数理解析研究所講究録
第 882 巻 1994 年 89-103

Jørgensen's inequality for classical Schottky groups of real type

Hiroki Sato

佐藤　宏樹　（静岡大学理学部）

Department of Mathematics, Faculty of Science
Shizuoka University

Abstract. In this paper we consider Jørgensen's inequality for Schottky groups of real type. Main theorems will be stated in §4. We will give examples in §9, each of which shows the lower bound in the Jørgensen inequality is best possible. Proofs of lemmas, propositions and theorems in this paper will be appeared in [8]

1. Notation and terminology.

Let $C_1, C_{g+1}; \cdots; C_g, C_{2g}$ be a set of $2g, g \geq 1$, mutually disjoint Jordan curves on the Riemann sphere which comprise the boundary of a 2g-ply connected region ω. Suppose there are g Möbius transformations $A_1, \cdots, A_g$ which have the property that A_j maps C_j onto C_{g+j} and $A_j(\omega) \cap \omega = \emptyset$, $1 \leq j \leq g$. Then the g necessarily loxodromic transformations A_g generate a *marked Schottky group* $G = < A_1, \cdots, A_g >$ of genus g with ω as a fundamental region. In particular, if all $C_j (j = 1, 2, \cdots, 2g)$ are circles, then we call $A_1, \cdots, A_g$ a *set of classical generators of* G. A *classical Schottky group* is a Schottky group for which there exists some set of classical generator.

Let Möb be the group of all Möbius transformations. We say two marked subgroups $G = < A_1, \cdots, A_g >$ and $G = < \hat{A}_1, \cdots, \hat{A}_g >$ of Möb to be *equivalent* if there exists a Möbius transformation T such that $\hat{A}_j = TA_jT^{-1}$ for $j = 1, 2$. The *Schottky space* (resp. *classical Schottky space*) of genus g, denoted by $\mathfrak{S}_g$ (resp. $\mathfrak{S}^0_g$), is the set of all equivalence classes of marked Schottky groups (resp. marked classical Schottky groups) of genus $g \geq 1$.

We denote by $\mathfrak{M}_2$ the set of all equivalence classes $[< A_1, A_2 >]$ of marked groups $< A_1, A_2 >$ generated by loxodromic transformations A_1

and A_2 whose fixed points are all distinct. Let $[<A_1, A_2>] \in \mathfrak{M}_2$. For $j = 1, 2$, let $\lambda_j(|\lambda_j| > 1), p_j$ and p_{2+j} be the multipliers, the repelling and the attracting fixed points of A_j, respectively. We define t_j by setting $t_j = 1/\lambda_j$. Thus $t_j \in D^* = \{z \mid 0 < |z| < 1\}$. We determine a Möbius transformation T by $T(p_1) = 0, T(p_3) = \infty$ and $T(p_2) = 1$, and define ρ by $\rho = T(p_4)$. Thus $\rho \in \mathbf{C} - \{0, 1\}$. We can define a mapping α of the space into $(D^*)^2 \times (\mathbf{C} - \{0, 1\})$ by setting $\alpha([<A_1, A_2>]) = (t_1, t_2, \rho)$. Then we say $[<A_1, A_2>]$ represents (t_1, t_2, ρ) and (t_1, t_2, ρ) corresponds to $[<A_1, A_2>]$ or $<A_1, A_2>$. Conversely, λ_1, λ_2 and p_4 are uniquely determined from a given point $\tau = (t_1, t_2, \rho) \in (D^*)^2 \times (\mathbf{C} - \{0, 1\})$ under the normalization condition $p_1 = 0, p_3 = \infty$ and $p_2 = 1$; we define $\lambda_j (j = 1, 2)$ and p_4 by setting $\lambda_j = 1/t_j$ and $p_4 = \rho$, respectively. We determine $A_1(z), A_2(z) \in$ Möb from τ as follows : The multiplier, the repelling and the attracting fixed points of $A_j(z)$ are λ_j , p_j and p_{2+j}, respectively. Thus we obtain a mapping β of $(D^*)^2 \times (\mathbf{C} - \{0, 1\})$ into $_2$ by setting $\beta(\tau) = [<A_1(z), A_2(z)>]$. Then we note that $\beta\alpha = \alpha\beta = id$. Therefore we identify $\mathfrak{M}_2$ with $\alpha(\mathfrak{M}_2)$. Similarly we can define the mapping α^* of $\mathfrak{S}_2$ or $\mathfrak{S}_2^0$ into $(D^*) \times (\mathbf{C} - \{0, 1\})$ by restricting α to this space, and identify $\mathfrak{S}_2$ (resp. $\mathfrak{S}_2^0$) with $\alpha^*(\mathfrak{S}_2)$ (resp. $\alpha^*(\mathfrak{S}_2^0)$). From now on we denote $\alpha(\mathfrak{M}_2), \alpha^*(\mathfrak{S}_2)$ and $\alpha(\mathfrak{S}_2^0)$ by $\mathfrak{M}_2, \mathfrak{S}_2$ and $\mathfrak{S}_2^0$, respectively.

A Möbius transformation $A(z) = (az + b)/(cz + d)$ is called a *real Möbius transformation* if $a, b, c, d \in R$ and $ad - bc \neq 0$. If $A_j(j = 1, 2, \cdots, g)$ are all real Möbius transformastions, then we call $G = <A_1, \cdots, A_g>$ a *marked group of real type*. In the case of $g = 2$, there are eight kinds of marked groups of real type as follows. Let (t_1, t_2, ρ) be the point in $\mathfrak{M}_2$, corresponding to $[G] = [<A_1, A_2>]$.

