)} w^i \) may be evaluated.
in two ways; by expressing \( h_\ast (\tilde{\chi}_j) \) in \( H_1(X) \) or by expressing the pull back of \( w^i \) in terms of the holomorphic differentials on \( \tilde{X} \). This leads us to the Hurwitz relation

\[
A\Pi = \tilde{\Pi}M,
\]

where \( A \in M(g, \tilde{g}; \mathbb{C}) \). The set of \( M \in M(2g, 2\tilde{g}; \mathbb{Q}) \) such that there exists \( A \in M(g, \tilde{g}; \mathbb{C}) \) with \( A\Pi = \tilde{\Pi}M \) will be called the space of Hurwitz relations. It is easy to see that it is a \( \mathbb{Q} \)-vector space.

**Lemma[4].** In the space of Hurwitz relations, \( \langle M, N \rangle = \text{tr}(\tilde{J}^tMJ^{-1}N) \) defines an inner product (\( 'M \) denotes transposition of \( M \)).

In particular, when \( M \) is a matrix representation of a holomorphic map \( h : \tilde{X} \to X \), \( \langle M, M \rangle = 2dg \), where \( d \) is the degree of the holomorphic map \( h \).

The Jacobian variety of \( X \) is \( J(X) = \mathbb{C}^g / \Gamma \), where \( \Gamma \) is the lattice (over \( \mathbb{Z} \)) generated by 2g-columns of \( \Pi \). Similarly for \( J(\tilde{X}) \).

For any holomorphic map \( h : \tilde{X} \to X \), there exists a homomorphism \( H : J(\tilde{X}) \to J(X) \) with \( \kappa \circ h = H \circ \tilde{\kappa} \), where \( \tilde{\kappa}, \kappa \) are canonical injections.

By an underlying real structure for \( J(X) \), we mean the real torus \( \mathbb{R}^{2g}/\mathbb{Z}^{2g} \) together with a map \( \mathbb{R}^{2g}/\mathbb{Z}^{2g} \to J(X) \) induced by a linear map \( \mathbb{R}^{2g} \ni x \mapsto \Pi x \in \mathbb{C}^g \). It is well-known that for any homomorphism \( H : J(\tilde{X}) \to J(X) \), there are \( A \in M(g, \tilde{g}; \mathbb{C}) \) and \( M \in M(2g, 2\tilde{g}; \mathbb{Z}) \) such that the following diagram is commutative (precisely, apart from an additive constant).

\[
\begin{array}{ccc}
\mathbb{R}^{2\tilde{g}} & \overset{\Pi}{\longrightarrow} & \mathbb{C}^g \\
\downarrow M & & \downarrow A \\
\mathbb{R}^{2g} & \overset{\Pi}{\longrightarrow} & \mathbb{C}^g \\
\end{array}
\]

\[
\longrightarrow J(\tilde{X}) \quad \longrightarrow J(X)
\]

In particular, if \( h \) is induced by a holomorphic map \( h : \tilde{X} \to X \), \( M \) is the matrix representation of \( h \).

Giving a nonconstant holomorphic map \( h : \tilde{X} \to X \), we denote by \( h^\ast (Q) \subset \tilde{X} \) a divisor of the preimages of \( Q \in X \) with multiplicities. Defining \( \tilde{\kappa}(h^\ast (Q)) \) by linearity, we get a holomorphic map \( X \to J(\tilde{X}) \), which can be extended to a homomorphism \( H^\ast : J(X) \to J(\tilde{X}) \). \( H^\ast \) is called the Rosati adjoint of \( H \). \( H^\ast \) is induced by the matrix \( M^\ast = \tilde{J}^tMJ^{-1} \) acting on the underlying real tori[4].

### 3. Statements

**Theorem 1.** Let \( \tilde{X}, X \) be compact Riemann surfaces of genera \( \tilde{g}, g(> 1) \). Let \( h_i : \tilde{X} \to X \) be a nonconstant holomorphic map, and \( M_i \in M(2g, 2\tilde{g}; \mathbb{Z}) \) be a matrix representation of \( h_i \) (\( i = 1, 2 \)). Suppose that there is an integer \( l > \sqrt{8(\tilde{g} - 1)} \) with \( M_1 \equiv M_2 \) (mod. \( l \)). Then \( h_1 = h_2 \).

Let \( m_i^j \) denote the j-th row vector of \( M_i \) (\( i = 1, 2 \)).

