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SEVERAL CHARACTERIZATIONS OF FUCHSIAN GROUPS OF DIVERGENCE TYPE

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Let Γ be a Fuchsian group, that is, a discrete isometry group of the hyperbolic plane. In the unit disk model Δ or the upper half plane model H of it, Γ is a subgroup of the Möbius transformations which acts properly discontinuously. We say Γ is of divergence type if $\sum_{\gamma\in\Gamma}(1-|\gamma(0)|)$ diverges in the model Δ. Otherwise, we say it is of convergence type. In this note, we survey several characterizations of Fuchsian groups of divergence type from a function theoretical viewpoint. In particular, we compose a neat explanation about Mostow rigidity theorem for such Fuchsian groups. For simplicity, we assume that Fuchsian groups are torsion-free; this is not essential at all.

We begin with a well-known result.

Proposition 1. *The following are equivalent:*
(1) Γ *is of divergence type;*
(2) *the Riemann surface* $W=\Delta/\Gamma$ *does not admit Green's function;*
(3) *the conical limit set of* Γ *has full measure on* $\mathbf{S}^1=\partial\Delta$*;*
(3') *the conical limit set of* Γ *has positive measure on* $\mathbf{S}^1$.

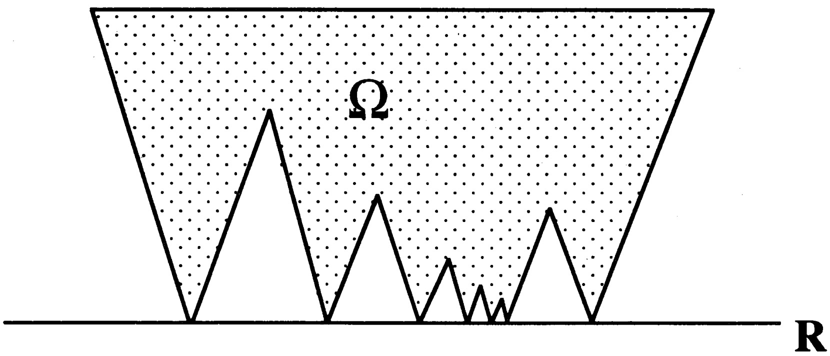
Here, we explain more about the above conditions. The condition (2) is equivalent to that the harmonic measure of the ideal boundary is zero. Of course, compact Riemann surfaces satisfy this condition. To explain the condition (3), we need define the conical limit set of Γ. Let $\Gamma(0)$ be the orbit of 0 by Γ. The definition below does not depend on the choice of the reference point, hence we may take 0. We say that $x\in\mathbf{S}^1$ is a conical limit point if $\Gamma(0)$ accummulates to x in some Stolz angular region with vertex x. The conical limit set is the set of the conical limit points and denoted by $\Lambda_\infty(\Gamma)$. The measure on $\mathbf{S}^1$ is the usual Lebesgue measure with normalization that the full measure is 1. We denote this measure by $\boldsymbol{m}$.

Proof of Proposition 1. (3′) ⇒ (1): Whenever $\sum(1-|\gamma(0)|)<\infty$, the Blaschke product

$$\sum_{\gamma\in\Gamma}\frac{z-\gamma(0)}{1-\overline{\gamma(0)}z}$$

converges and determines a bounded analytic function $B(z)$ which takes zero at $\{\gamma(0)\}_{\gamma\in\Gamma}$. It has the non-tangential limit on $\mathbf{S}^1$. If $\boldsymbol{m}(\Lambda_\infty(\Gamma))>0$, then the limit must vanish on the set of positive measure. This implies that $B(z)\equiv 0$, which is a contradiction.

(1) ⇒ (2): Assume that W has Green's function g. Let $G(z)$ be the lift of g to Δ, more precisely, $G=g\circ\pi$, where π is the universal covering that maps the pole of g to the origin. Further, let $B(z)=exp\{-G(z)-iG^*(z)\}$, where $G^*(z)$ is the harmonic conjugate to $G(z)$. Then $B(z)$ is a bounded analytic function whose zeros are $\{\gamma(0)\}_{\gamma\in\Gamma}$. Hence, it satisfies the Blaschke condition: $\sum(1-|\gamma(0)|)<\infty$.



figure

(2) ⇒ (3): Assume that $\boldsymbol{m}(\Lambda_\infty(\Gamma))<1$. We use the upper half plane model. Let $\pi : H\to W$ be the universal cover, and K a small closed disk in W. Then, we can take a “saw region” Ω with the same edge angle (see figure) such that $\Omega\cap\pi^{-1}(K)=\emptyset$ and $\boldsymbol{m}(\bar\Omega\cap\mathbf{R})>0$. This Ω is bounded by a rectifiable Jordan curve, hence the harmonic measure and the linear measure on $\partial\Omega$ are mutually absolutely continuous. This implies that the subregion Ω in the hyperbolic plane has positive harmonic measure on the ideal boundary, and so does the larger region $H-\pi^{-1}(K)$. Therefore, the ideal boundary of $W-K$ is of positive harmonic measure, which is equivalent to that W admits Green's function.