DEFINITION 1.1 (cf.[4])

(1) G is of the first type (Type I) if $t_1 > 0, t_2 > 0, \rho > 0$.
(2) G is of the second type (Type II) if $t_1 > 0, t_2 < 0, \rho > 0$.
(3) G is of the third type (Type III) if $t_1 > 0, t_2 < 0, \rho < 0$.
(4) G is of the fourth type (Type IV) if $t_1 > 0, t_2 > 0, \rho < 0$.
(5) G is of the fifth type (Type V) if $t_1 < 0, t_2 > 0, \rho > 0$.
(6) G is of the sixth type (TypeVI) if $t_1 < 0, t_2 < 0, \rho > 0$.
(7) G is of the seventh type (Type VII) if $t_1 < 0, t_2 < 0, \rho < 0$.
(8) G is of the eighth type (type VIII) if $t_1 < 0, t_2 > 0, \rho < 0$.

For each $k = I, II, \cdots, VIII$, we call the set of all equivalence classes of marked groups (resp. marked Schottky groups and marked classical Schottky groups) of Type k the real space (resp.the real Schottky space and the real classical Schottky space) of Type k, and denote them by $R_k\mathfrak{M}_2$ (resp. $R_k\mathfrak{S}_2$ and $R_k\mathfrak{S}_2^0$).

2. The Nielsen transformations.

THEOREM A (Neumann [3]). *The group Φ_2 of automorphisms of $G = <A_1, A_2>$ has the following presentation:*

$$\Phi_2 = < N_1, N_2, N_3 \mid (N_2N_1N_2N_3)^2 = 1,$$
$$N_3^{-1}N_2N_3N_2N_1N_3N_1N_2N_1 = 1, N_1N_3N_1N_3 = N_3N_1N_3N_1 >,$$

where $N_1 : (A_1, A_2) \mapsto (A_1, A_2^{-1})$, $N_2 : (A_1, A_2) \mapsto (A_2, A_1)$ and $N_3 : (A_1, A_2) \mapsto (A_1, A_1A_2)$.

We call the mapping N_1, N_2 and N_3 the Nielsen transformations. In the following propositions X denotes the spaces $\mathfrak{M}_2, \mathfrak{S}_2$ or $\mathfrak{S}_2^0$.

PROPOSITION 2.1. $N_1(R_kX) = R_kX$ *for each* $k = I, II, \cdots, VIII$.

PROPOSITION 2.2.
(i) $N_2(R_kX) = R_kX$ *for* $k = I, IV, VI, VII$.
(ii) $N_2(R_{II}X) = R_VX$ *and* $N_2(R_VX) = R_{II}X$.
(iii) $N_2(R_{III}X) = R_{VIII}X$ *and* $N_3(R_{VIII}X) = R_{III}X$.

PROPOSITION 2.3.
(i) $N_3(R_kX) = R_kX$ *for* $k = I, II, III, IV$.
(ii) $N_3(R_VX) = R_{VII}X$ *and* $N_3(R_{VII}X) = R_VX$.
(iii) $N_3(R_{VI}X) = R_{VIII}X$ *and* $N_3(R_{VIII}X) = R_{VI}X$.

3. Fundamental regions.

The Schottky modular group of genus two, which is denoted by $\mathrm{Mod}(\mathfrak{S}_2)$, is the set of all equivalence classes of orientation preserving automorphisms of $\mathfrak{S}_2$. We denote by $\mathrm{Mod}(R_k\mathfrak{S}_2^0)$ the restriction of $\mathrm{Mod}(\mathfrak{S}_2)$ to $R_k\mathfrak{S}_2^0$ for $k = I, II, \cdots, VIII$. We denote by $F_k(\mathrm{Mod}(\mathfrak{S}_2^0))$ fundamental regions for $\mathrm{Mod}(R_k\mathfrak{S}_2^0)$ in R^3.

PROPOSITION 3.1 (Sato[4]).

$$F_I(\mathrm{Mod}(\mathfrak{S}_2^0)) = \{(t_1, t_2, \rho) \in R_I\mathfrak{S}_2^0 \mid \rho(t_1, t_2)^{-1} < \rho$$
$$< \rho(t_1, t_2), \rho \neq 1, 0 < t_2 < 1, 0 < t_1 < 1\},$$

where $\rho(t_1,t_2)=(1+\sqrt{t_1}t_2)/(\sqrt{t_1}+t_2)$.

PROPOSITION 3.2 (Sato[5]).

$$F_{II}(\mathrm{Mod}(\mathfrak{G}_2^0))=\{(t_1,t_2,\rho)\in R_{II}\mathfrak{G}_2^0 \mid (1+\sqrt{t_1}t_2)/(\sqrt{t_1}+t_2) < \rho < ((1-\sqrt{t_1}t_2)/(\sqrt{t_1}-t_2))^2, -1<t_2<0, 0<t_1<1\}.$$

PROPOSITION 3.3 (Sato[7]).

$$F_{III}(\mathrm{Mod}(\mathfrak{G}_2^0))=\{(t_1,t_2,\rho)\in R_{III}\mathfrak{G}_2^0 \mid \rho^*(T_1,T_2)<\rho < -1,\ \ t_2^*(t_1,\rho)<t_2<0,\ \ 0<t_1<1\},$$

where $\rho^*(T_1,T_2)=(4-T_1T_2+((4-T_1^2)(4-T_2^2))^{1/2})/2(T_2-T_1)$, $T_1=t_1+1/t_1$, $T_2=t_2+1/t_2$, *and* $t^*(t_1,t_2)$ *is* t_2 *satisfying the equation*

$$(1+t_1)(\sqrt{-\rho}+1/\sqrt{-\rho})=(1-t_1)(\sqrt{-t_2}+1/\sqrt{-t_2})$$

PROPOSITION 3.4 (Sato[4]).

