ON PARTIALLY CONFORMAL QC
DEFORMATIONS (Complex Analysis on Hyperbolic 3-Manifolds)

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ON PARTIALLY CONFORMAL QC DEFORMATIONS

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1. Let $M(R)$ be the Banach space of all Beltrami differentials $\mu = \mu(z) \frac{dz}{dz}$ on a Riemann surface $R$ with norm $\|\mu\|_\infty := \text{ess sup} |\mu(z)|$. We denote by $M(R)_1$ the open unit ball of $M(R)$. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. For each $\mu \in M(\mathbb{D})_1$, there is a unique normalized quasiconformal self-mapping $W^\mu$ of $\mathbb{D}$ whose Beltrami coefficient $\mu(W^\mu) := W^\mu_z/W^\mu_{\overline{z}}$ is $\mu$, that is, $W^\mu: \mathbb{D} \to \mathbb{D}$ is a homeomorphism whose generalized derivatives satisfy the Beltrami equation $f_z = \mu f_z$, and its continuous extension to the closed unit disk $\mathbb{D}$ fixes 1, $i$ and $-1$. Two elements $\mu$ and $\nu$ in $M(\mathbb{D})_1$ are said to be equivalent if $W^\mu$ and $W^\nu$ have the same boundary values. Let $R$ be a hyperbolic Riemann surface and $\pi: \mathbb{D} \to R$ be a universal covering mapping. We define $\mu$, $\nu \in M(R)_1$ are equivalent when so are their pull-backs $\pi^*\mu$ and $\pi^*\nu$, and quasiconformal mappings $f: R \to f(R)$ and $g: R \to g(R)$ are equivalent if so are their Beltrami coefficients $\mu(f)$ and $\mu(g)$. It is known that $f$ and $g$ are equivalent if and only if there is a conformal mapping $h: f(R) \to g(R)$ such that $h \circ f$ is homotopic to $g$ modulo the border of $R$. The Teichmüller space $T(R)$ of $R$ is the quotient space of $M(R)_1$ with respect to this equivalence relation. We denote by $[\mu]$ the equivalence class containing $\mu$, and identify it with the marked Riemann surface $[f(R), f], \mu(f) = \mu$.

Let $V$ be a measurable subset of $R$ and set

$$M(V)_1 := \{\mu \in M(R)_1: \mu|_{R \backslash V} = 0\}.$$ 

A quasiconformal mapping $f$ is 'conformal' outside $V$ if $\mu(f) \in M(V)_1$, so we say $[f(R), f]$ is a partially conformal qc deformation of $[R, \text{id}_R]$. A family of partially conformal qc mappings is useful to investigate Teichmüller spaces and extremal problems on them (see for example Krushkal [5], Gardiner [2], [3], Reich [10] and Fehlmann-Sakan [1]).

2. We summarize some known facts. First of all, in general, $[M(V)_1] \neq T(R)$ (cf. Savin [11]). For example, if $R \backslash V$ is an incompressible annular domain, then $[M(V)_1] \neq T(R)$. But if $R \backslash V$ is a topological disk, then $[M(V)_1] = T(R)$.

If $R$ is of finite conformal type, that is, $R$ is a Riemann surface obtained by removing a finite number of punctures from a compact one, then $[M(V)_k]$ is a
neighborhood of the origin $[0]$ of $T(R)$ for any $V$ with positive measure and any $0 < k \leq 1$. This is a classical result. While there are $R$ of infinite conformal type and a subset $V$ of $R$ with positive measure such that $[M(V)_1]$ is not a neighborhood of $[0]$ (Oikawa [9]).

A general necessary condition for $V$ to insure that $[M(V)_1]$ becomes a neighborhood of $[0]$ is

$$r(V) := \inf \left\{ \iint_V |\phi| \, dx \, dy : \phi \in A_2^1(R), \|\phi\|_1 = 1 \right\} > 0.$$  

Moreover, when $R = \mathbb{D}$, the condition (1) is equivalent to a simple geometric one:

$$\inf \{ \text{Area}(V \cap \Delta(z;\rho)) : z \in \mathbb{D} \} > 0 \quad \text{for some } \rho > 0$$

where $\Delta(z;\rho)$ is the hyperbolic disk with center $z$ and radius $\rho$, and Area means its hyperbolic area (Ohtake [7]).

