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On the topology of the space of representations

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Abstract

We give a proof that the set of discrete and faithful SL_2 representations of the fundamental group of a topologically finite Riemann surface is closed in the set of all SL_2 representations. This result is well known.

1 Introduction

If we treat Teichmüller space as Fricke moduli, we first study the set of discrete and faithful $SL_2(\mathbf{R})$ or $SL_2(\mathbf{C})$ representations of the fundamental group of a topologically finite Riemann surface. And making it as a topological space we consider it as a subspace of all $SL_2(\mathbf{R})$ or $SL_2(\mathbf{C})$ representations of the fundamental group of a topologically finite Riemann surface. Then it is well known (for example [MS]) that it is a closed subset. The key idea to prove this result is using so called Jørgensen's inequalities [J]. In this article we show a proof of this result in the elementary style. It should be remarked that if we consider the set of discrete and faithful representations as a subset of the more restricted space of representations, we can also show the openness of it for the case of $SL_2(\mathbf{R})$ representations by using the rigidity of the finite sided fundamental domains (for example [M]. for compact case see [W]). Fron this topological properties Teichmüller space of a topologically finite Riemann surface has the structure of a semi algebraic set.

2 Preliminaries

In this paper Γ means a non abelian free group of finite rank or surface group.

Assertion 1 Γ *is torsion free.*

Proof.
Because of the uniformization theorem, Γ can be considered as a discrete subgroup of $PSL_2(\mathbf{R})$ the analytic automorphism group of the upper half plane H and Γ acts on H fixed point freely. Hence any non identity element of Γ is hyperbolic or parabolic element of $PSL_2(\mathbf{R})$ and this shows that Γ is torsion free.

Assertion 2 *For any* $\alpha, \beta \in \Gamma$ *with* $\alpha\beta \neq \beta\alpha$, *put* $G := < \alpha, \beta >$ *the subgroup of* Γ *generated by* α *and* β. *Then* G *is a free group of rank two.*

Proof.
As in the proof of Assertion 1, we may assume that Γ is a discrete subgroup of $PSL_2(\mathbf{R})$ and acts on H fixed point freely. Then the quotient space $G\backslash H$ has the structure of a Riemann surface and its first homology group $H_1(G\backslash H, \mathbf{Z})$ is a quotient group of $\mathbf{Z}^2$. If $G \backslash H$ is compact, then $H_1(G \backslash H, \mathbf{Z})$ is isomorphic to $\mathbf{Z}^{2g}$ where $g\,(\geq 2)$ is the genus of this surface. Hence $G \backslash H$ is an open Riemann surface and its fundamental group G is a free group. Because G is generated by two elements which are not commutative, it is a free group of rank two.

Assertion 3 *The center of* Γ *is trivial.*

Proof.
Let γ be a center of Γ. If there exists a hyperbolic element α of Γ (where we take some realization $\Gamma \subset PSL_2(\mathbf{R})$), we may assume by conjugation that the representative of α in $SL_2(\mathbf{R})$ is a diagonal matrix. Then the assumption $\gamma\alpha = \alpha\gamma$ implies that γ has also a diagonal matrix as its representative. Since Γ is non abelian, there exists $\beta \in \Gamma$ whose representative is not diagonal and $\gamma\beta = \beta\gamma$ shows that γ must be an identity. Similar argument holds for the case that α is parabolic.

3 Results

Theorem 1 *A representation* $\rho : \Gamma \to SL_2(\mathbf{C})$ *is discrete (i.e.* $\rho(\Gamma) \subset SL_2(\mathbf{C})$ *is a discrete subgroup) and faithful (i.e.* ρ *is injective) if and only if the following inequalities hold:*

for any $\alpha, \beta \in \Gamma$ *with* $\alpha\beta \neq \beta\alpha$, *put* $A := \rho(\alpha)$ *and* $B := \rho(\beta)$. *Then* $A, B \in SL_2(\mathbf{C})$ *satisfy so called Jørgensen's inequality*