$$F_{IV}(\mathrm{Mod}(\mathfrak{G}_2^0))=\{(t_1,t_2,\rho)\in R_{IV}\mathfrak{G}_2^0 \mid \rho^*(t_1,t_2)<\rho < 1/\rho*(t_1,t_2), t_2<t_1,\ \ 0<t_2<t_2^*(t_1,\rho),\ \ 0<t_1<1\},$$

where $\rho^*(t_1,t_2)=(1-\sqrt{t_1}t_2)(t_2-\sqrt{t_1})$ *and* $t_2^*(t_1,\rho)$ *is* t_2 *satisfying the equation*

$$2\sqrt{t_1}\sqrt{t_2}(1-\rho)=\sqrt{-\rho}(1-t_1)(1-t_2).$$

PROPOSITION 3.5 (Sato[5]).

$$F_V(\mathrm{Mod}(\mathfrak{G}_2^0))=\{(t_1,t_2,\rho)\in R_V\mathfrak{G}_2^0 \mid (1-t_1t_2)/(t_2-t_1)<\rho < ((1-\sqrt{t_2}t_1)/(\sqrt{t_2}-t_1))^2, 0<t_2<1, -1<t_1<0\}.$$

PROPOSITION 3.6 (Sato[7]).

$$F_{VI}(\mathrm{Mod}(\mathfrak{G}_2^0))=\{(t_1,t_2,\rho)\in R_{VI}\mathfrak{G}_2^0 \mid -(1+t_1\sqrt{\rho})/(\sqrt{\rho}+t_1) < t_2<0, 1<\rho<1/t_1^2, t_2<t_1, -1<t_1<0\}.$$

PROPOSITION 3.7 (Sato[5]).

$$F_{VII}(\mathrm{Mod}(\mathfrak{S}_2^0)) = \{(t_1, t_2, \rho) \in R_{VII}\mathfrak{S}_2^0 \mid$$

$$(\sqrt{-t_1} + \sqrt{-t_2})/(1 - \sqrt{-t_1}\sqrt{-t_2}) < \sqrt{-\rho}$$
$$< (1 - \sqrt{-t_1}\sqrt{-t_2})/(\sqrt{-t_1} + \sqrt{-t_2}), t_2 < t_1, -1 < t_1 < 0\}.$$

PROPOSITION 3.8 (Sato[7]).

$$F_{VIII}(\mathrm{Mod}(\mathfrak{S}_2^0)) = \{(t_1, t_2, \rho) \in R_{VIII}\mathfrak{S}_2^0 \mid 0 < t_2$$
$$< \frac{(\sqrt{-\rho} - \sqrt{-t_1})(1 - \sqrt{-t_1}\sqrt{-\rho})}{(\sqrt{-\rho} + \sqrt{-t_1})(1 + \sqrt{-t_1}\sqrt{-\rho})},$$
$$1/t_1 < \rho < -1, -1 < t_1 < 0\}.$$

4. Main theorems.

Let G be a marked two-generator group generated by Möbius transformations A_1 and A_2 : $G =< A_1, A_2 >$. The number

$$J(G) := \mid \mathrm{tr}^2(A_1) - 4 \mid + \mid \mathrm{tr}(A_1 A_2 A_1^{-1} A_2^{-1}) - 2 \mid$$

is called *Jørgensen's number* of G , where tr is the trace.

THEOREM 1. (Gilman[1], Sato[6]). *If $G =< A_1, A_2 >\in R_I\mathfrak{S}_2^0$, then $J(G) > 16$. The lower bound is the best possible.*

THEOREM 2. *If $G =< A_1, A_2 >\in R_{II}\mathfrak{S}_2^0$,then $J(G) > 16$. The lower bound is the best possible.*

THEOREM 3. *If $G =< A_1, A_2 >\in R_{III}\mathfrak{S}_2^0$, then $J(G) > 4$. The lower bound is the best possible.*

THEOREM 4 (Gilman[1], Sato[6]). *If $G =< A_1, A_2 >\in R_{IV}\mathfrak{S}_2^0$, then $J(G) > 4$. The lower bound is the best possible.*

THEOREM 5. *If $G =< A_1, A_2 >\in R_V\mathfrak{S}_2^0$, then $J(G) > 4(1 + \sqrt{2})^2$. The lower bound is the best possible.*

THEOREM 6. *If $G =< A_1, A_2 >\in R_{VI}\mathfrak{S}_2^0$, then $J(G) > 16$. The lower bound is the best possible.*

THEOREM 7. *If* $G = < A_1, A_2 > \in R_{VII}\mathfrak{G}_2^0$ *, then* $J(G) > 4(1+\sqrt{2})^2$*. The lower bound is the best possible.*

THEOREM 8. *If* $G = < A_1, A_2 > \in R_{VIII}\mathfrak{G}_2^0$ *, then* $J(G) > 16$*. The lower bound is the best possible.*

5. Lemmas.

We define functions $t_2 = t_2(t_1, \rho; k)$ $(k = II, III, V, VI, VII, VIII)$ as follows:

(i) $t_2(t_1, \rho; II) = (\sqrt{t_1}\sqrt{\rho} - 1)/(\sqrt{\rho} - \sqrt{t_1})$
$(1 < \rho < 1/t_1, 0 < t_1 < 1)$.