On the other hand, a known sufficient condition is as follows. Set

$$\omega(z) := \sup \{ \lambda(z)^{-2} |\phi(z)| : \phi \in A_2^1(R), \|\phi\|_1 = 1 \}.$$  

It is not difficult to see that the function $\omega$ on $R$ is continuous and vanishing at the punctures of $R$. If $V$ has positive measure and if

$$\iint_V \max \{ \omega(z)^2, 1 \} \, dx \, dy < \infty,$$

then $[M(V)_k]$ contains $[0]$ in its interior for any $0 < k \leq 1$ (Ohtake [6]).

3. We give here a quantitative version of the necessary condition (1) above.

**Theorem 1.** If $[M(R)_k] \subset [M(V)_{k'}]$, then we have

$$r(V) \geq \frac{K - 1}{K' - 1}.$$  

where $K := (1 - k)/(1 + k)$, $K' := (1 - k')/(1 + k')$.

**Proof.** Take arbitrary $0 < t < k$ and $\phi \in A_2^1(R)$ with $\|\phi\|_1 = 1$. Let $f_0 : R \to R_0$ be a quasiconformal mapping whose Beltrami coefficient is $t\overline{\phi}/|\phi|$ and $\psi \in A_2^1(R_0)$ be the terminal differential of the Teichmüller mapping $f_0$ (cf. Lehto [4]). Then $f_0^{-1} : R_0 \to R$ is a Teichmüller mapping with $\mu(f_0^{-1}) = -k\overline{\psi}/|\psi|$. By assumption, there is a quasiconformal mapping $f : R \to R_0$ which is equivalent to $f$ and whose Beltrami coefficient $\mu(f)$ is in $M(V)_{k'}$. Applying Reich-Strebel inequality (Strebel [12], [13]) to $-\psi$ and $f \circ f_0^{-1}$ equivalent to the identity mapping of $R_0$, we have

$$\|\psi\|_1 \leq \iint_{R_0} |\psi| \frac{1 + \mu(f \circ f_0^{-1})}{1 - |\mu(f \circ f_0^{-1})|^2} \, dudv.$$
Since
\[ K(f_0)|\phi(z)| \, dxdy = |\psi(w)| \, dudv, \quad w = f_0(z) \]
\[ \frac{\overline{\psi}(w)}{|\psi(w)|} = \frac{p(z)}{|\overline{p}(z)|} \cdot \frac{\overline{\phi}(z)}{|\phi(z)|}, \quad p = (f_0)_{\overline{z}} \]
\[ \mu(f \circ f_0^{-1})(w) = \frac{\mu(f)(z) - \mu(f_0)(z)}{1 - \overline{\mu}(f_0)(z) \mu(f)(z)} \cdot \frac{p(z)}{|\overline{p}(z)|} \]
change of variable gives us
\[ K(f_0) \leq K(f_0) \int \int_R \frac{|1 - \mu(f_0) \frac{\phi}{|\phi|}|^2 |1 + \mu(f) \frac{\phi}{|\phi|} \cdot \frac{1 - \overline{\mu}(f_0) \overline{\phi}/|\phi|}{1 - \mu(f_0) \phi/|\phi|}|^2}{(1 - |\mu(f_0)|^2)(1 - |\mu(f)|^2)} |\phi| \, dxdy \]
\[ = \int \int_{R} \frac{|1 + \mu(f) \phi/|\phi||^2}{1 - |\mu(f)|^2} |\phi| \, dxdy \]
\[ \leq K' \int \int_{V} |\phi| \, dxdy + \int \int_{R \setminus V} |\phi| \, dxdy \]
\[ = (K' - 1) \int \int_{V} |\phi| \, dxdy + 1. \]
Letting \( t \to k \), we have a desired inequality (2).

We can show a partial converse of Theorem 1. Its proof and the details are omitted and will appear elsewhere.

**Theorem 2.** For \( A > 0 \) and \( l > 0 \), there are positive constants \( C \) and \( t_0 \leq 1 \) such that if a Riemann surface \( R \) has hyperbolic area less than \( A \) and if the length of each closed geodesic of \( R \) is not shorter than \( l \), then
\[ [M(R)_t] \subset [M(V)_{Ct/r(V)^2}] \quad \text{for any } 0 \leq t \leq t_0. \]
where the constants \( C \) and \( t_0 \) depend only on \( A \), \( l \) and \( r(V) \) but not on \( R \) nor \( V \).

**References**


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