This construction of the saw region is important for our survey; we precisely state this result as a lemma. A proof can be found, for example, in [7].

Lemma 1. *Let Γ be a Fuchsian group such that $\boldsymbol{m}(\mathbf{S}^1 - \Lambda_\infty(\Gamma)) > 0$, and K a compact set in the hyperbolic surface $\Delta/\Gamma = \pi(\Delta)$. Then, for any positive measure subset A of $\mathbf{S}^1 - \Lambda_\infty(\Gamma)$, there is a simply connected subregion Ω of Δ such that $\Omega \cap \pi^{-1}(K) = \emptyset$, $\boldsymbol{m}(\bar{\Omega} \cap A) > 0$ and the ideal boundary has positive harmonic measure.*

Now, we introduce other conditions. The first one is concerning a certain property of the normal subgroups.

(4) every non-trivial normal subgroup of Γ is conservative.

We define conservative Fuchsian groups here. In a Riemann surface $W = \Delta/\Gamma$, we consider simply connected subregions ω. We say Γ (or W) is conservative if the harmonic measure of the ideal boundary of any such ω is zero. This definition is equivalent to the usual ones in terms of wandering sets, the horocyclic limit set, the Dirichlet fundamental region or growth of the hyperbolic area. See [9] for these definitions and [7] for the equivalence.

Proposition 2 [6]. *The conditions (3) and (4) are equivalent.*

Proof. (3) $\Rightarrow$ (4): Assume that a non-trivial normal subgroup G of Γ is not conservative. Then, there is a simply connected subregion ω in Δ/G which has positive harmonic measure on the ideal boundary. Let Ω be a connected component of the inverse image of ω by the universal cover. Since Ω also has positive harmonic measure on the ideal boundary, we see that $\boldsymbol{m}(\bar{\Omega} \cap \mathbf{S}^1) > 0$. Moreover, McMillan's twist point theorem (cf. [8]) shows that at almost all points of $\bar{\Omega} \cap \mathbf{S}^1$, Ω is tangent to $\mathbf{S}^1$. Since the conical limit set of Γ is of full measure, we can take $x \in \Lambda_\infty(\Gamma)$ where Ω is tangent to $\mathbf{S}^1$ and a sequence $\{\gamma_n(0)\}$ which converges to x non-tangentially. Then, the injective radii of the universal cover $\Delta \to \Delta/G$ at $\{\gamma_n(0)\}$ grow to infinity. But, they must be the same because they project to the same point on Δ/Γ and the covering $\Delta/G \to \Delta/\Gamma$ is normal. This contradiction proves that (3) implies (4).

(4) $\Rightarrow$ (3): Assume that the Riemann surface Δ/Γ admits Green's function. We can take a subregion X in it whose relative boundary consists of one simple loop, fundamental group is non-trivial, and complement has

positive harmonic measure on the ideal boundary. We remove X from Δ/Γ and fill the hole with a disk K, then consider the universal cover of the resulting Riemann surface which has Green's function, too. As in Lemma 1, we take a simply connected subregion Ω in Δ. Replacing the inverse images of K with the copies of X, we obtain a non-universal normal cover of Δ/Γ containing Ω whose ideal boundary has positive harmonic measure. Hence, the normal cover is not conservative, which completes the proof.

Remark. As a corollary to the above proof, we see that: *if a hyperbolic Riemann surface has a upper bound of the injective radii, then it is conservative.* The converse is not true. Actually, there is a Riemann surface with unbounded injective radii even if it does not admit Green's function.

Next, we consider the Mostow rigidity of Fuchsian groups. The original rigidity theorem works for hyperbolic discrete groups of the dimension greater than 2. It says that an automorphism of the sphere at infinity $\mathbf{S}^{n-1}$ $(n \geq 3)$ compatible with a cocompact discrete group must be a Möbius transformation. This statement fails for Fuchsian groups. Indeed, we have $6g-6$ dimensional deformation space of a cocompact Fuchsian group of genus g up to conjugation of Möbius transformations. However, if we assume certain regularity for the automorphism, we still sustain the rigidity theorem. The statements are as follows:

(5) every automorphism of $\mathbf{S}^1$ compatible with Γ but not singular with respect to $\boldsymbol{m}$ is a Möbius transformation;
(5′) every automorphism of $\mathbf{S}^1$ compatible with Γ and absolutely continuous with respect to $\boldsymbol{m}$ is a Möbius transformation.