(ii) $t_2(t_1, \rho; III) = t_2^*(t_1, \rho)$ $(0 < t_1 < 1)$, where $t_2^*(t_1, \rho)$ is t_2 satisfying the equation

$$(1 + t_1)(\sqrt{-\rho} + 1/\sqrt{-\rho}) = (1 - t_1)(\sqrt{-t_2} + 1/\sqrt{-t_2}).$$

(iii) $t_2(t_1, \rho; V) = (1 + t_1\sqrt{\rho})/(\sqrt{\rho} + t_1)$
$(1 < \rho < 1/t_1^2, \ -1 < t_1 < 0)$.

(iv) $t_2(t_1, \rho; VI) = -(1 + t_1\sqrt{\rho})/(\sqrt{\rho} + t_1)$
$(1 < \rho < 1/t_1^2, \ -1 < t_1 < 0)$.

(v) $t_2(t_1, \rho; VII) = -\{(1 - \sqrt{-\rho}\sqrt{-t_1})/(\sqrt{-\rho} + \sqrt{-t})\}^2$
$(1/t_1 < \rho < -1, \ -1 < t_1 < 0)$.

(vi)

$$t_2(t_1, \rho; VIII) = \frac{(\sqrt{-\rho} - \sqrt{-t_1})(1 - \sqrt{-t_1}\sqrt{-\rho})}{(\sqrt{-\rho} + \sqrt{-t_1})(1 + \sqrt{-t_1}\sqrt{-\rho})}$$

$$(1/t_1 < \rho < -1, -1 < t_1 < 0).$$

We introduce some regions as follows. Let $\tau = (t_1, t_2, \rho) \in R^3$.

$M_{II} := \{\tau \in R^3 \mid t_2(t_1, \rho : II) < t_2 < 0, 1 < \rho < 1/t_1, 0 < t_1 < 1\}$.

$M_{III} := \{\tau \in R^3 \mid t_2(t_1, \rho : III) < t_2 < 0, 0 < t_1 < 1\}$.

$M_V := \{\tau \in R^3 \mid 0 < \sqrt{t_2} < t_2(t_1, \rho : V), 1 < \rho < 1/t_1^2, -1 < t_1 < 0\}$.

$M_{VI} := \{\tau \in R^3 \mid t_2(t_1, \rho : VI) < t_2 < 0, 1 < \rho < 1/t_1^2, -1 < t_1 < 0\}$.

$M_{VII} := \{\tau \in R^3 \mid t_2(t_1, \rho : VII) < t_2 < 1/t_2(t_1, \rho : VII), 1/t_1 < \rho < t_1, -1 < t_1 < 0\}$.

$M_{VIII} := \{\tau \in R^3 \mid 0 < t_2 < t_2(t_1, \rho : VIII), 1/t_1 < \rho < -1, -1 < t_1 < 0\}$.

LEMMA 5.1. *For each* $k = II, III, V, VI, VII, VIII$

$$F_k(\mathrm{Mod}(\mathfrak{S}_2^0)) \subseteq M_k \subseteq R_k\mathfrak{S}_2^0.$$

THEOREM B (Jørgensen[2]). *Suppose that the Möbius transformations A and B generate a non-elementary discrete group G.Then*

$$J(G) := \mid \mathrm{tr}^2(A) - 4 \mid + \mid \mathrm{tr}(ABA^{-1}B^{-1}) - 2 \mid \geq 1.$$

The lower bound is the best possible.

PROPOSITION 5.1. *Let* $G =< A_1, A_2 >$ *be a non-elementary discrete group and let* $\tau = (t_1, t_2, \rho)$ *be the point corresponding to* $< A_1, A_2 >$. *Then*

$$J(\tau) = \frac{\mid 1 - t_1 \mid^2}{\mid t_1 \mid} + \frac{\mid 1 - t_1 \mid^2 \mid 1 - t_2 \mid^2 \mid \rho \mid}{\mid t_1 \mid \quad \mid t_2 \mid \quad \mid \rho - 1 \mid^2} \geq 1.$$

REMARK. If $\tau = (t_1, t_2, \rho)$ corresponds to $G =< A_1, A_2 >$, then $J(G) = J(\tau)$.

For $G =< A_1, A_2 >$ we set

$$J_1(G) := \mid \mathrm{tr}(A)^2 - 4 \mid, \qquad J_1(\tau) := \mid 1 - t_1 \mid^2 / \mid t_1 \mid,$$

$$J_2(G) := \mid \mathrm{tr}(A_1 A_2 A_1^{-1} A_2^{-1}) - 2 \mid$$

and

$$J_2(\tau) := \frac{\mid 1 - t_1 \mid^2 \mid 1 - t_2 \mid^2 \quad \mid \rho \mid}{\mid t_1 \mid \quad \mid t_2 \mid \quad \mid \rho - 1 \mid^2}.$$

LEMMA 5.2. $J_2(G)$ *is* Φ_2*-invariant, that is,* $J_2(N_j(G)) = J_2(G)$ $(j = 1, 2, 3)$.

LEMMA 5.3. $J_1(G)$ *and* $J(G)$ *are invariant under the Nielsen transformations* N_1 *and* N_3 *, that is,*

(i) $J_1(N(G)) = J_1(G)$ *and* $J_1(N_3(G)) = J_1(G)$.

(ii) $J(N_1(G)) = J(G)$ *and* $J(N_3(G)) = J(G)$.