$$|tr^2 A - 4| + |tr[A,B] - 2| \geq 1.$$

Proof.
(if) We assume that ρ is not faithful. Then there exists $\alpha(\neq id.) \in \Gamma$ such that $A = \rho(\alpha) = E$ (identity matrix). As the center of Γ is trivial by Assertion 3, there exists $\beta \in \Gamma$ with $\alpha\beta \neq \beta\alpha$ and put $B = \rho(\beta)$. Then

$$|tr^2 A - 4| + |tr[A,B] - 2| = 0$$

a contradiction. Hence ρ is faithful. Next suppose that ρ is faithful but not discrete. Then there exists a sequence $(X_n)_{n \geq 1} \subset \rho(\Gamma)$ with $X_n \rightarrow E$ in $SL_2(\mathbf{C})$. Then for any $B \in \rho(\Gamma)$, $|tr^2 X_n - 4| \rightarrow 0$ and $|tr[X_n, B] - 2| \rightarrow 0$. Therefore there exists $N(B) \in \mathbf{N}$ depending only on B such that for $n \geq N(B)$

$$|tr^2 X_n - 4| + |tr[X_n, B] - 2| < 1.$$

Hence if there exists $n \geq N(B)$ such that $X_n B \neq B X_n$ then because of the faithfulness of ρ, we put $A = X_n$ and get a contradiction of the assumption. Therefore we may assume in the following that $X_n B = B X_n$ for all $n \geq N(B)$. If B is not parabolic, we may suppose that B is a diagonal matrix by conjugation. Then X_n is also diagonal. By Assertion 3, there exists $C \in \rho(\Gamma)$ such that $CB \neq BC$ in other words C is not diagonal, hence $CX_n \neq X_n C$ for any $n \geq N(B)$. Then there exists $N(C) \in \mathbf{N}$ depending only C such that for $n \geq N(C)$

$$|tr^2 X_n - 4| + |tr[X_n, C] - 2| < 1.$$

and $CX_n \neq X_n C$ which contradicts the assumption. Similar argument holds for the case that B is parabolic and we conclude that ρ is discrete and faithful.
(only if) Because ρ is faithful and by Assertion 2, $< A, B >$ is a free subgroup of $SL_2(\mathbf{C})$. Assume that

$$|tr^2 A - 4| + |tr[A,B] - 2| < 1$$

and put $B_0 := B$, $B_{n+1} := B_n A B_n^{-1}$ $(n = 0, 1, 2, \cdots)$. Then the completely same proof of Lemma 1 of [J] shows that B_{n+1} converges to A in $SL_2(\mathbf{C})$ but the discreteness of ρ means that $< A, B >$ is a discrete subgroup of $SL_2(\mathbf{C})$ hence $B_{n+1} = A$ for sufficiently large $n \in \mathbf{N}$. But this contradicts that $< A, B >$ is a free group of rank two.

Corollary 1 *A representation $\rho : \Gamma \to SL_2(\mathbf{R})$ is discrete and faithful if and only if the following inequalities hold:*
for any $\alpha, \beta \in \Gamma$ with $\alpha\beta \neq \beta\alpha$, put $A := \rho(\alpha)$ and $B := \rho(\beta)$. Then $A, B \in SL_2(\mathbf{R})$ satisfy so called Jørgensen's inequality

$$|tr^2 A - 4| + |tr[A, B] - 2| \geq 1.$$

Proof.
(if) We can use the same argument in the proof of the above Theorem to show the faithfulness of ρ. Then $\rho(\Gamma)$ is a non abelian subgroup of $SL_2(\mathbf{R})$, we can find a hyperbolic or parabolic element of $\rho(\Gamma)$ and we can prove the discreteness of ρ in the same way.
(only if) The natural inclusion $\mathbf{R} \subset \mathbf{C}$ induces the discrete and faithful representation

$$\rho : \Gamma \to SL_2(\mathbf{R}) \subset SL_2(\mathbf{C}).$$

Corollary 2 *The set of discrete and faithful $SL_2(\mathbf{C})$ (resp. $SL_2(\mathbf{R})$) representations of the fundamental group of a Riemann surface of topologically finite type is closed in the set of all $SL_2(\mathbf{C})$ (resp. $SL_2(\mathbf{R})$) representations.*

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