Not only cocompact Fuchsian groups but also those of divergence type satisfy this property. In fact, it is well-known that a Fuchsian group Γ of divergence type acts on $\mathbf{S}^1 \times \mathbf{S}^1$ ergodically with respect to the product measure $\boldsymbol{m} \times \boldsymbol{m}$, and vice versa. Then, as Kuusalo [5] and Sullivan [9] showed, derivatives of the strictly increasing function $\mathbf{R} \to \mathbf{R}$ which is regarded as the boundary homeomorphism compatible with Γ must be the same almost everywhere, thus (5) follows. But, we do not prove the characterization of $\mathbf{S}^1 \times \mathbf{S}^1$ ergodicity in this note; instead of this, we shall prove (5) from the condition (3). Originally, this is due to Agard [1], however, we introduce a simple proof below from a recent work of Ivanov [4].

Conversely, Astala-Zinsmeister [2] proved that for any Fuchsian group Γ

of convergence type, there is a non-trivial absolutely continuous automorphism compatible with Γ, namely, that (5') implies (1). Tukia [10] has given a simple proof of this result. Recently, Hamilton [3] has improved Tukia's argument as the following Lemma 2. We are able to derive it immediately from the previous Lemma 1.

Lemma 2. *Let Γ be a Fuchsian group of convergence type, and f a quasiconformal deformation of the Riemann surface $\Delta/\Gamma = \pi(\Delta)$ which is conformal out of a compact subset K of Δ/Γ. We lift f to Δ as an automorphism $\tilde{f}$ and extend it to $\mathbf{S}^1$. Then, $\tilde{f}$ is absolutely continuous with respect to $\boldsymbol{m}$.*

Proof. Assume that $\tilde{f}$ is not absolutely continuous. Then, there is a measurable set $A \subset \mathbf{S}^1$ such that $\boldsymbol{m}(A) = 0$ and $\boldsymbol{m}(\tilde{f}(A)) > 0$. The quasiconformal deformation $\Gamma' = \tilde{f}\Gamma\tilde{f}^{-1}$ is also a Fuchsian group of convergence type, because the property (2) is invariant under the deformation on a compact subset. Thus, we may apply Lemma 1 to Γ'. We have a simply connected subregion Ω' such that $\Omega' \cap {\pi'}^{-1}(f(K)) = \emptyset$, $\boldsymbol{m}(\overline{\Omega'} \cap \tilde{f}(A)) > 0$ and the ideal boundary has positive harmonic measure, where $\pi' : \Delta \to \Delta/\Gamma'$ is the universal cover. Consider the inverse image under $\tilde{f}$. Then so does the ideal boundary of $\tilde{f}^{-1}(\Omega')$ because $\tilde{f}^{-1}$ is conformal in Ω'. Therefore, $\overline{\tilde{f}^{-1}(\Omega')} \cap \mathbf{S}^1$ must be of positive measure, but this contradicts that $\boldsymbol{m}(A) = 0$. We have done the proof of Lemma 2.

Proposition 3. *The conditions (3) and (5) are equivalent.*

Proof. (3) $\Rightarrow$ (5): Suppose that h is a non-singular automorphism of the cicle at infinity compatible with Γ. Using the upper half plane model, we may assume that h is a strictly increasing function $\mathbf{R} \to \mathbf{R}$. Since the conical limit set of Γ is of full measure, there is a conical limit point x where the derivative of h is not zero. Without loss of generality, we assume $x = 0$. Let $\{\gamma_n\}_{n\in\mathbf{N}}$ be a sequence in Γ such that $\gamma_n(i)$ converges to 0 conically, and set $\lambda_n = Im\gamma_n(i)$. Then, the hyperbolic distance $d(\gamma_n i, \lambda_n i)$ is bounded from the above, hence there is a subsequence of $\{{\gamma_n}^{-1}\lambda_n(\cdot)\}$ which converges to a Möbius transformation. If $h'(0) = \delta > 0$, for each $x \in \mathbf{R}$,

$$\lim_{n\to\infty} \frac{h(\lambda_n x) - h(0)}{\lambda_n} = \delta x.$$

By compatibility, $h\gamma_n h^{-1}$ is Möbius, and so is

$$g_n(x) = \frac{h\gamma_n h^{-1}(x) - h(0)}{\lambda_n}.$$

Then,

$$g_n h\{\gamma_n^{-1}\lambda_n(x)\} = g_n \circ (h\gamma_n^{-1}h^{-1}) \circ h(\lambda_n x) = \frac{h(\lambda_n x) - h(0)}{\lambda_n},$$

where the right side converges to a Möbius transformation $\delta(x)$ and the subsequence of $\{\gamma_n^{-1}\lambda_n(x)\}$ to another one. Therefore, h must be Möbius.

(5′) $\Rightarrow$ (3): Let Γ be any non-elementary Fuchsian group of convergence type. Take a simple closed geodesic on the Riemann surface $W = \Delta/\Gamma$, and do quasiconformal deformation f of W so that the geodesic may be shorter, but conformally out of an annular neighborhood K of it. Then, Lemma 2 shows that the lift $\tilde{f}$ extends absolutely continuously to $\mathbf{S}^1$. And this $\tilde{f}$ is not a Möbius transformation because W and $f(W)$ are not conformally equivalent.

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