PROPOSITION 5.2. *For* $k = II, III, V, VI, VII, VIII$

$$\inf\{J(G) \mid G \in F_k(\mathrm{Mod}\mathfrak{S}_2^0\} \geq \inf\{J(G) \mid G \in M_k\} \geq \inf\{J(G) \mid G \in R_k\mathfrak{S}_2^0\}.$$

LEMMA 5.4. *If for each* $k = II, III, V, VI, VII, VIII$, $\tau = (t_1, t_2, \rho) \in M_k$ *and* $\tau_0 = (t_1, t_{20}, \rho) \in \partial M_k$, *then* $J(\tau_0) < J(\tau)$.

6. Proofs of Theorems 2 and 3.

LEMMA 6.1. *Let*

$$f(x,y) = \frac{(1-x^2)^2\ (1-x)^2y^2}{x^2\ (1-y)^2(1-xy)(y-x)}$$

Then $f(x,y) > 16$ *for* $1 < y < 1/x$ *and* $0 < x < 1$.

LEMMA 6.2. *There exists a sequence* $\{\tau_n\}$ $(\tau_n = (t_{1n}, t_{2n}, \rho_n))$ *in* M_{II} *converging to* $(1, t_{20}, 1) \in \partial M_{II}$ *such that* $\lim_{n\to\infty} J(\tau_n) = 16$.

REMARK. For the sequenne $\{\tau_n\}$ in Lemma 6.2, $\lim_{n\to\infty} J_1(\tau_n) = 0$, that is, $\lim_{n\to\infty} J(\tau_n) = \lim_{n\to\infty} J_2(\tau_n) = 4$.

Proof of Theorem 2. By Proposition 3.2 we have that for any $\tau \in R_{II}\mathfrak{G}_2^0$ there exists $\phi \in \mathrm{Mod}_{II}(\mathfrak{G}_2^0)$ such that $\phi(\tau) \in M_{II}$.Then by Lemmas 5.2, Proposition 5.2 and the above remark, $J(\tau) = J_1(\tau) + J_2(\tau) \geq J_2(\tau) = J_2(\phi(\tau)) = \lim_{n\to\infty} J_2(\tau_n) = \lim_{n\to\infty} J(\tau_n) = 16$, where $\{\tau_n\}$ is the sequence in Lemma 6.2. q.e.d.

LEMMA 6.3. *On the boundary surface of* M_{III} *defined by the equation*

$$(1+t_1)(\sqrt{-\rho} + 1/\sqrt{-\rho}) = (1-t_1)(\sqrt{-t_2} + 1/\sqrt{-t_2})$$

, *the Jørgensen number* $J(\tau)$ *is* $J(\tau) = 2(1+t_1^2)/t_1 > 4$.

LEMMA 6.4. *There exists a sequence* $\{\tau_n\}$ $(\tau_n = (t_{1n}, t_{2n}, \rho_n))$ in M_{III} converging to $(1, t_{20}, -1) \in \partial M_{III}$ *such that* $\lim_{n\to\infty} J(\tau_n) = 4$.

REMARK. For the above sequence $\{\tau_n\}$, $\lim_{n\to\infty} J_1(\tau_n) = 0$ and so $\lim_{n\to\infty} J_2(\tau_n) = \lim_{n\to\infty} J(\tau_n) = 4$.

Proof of Theorem 3. We can prove Theorem 3 by the same method as in the proof of Theorem 2.

7. Proofs of Theorems 5 and 8.

PROPOSITION 7.1 (Sato[5]). *The group* $\mathrm{Mod}(R_V \; {}^0_2)$ *is generated by* $[N_3^2]$ *and* $[N_2N_3N_2]$.

PROPOSITION 7.2.

$$\inf\{J(G) \mid G \in M_V\} = \inf\{J(G) \mid G \in R_V\mathfrak{S}{}^0_2\}.$$

We can prove Proposition 7.2 by using the following Lemmas. Throughout this section let $\varphi = N_3^2$ and $\psi = N_2N_3N_2$.

LEMMA 7.1.

$$J(\varphi^{m_k}\psi^{n_{k-1}}\cdots\psi^{n_1}\varphi^{m_1}(G)) = J(\psi^{n_{k-1}}\cdots\psi^{n_1}\varphi^{m_1}(G))$$

$(m_1 \geq 1, m_i, n_j \in Z \;\; (i = 2, 3, \cdots, k; j = 1, 2, \cdots, k-1), k \geq 1)$.

LEMMA 7.2.

$$J(\psi\varphi^{m_k}\psi^{n_{k-1}}\cdots\psi^{n_1}\varphi^{m_1}((G)) \geq J(\varphi^{m_k}\cdots\psi^{n_1}\varphi^{m_1}(G))$$

$$(m_1 \geq 1, m_i \in Z \;\; (2 \leq i \leq k), n_j \in Z \;\; (1 \leq j \leq k-1), k \geq 1).$$

LEMMA 7.3.

$$J(\psi^{(n+1)}\varphi^{m_k}\cdots\psi^{n_1}\varphi^{m_1}(G)) > J(\psi^n\varphi^{mk}\cdots\psi^{n_1}\varphi^{m_1}(G))$$

$$(m_1 \geq 0, m_i \geq 1 \;\; (2 \leq i \leq k), n_j \leq 1 \;\; (1 \leq j \leq k-1), k \geq 1)$$

LEMMA 7.4.