**Theorem 2.** Let \( h_1, h_2 \), and \( M_1, M_2 \) be as in Theorem 1. Suppose that there is an integer \( l > 8(\tilde{g} - 1) \) with \( m_1^j \equiv m_2^j \) (mod. \( l \)) for every \( j \in \{1, \ldots, g\} \). Then \( h_1 = h_2 \).

It is already known that \( M_1 = M_2 \) implies \( h_1 = h_2 \) (see [3]).
Theorem 3. Let $X_1, X_2$ be compact Riemann surfaces of genus $g > 1$. Let $h_i : \tilde{X} \to X_i$ be a nonconstant holomorphic map, and $M_i$ be a matrix representation of $h_i (i = 1, 2)$. Suppose that there is an integer $l > \sqrt{8}(\tilde{g} - 1)$ with $M_1 \equiv M_2 \pmod{l}$. Then $X_1, X_2$ are conformally equivalent and there exists a conformal map $f : X_1 \to X_2$ with $f \circ h_1 = h_2$.

Only outlines of the proofs are given here. For complete proofs, see [5] which will be published elsewhere.

As we have seen in the lemma before, we have an inner product in the space of Hurwitz relations. Therefore, we may induce a distance in it. Using this distance, we have Theorem 1 and 2. To get Theorem 3, we use the Rosati adjoint. Let $G_i = M_i^* M_i = \tilde{J}^t M_i J^{-1} M_i (i = 1, 2)$. Then we have endomorphisms of $J(\tilde{X})$ with the matrices $G_1, G_2$ acting on the underlying real tori. If $G_1 = G_2$, then the targets $X_1, X_2$ are conformally equivalent. Using the distance induced by the inner product, we have Theorem 3.

Next we will give an bound on $n$ which was defined in section 1. Let

$$S_g = \{X \in S(\tilde{X})|\text{genus } g\},$$

and

$$Hol_g(\tilde{X}) = \bigcup_{X \in S_g} \{h : \tilde{X} \to X|\text{nonconstant holomorphic}\}.$$ 

Let $F_l = \mathbb{Z}/(l)$, where $l$ is a prime number $> \sqrt{8}(\tilde{g} - 1)$. By Theorem 1 and 3, we have an injection $Hol_g(\tilde{X}) \to M(2g, 2\tilde{g}; F_l)$. Thus we consider each matrix representation in $M(2g, 2\tilde{g}; F_l)$, for the convenience of calculation. Let $h_i$ be an element of $Hol_g(\tilde{X})$ and $M_i \in M(2g, 2\tilde{g}; F_l)$ a matrix representation of $h_i (i = 1, 2)$. If there exists $S \in Sp(2g; F_l)$ with $M_2 = SM_1$, then targets of $h_1, h_2$, say $X_1, X_2$, are conformally equivalent and there is a conformal map $f : X_1 \to X_2$ with $f \circ h_1 = h_2$ ($Sp$ denotes symplectic groups). $M_i$ satisfies $M_i \tilde{J}^t M_i = d_i J$, where $d_i$ is the degree of $h_i$. Therefore, we have

$$\#Hol_g(\tilde{X}) \leq \sum_d \#\{M \in M(2g, 2\tilde{g}; F_l)|MJ^t M = dJ\} \times 84(g - 1)/\#Sp(2g; F_l),$$

where $d$ runs through all considerable numbers as degrees of holomorphic maps. We have

$$\#Sp(2g; F_l) = l^{2g}(l^2 - 1)(l^4 - 1) \ldots (l^{2g} - 1)$$

(see [1]), and we may take $l$ with $\sqrt{8}(\tilde{g} - 1) < l < 2\sqrt{8}(\tilde{g} - 1)$. Consequently,

$$n \leq 42(\tilde{g} - 1)(\tilde{g} - 2)2^{2\tilde{g}}(4\sqrt{2}(\tilde{g} - 1))^{\tilde{g}^2 + \tilde{g}/2} + 84(\tilde{g} - 1).$$

Howard and Sommese [2] showed that

$$n \leq (2\sqrt{6}(\tilde{g} - 1) + 1)^{2\tilde{g}^2 + 2\tilde{g}}(\tilde{g} - 1)(\sqrt{2})^{\tilde{g}(\tilde{g} - 1)} + 84(\tilde{g} - 1).$$

It is easy to see that our bound is smaller for every $\tilde{g} > 1$. 


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DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY,
OHOKAYAMA, MEGURO, TOKYO,
152, JAPAN