$$J(\psi^{-1}\varphi^{m_k}\psi^{n_{k-1}}\cdots\psi^{n_1}\varphi^{m_1}(G)) \geq J(\varphi^{m_k}\psi^{n_{k-1}}\cdots\psi^{n_1}\varphi^{m_1}(G))$$

$$(m_1 \geq 1, m_i, n_j \in Z \;\; (2 \leq i \leq k, 1 \leq j \leq k-1), k \geq 1).$$

LEMMA 7.5.

$$\begin{aligned} &J(\psi^{-(n+1)}\varphi^{m_k}\psi^{n_{k-1}}\cdots\psi^{n_1}\varphi^{m_1}(G)) \\ &\geq J(\psi^{-n}\varphi^{m_k}\psi^{n_{k-1}}\cdots\psi^{n_1}\varphi^{m_1}(G)) \end{aligned}$$

$$(n \geq 0, m_1 \geq 1, m_i, n_j \in Z \;\; (2 \leq i \leq k, 1 \leq j \leq k-1), k \geq 1).$$

LEMMA 7.6. (i) $N_2^2 = 1$.
(ii) $\psi^{-n}N_1 = N_1\psi^n$.

(iii) $\varphi^{-m}N_1 = N_1\varphi^m$.

We set $M_V(1) = M_V$ and $M_V(-1) = N_1(M_V)$.

LEMMA 7.7. *Let* $\phi \in \Phi_2$.
(i) $\phi^{-1}(M_V(1)) = M_V(-1)$.
(ii) $\phi^{-m}N_1(M_V(1)) = N_1\phi^m(M_V(1))$.
(iii) $\phi^{-m}(M_V(1)) = \phi^{-(m-1)}(N_1(M_V(1))$.

LEMMA 7.8. (i) $J(G) = J(N_1(G))$.
(ii) $\inf\{J(G) \mid G \in M_V(1)\} = \inf\{J(G) \mid G \in M_V(-1)\}$.

LEMMA 7.9.

$$\begin{aligned}&\psi^{n_k}\varphi^{m_{k-1}}\cdots\psi^{n_1}\varphi^{-(m_1-1)}(M_V(1))\\ &\quad= N_1\psi^{-n_k}\cdots\psi^{-n_1}\varphi^{(m_1-1)}(M_V(1)) \quad (m_1 \geq 1).\end{aligned}$$

LEMMA 7.10. (i)

$$\inf\{J(\varphi^{-m}(G)) \mid G \in M_V(1)\} = \inf\{J(\varphi^{(m-1)}(G) \mid G \in M_V(1)\} \quad (m \geq 1).$$

(ii)

$$\begin{aligned}&\inf\{J(\varphi^{m_k}\psi^{n_{k-1}}\cdots\psi^{n_1}\varphi^{-m_1}(G) \mid G \in M_V(1)\}\\ &\quad= \inf\{J(\varphi^{-m_k}\psi^{-n_{k-1}}\cdots\psi^{-n_1}\varphi^{m_1-1}(G) \mid G \in M_V(1)\} \quad (m_1 \geq 1)\end{aligned}$$

(iii)

$$\begin{aligned}&\inf\{J(\psi^{n_k}\varphi^{m_k}\cdots\psi^{n_1}\varphi^{-m_1}(G) \mid G \in M_V(1)\}\\ &\quad= \inf\{J(\psi^{-n_k}\varphi^{-m_k}\cdots\psi^{-n_1}\varphi^{(m_1-1)}(G) \mid G \in M_V(1)\} \quad (m_1 \geq 1)\end{aligned}$$

LEMMA 7.11. $J(G) \leq J(\psi(G))$ for $G \in M_V(1)$.

LEMMA 7.12. $J(\psi^{n+1}(G)) \geq J(\psi^n(G))$ $(n \geq 1)$ *for* $G \in M_V(1)$.

LEMMA 7.13.
(i) $J(\varphi^{m_k}\psi^{n_{k-1}}\cdots\varphi^{m_2}\psi^{n_1}(G)) = J(\psi^{n_{k-1}}\cdots\varphi^{m_2}\psi^{n_1}(G))$

$(n_1 \geq 1)$.

(ii)

$$J(\psi^{\pm 1}\varphi^{m_k}\psi^{n_{k-1}}\cdots\varphi^{m_2}\psi^{n_1}G))$$
$$\geq J(\varphi^{m_k}\psi^{n_{k-1}}\cdots\varphi^{m_2}\psi^{n_1}(G)) \quad (n_1 \geq 1)$$

(iii)

$$J(\psi^{\pm(n+1)}\varphi^{m_k}\psi^{n_{k-1}}\cdots\varphi^{m_2}\psi^{n_1}(G))$$
$$\geq J(\psi^{\pm n}\varphi^{m_{k-1}}\cdots\varphi^{m_2}\psi^{n_1}(G)) \quad (n_1 \geq 1)$$

LEMMA 7.14. $J(\psi^{-n}(N_1(G)) \geq J(N_1(G)) = J(G) \quad (n \geq 1)$ *for* $G \in M_V(1)$.

LEMMA 7.15.
(i) $\psi^{-1}(M_V(1)) = M_V(-1)$.
(ii) $\inf\{J(\psi^{-n}(G)) \mid G \in M_V(1)\} = \inf\{J(\psi^{(n-1)}(G)) \mid G \in M_V(1)\}$ $(n_1 \geq 1)$.
(iii)

$$\inf\{J(\varphi^{m_k}\psi^{n_{k-1}}\cdots\varphi^{m_2}\psi^{n_1}(G)) \mid G \in M_V(1)\}$$
$$= \inf\{J(\varphi^{-m_k}\psi^{-(n_{k-1})}\cdots\varphi^{-m_2}\psi^{(n_1-1)}(G) \mid G \in M_V(1)\}.$$
$$(n_1 \geq 1)$$

(iv)

$$\inf\{J(\psi^{n_k}\varphi^{m_k}\cdots\varphi^{m_2}\psi^{n_1}(G)) \mid G \in M_V(1)\}$$
$$= \inf\{J(\psi^{-n_k}\varphi^{-m_k}\cdots\varphi^{-m_2}\psi^{(n_1-1)}(G) \mid G \in M_V(1)\}$$
$$(n_1 \geq 1).$$

PROPOSITION 7.3. $\inf\{J(G) \mid G \in M_V(1)\} = 4(1+\sqrt{2})^2$.

Proof of Theorem 5. We can prove Theorem 5 by Propositions 7.1 , 7.2 and 7.3.

PROPOSITION 7.4 (Sato[7]). *The group* $\mathrm{Mod}(R_{VIII}\,\mathfrak{G}^{\,0}_{\,2})$ *is generated by* $[N_3^2]$ *and* $[N_2N_3N_2]$.

PROPOSITION 7.5.

$$\inf\{J(G) \mid G \in M_{VIII}(1)\} = \inf\{J(G) \mid G \in R_{VIII}\mathfrak{G}^{\,0}_{\,2}\}.$$

We can similarly prove Proposition 7.5 to Proposition 7.2.

PROPOSITION 7.6. $\inf\{J(G) \mid G \in M_{VIII}(1)\} = 16$.

Proof of Theorem 8. We can prove Theorem 8 by using Propositions 7.4 , 7.5 and 7.6.

8. Proofs of Theorems 6 and 7.

PROPOSITION 8.1 (Sato[7]). *The group* $\mathrm{Mod}(R_{VI}\mathfrak{S}_2^0)$*is generated by* $[N_3^2]$ *and* $[N_1N_2]$.

PROPOSITION 8.2.

$$\inf\{J(G) \mid G \in R_{VI}\mathfrak{S}_2^0\} = \inf\{J(G) \mid G \in M_{VI}\}$$

We can prove Proposition 8.2 by using the following Lemma 8.1 through Lemma 8.6. Let $\phi_1, \phi_2 \in \Phi_2$. We say ϕ_1 and ϕ_2 are equivalent if $\phi_1(G)$ is equivalent to $\phi_2(G)$, and denote by $\phi_1 \sim \phi_2$. We set $M_{VI}(1) = M_{VI}$ and $M_{VI}(-1) = N_1(M_{VI})$. We set $\varphi = N_3^2$ and $\chi = N_1N_2$.

LEMMA 8.1. (i) $N_1N_2 \sim N_2N_1$ *and* $N_1N_2N_1N_2 \sim 1$.
(ii)

$$\chi^n \sim \begin{cases} N_1N_2 & \textit{if } n \textit{ is odd} \\ 1 & \textit{if } n \textit{ is even.} \end{cases}$$

(iii) $\chi^{-1} \sim \chi$.
(iv) $\chi N_1 = N_1\chi^{-1}$.
(v) $\varphi N_1 = N_1\phi^{-1}$.
(vi)

$$\chi^n(M_{VI}(1) = \begin{cases} M_{VI}(-1) & \textit{if } n \textit{ is odd} \\ M_{VI}(1) & \textit{if } n \textit{ is even.} \end{cases}$$

LEMMA 8.2. (i) $J(\varphi^m(G)) = J(G)$ ($m \in Z$) *for* $G \in M_{VI}(1)$.
(ii) $\inf\{J(G) \mid G \in \chi^n(M_{VI}(1))\} = \inf\{J(G) \mid G \in M_{VI}(1)\}$ $(n \in Z)$.

LEMMA 8.3. *Let* $G \in M_{VI}(1)$.

(i) $J(\chi\varphi(G)) \geq J(G)$.
(ii) $J(\chi^n\varphi^m(G)) \geq J(G) \quad (n \geq 1,\ m \geq 1)$.

LEMMA 8.4. (i)

$$J(\varphi^{m_k}\chi^{n_{k-1}}\cdots\chi^{n_1}\varphi^{m_1}(G)) = J(\chi^{n_k-1}\cdots\chi^{n_1}\varphi^{m_1}(G))$$

$$(m_1 \geq 1, m_i \in Z\ \ (2 \leq i \leq k), n_j \in Z\ \ (1 \leq j \leq k-1)).$$

(ii) $J(\chi^{n_k}\varphi^{m_k}\cdots\chi^{n_1}\varphi^{m_1}(G)) \geq J(\chi^{n_{k-1}}\cdots\chi^{n_1}\varphi^{m_1}(G))$
$(m_1 \geq 1\ \ m_i \in Z\ \ (2 \leq i \leq k), n_j \in Z\ \ (1 \leq j \leq k))$.

LEMMA 8.5. (i)

$$\begin{aligned}&\inf\{J(\chi^n\varphi^{-m}(G) \mid G \in M_{VI}(1)\}\\ &\quad = \inf\{J(\chi^{-n}\varphi^{(m-1)}(G)) \mid G \in M_{VI}(1)\}\end{aligned}$$

$$(m \geq 1, n \in Z).$$

(ii)

$$\begin{aligned}&\inf\{J(\varphi^{n_k}\chi^{n_{k-1}}\cdots\chi^{n_1}\varphi^{m_1}(G) \mid G \in M_{VI}(1)\}\\ &\quad = \inf\{J(\varphi^{-m_k}\chi^{-n_{k-1}}\cdots\chi^{-n_1}\varphi^{(m_1-1)}(G)) \mid G \in M_{VI}(1)\}\end{aligned}$$

$$(m_1 \geq 1, m_i \in Z\ \ (2 \leq i \leq k), n_j \in Z\ \ (1 \leq j \leq k-1)).$$

LEMMA 8.6. (i)

$$\begin{aligned}&\inf\{J(\varphi^{m_k}\chi^{n_{k-1}}\cdots\varphi^{m_2}\chi^{n_1}(G)) \mid G \in M_{VI}(1)\}\\ &\quad = \inf\{J(\varphi^{m_k}\chi^{n_{k-1}}\cdots\varphi^{m_2}(G) \mid G \in M_{VI}(1)\}\end{aligned}$$

$$(m_i \in Z\ \ (2 \leq i \leq k), n_j \in Z\ \ (1 \leq j \leq k-1)).$$

(ii)

$$\begin{aligned}&\inf\{J(\chi^{n_k}\varphi^{m_k}\cdots\chi^{n_2}\varphi^{m_2}\chi^{n_1}(G)) \mid G \in M_{VI}(1)\}\\ &\quad = \inf\{J(\chi^{n_k}\varphi^{m_k}\cdots\chi^{n_2}\varphi^{m_2}(G)) \mid G \in M_{VI}(1)\}\end{aligned}$$

$$(m_i \in Z\ \ (2 \leq i \leq k), n_j \in Z\ \ (1 \leq j \leq k)).$$

PROPOSITION 8.3. $\inf\{J(G) \mid G \in M_{VI}\} = 16$.

Proof of Theorem6. We can prove Theorem 6 by Propositions 8.1, 8.2 and 8.3.

PROPOSITION 8.4 (Sato[5]). *The group* $\mathrm{Mod}(R_{VII}\mathfrak{S}_2^0)$ *is generated by* $[N_3^2]$ *and* $[N_1N_2]$.

PROPOSITION 8.5.

$$\inf\{J(G) \mid G \in R_{VII}\mathfrak{S}_2^0\} = \inf\{J(G) \mid G \in M_{VII}\}.$$

We can similarly prove this Proposition to Proposition 8.2.

PROPOSITION 8.6. $\inf\{J(G) \mid G \in M_{VII}\} = 4(1+\sqrt{2})^2$.

Proof of Theorem 7. We can prove Theorem 7 by Propositions 8.4, 8.5 and 8.6.

9. Examples.

Let $\{\tau_n = (t_{1n}, t_{2n}, \rho_n)\}$ $(n = 1, 2, 3, \cdots)$ be a sequence of points in R^3 and let $G_n = < A_{1n}, A_{2n} >$ be the groups representing τ_n .

EXAMPLE 1 (Type II). Let $t_n = (1 - \sqrt{2}/\sqrt{3n})^2, t_{2n} = -(\sqrt{2} - 1)^2 + (3 - \sqrt{3})/2n$ and $\rho_n = 1/\sqrt{3n} + 1$ $(n = 2, 3, 4, \cdots)$. Then (i) $G_n \in R_I\mathfrak{S}_2^0$ and (ii) $\lim_{n\to\infty} J(G_n) = 16$.

EXAMPLE 2 (Type III). Let $t_{1n} = ((n-2)/(n+2))^2, t_{2n} = -1/n^2$ and $\rho = -1$ $(n = 3, 4, 5, \cdots)$. Then (i) $G_n \in R_{III}\mathfrak{S}_2^0$ and (ii) $\lim_{n\to\infty} J(G_n) = 4$.

EXAMPLE 3 (Type V). Let $t_{1n} = -(1 + \sqrt{2} - \sqrt{2 + 2\sqrt{2}}) + 1/n, t_{2n} = (1 - 2/n)^2$ and $\rho_n = (1 + 1/n)^2$ $(n = 3, 4, 5, \cdots)$. Then (i) $G_n \in R_V\mathfrak{S}_2^0$ and (ii) $\lim_{n\to\infty} J(G_n) = 4(1 + \sqrt{2})^2$.

EXAMPLE 4 (Type VI). Let $t_{1n} = -(3-2\sqrt{2})+1/n$, $t_{2n} = -(5-2\sqrt{6}) + 1/n$ and $\rho_n = 7 + 4\sqrt{3}$ $(n = 1, 2, 3, \cdots)$. Then (i) $G_n \in R_{VI}\mathfrak{S}_2^0$ and (ii) $\lim_{n\to\infty} J(G_n) = 16$.

EXAMPLE 5 (Type VII). Let $t_{1n} = -(\sqrt{-t_{10}} - 1/n)^2, t_{2n} = t_{20}$ and $\rho = -1$ $(n = 1, 2, 3, \cdots)$, where $t_{10} = -(1 + \sqrt{2}) + \sqrt{2 + 2\sqrt{2}}$ and $t_{20} = -((1 - \sqrt{-t_{10}})/(1 + \sqrt{-t_{10}}))^2$. Then (i) $G_n \in R_{VII}\mathfrak{S}_2^0$ and (ii) $\lim_{n\to\infty} J(G_n) = 4(1 + \sqrt{2})^2$.

EXAMPLE 6 (Type VIII). Let $t_{1n} = -(3 - 2\sqrt{2}) + 1/n, t_{2n} = 3 - 2\sqrt{2} - 1/n$ and $\rho = -1$ $(n = 1, 2, 3, \cdots)$. Then (i) $G_n \in R_{VIII}\mathfrak{S}_2^0$

and (ii) $\lim_{m\to\infty} J(G_n) = 16$.

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